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ON COMMUTATIVE SUBIDEAL SERIES OF SEMIGROUPS

EDMUND SWYLAN

Abstract. A semigroup with a commutative subideal series strictly shorter than any of its commutative ideal series is exhibited.

We recall some well-known facts about groups. D. J. S. Robinson's textbook [2] is one of the many that can be used for reference.

Let G be a group and $n \in \mathbb{N}$. A finite sequence $(G_0, \ldots, G_n)$ of G's subgroups is said to be a subnormal series of G iff $G_0 = G$, $G_n = \mathbf{1}$ (the trivial group), and if $n \geq 1$ then for any $i \in \{0, \ldots, n-1\}$ G_{i+1} is a normal subgroup of G_i: $G_{i+1} \lhd G_i$. A subnormal series $(G_0, \ldots, G_n)$ of G is said to be normal iff $(\forall i \in \{0, \ldots, n\})(G_i \lhd G)$. A subnormal series $(G_0, \ldots, G_n)$ of G is said to be Abelian iff $n = 0$ or $n \geq 1$ and for any $i \in \{0, \ldots, n-1\}$ G_i/G_{i+1} is Abelian. If a group has an Abelian subnormal series, ie is soluble, it has also an Abelian normal series and, moreover, its shortest Abelian normal series are no longer than its shortest Abelian subnormal ones.

In [1] L. Martinov among other things substituted in the above cast semigroups for groups and (two-sided) ideals for normal subgroups.

Let S be a semigroup and $n \in \mathbb{N}$. A finite sequence $(S_0, \ldots, S_n)$ of S's subsemigroups is said to be a subideal series of S iff $S_0 = S$, $S_n = \emptyset$ or $S_n = \mathbf{1}$, and if $n \geq 1$ then for any $i \in \{0, \ldots, n-1\}$ S_{i+1} is an ideal of $S_i : S_{i+1} \blacktriangleleft S_i)$. A subideal series $(S_0, \ldots, S_n)$ of S is said to be ideal iff $(\forall i \in \{0, \ldots, n\})(S_i \blacktriangleleft S)$. A subideal series $(S_0, \ldots, S_n)$ of S is said to be commutative iff $n = 0$ or $n \geq 1$ and for any $i \in \{0, \ldots, n-1\}$ the Rees quotient S_i/S_{i+1} is commutative; by definition $T/\emptyset := T$ for any semigroup T. L. Martinov has shown that a semigroup has a commutative ideal series iff it has a commutative subideal one.

It occurred to us to see if it was possible to design a semigroup with a commutative subideal series strictly shorter than any of its commutative ideal series. It is and here is our construction.

Let S be the semigroup with the following multiplication table:

	0	1	2	3
0	0	1	2	3
1	0	1	2	3
2	2	2	2	3
3	3	3	2	3

$T := (\mathbb{N}_+, +)/\{13, 14, \ldots\}$, and $U := S \times T$. We write the U's elements down as

$$(0,1), \ldots, (0,13), (1,1), \ldots, (1,13),$$
$$(2,1), \ldots, (2,13), (3,1), \ldots, (3,13).$$

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We define a $\rho \subseteq U \times U$:

$$(\forall s, s' \in S)(\forall t, t' \in T)(((s,t),(s',t')) \in \rho :\Longleftrightarrow \\ (t = t' \ \& \ (s = s' \vee (\{s,s'\} = \{0,1\} \ \& \ t \in \{6,\dots,13\}) \vee \\ (\{s,s'\} = \{2,3\} \ \& \ t = 13)))).$$

ρ is a congruence on U. $V := U/\rho$. We write the V's elements down as

$$(0,1),\dots,(0,5),(1,1),\dots,(1,5)(\dagger,6),\dots,(\dagger,13), \\ (2,1),\dots,(2,12),(3,1),\dots,(3,12),(\ddagger,13).$$

We finally observe that $(2,1),(3,1) \in V - V^2$ and define a subsemigroup of V : $W := V - \{(2,1),(3,1)\}$.

For any semigroup X and any its subsemigroup Y

$$Y'_X := X^{\mathbf{1}}\{yy' : y, y' \in Y \ \& \ yy' \neq y'y\}X^{\mathbf{1}} \text{ and } Y' := Y'_Y.$$

The roles of these constructs are similar to that of derived subgroup. If a semigroup X has a commutative subideal series then $(X, X', (X')', \dots)$ ends in $\emptyset$ and, appropriately curtailed, becomes a shortest commutative subideal series of X, while $(X, X'_X, (X'_X)'_X, \dots)$ also ends in $\emptyset$ and yields a shortest commutative ideal series of X.

$$\begin{aligned}
W' = W'_W \quad &= \{(0,2),\dots,(0,5),(1,2),\dots,(1,5),(\dagger,6),\dots,(\dagger,13), \\
&\quad (2,4),\dots,(2,12),(3,4),\dots,(3,12),(\ddagger,13)\}, \\
(W')' \quad &= \{(0,4),(0,5),(1,4),(1,5),(\dagger,6),\dots,(\dagger,13), \\
&\quad (2,8),\dots,(2,12),(3,8),\dots,(3,12),(\ddagger,13)\}, \\
(W'_W)'_W \quad &= (W')' \cup \{(2,6),(2,7),(3,6),(3,7)\}, \\
((W')')' \quad &= \emptyset, \\
((W'_W)'_W)'_W \quad &= \{(2,12),(3,12),(\ddagger,13)\}.
\end{aligned}$$

References

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3 Vecsaules Street
Riga LV 1004
Latvia
E-mail address: Edmund.Swylan@ponymail.com