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ON SOME SEQUENCES DERIVED FROM THE POISSON DISTRIBUTION

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Abstract. In this note we give a new solution and some generalizations of the problem raised by Professor Zoltán László concerning the limit of the sequence $(a_n)_{n\geq 1}$ with $a_n = P(U_n \leq n)$, $n \in \mathbf{N}$ and $U_n \sim Po(n)$.

1. Introduction

In their paper [2], László and Vörös consider the sequence

$$a_n := \frac{\sum\limits_{i=0}^{n} \frac{n^i}{i!}}{e^n}$$

as a reformulation of the case $\theta = x$ in the following result concerning the Poisson distribution which can be found in the book of Feller [1] on p. 229 (or p. 288 for the Russian edition from 1967):

$$\lim_{\lambda\to\infty} e^{-\lambda\theta} \sum_{k\leq\lambda x} \frac{(\lambda\theta)^k}{k!} = \begin{cases} 0, & \text{if } \theta > x \\ 1, & \text{if } \theta < x, \end{cases} \qquad \forall\,\theta, x > 0. \tag{1}$$

They were the firsts to show that for $\theta = x$ the above limit is $\frac{1}{2}$. This problem was raised by Professor Zoltán László a few years ago. The proof in [2] uses analytical means.

More appropriate seems to be for such a problem the framework of classical theory of probability and we shall show that it is very easy to derive this limit from the Central Limit Theorem. In this way we can also give some other generalizations of this problem.

A reformulation of the result obtained by László and Vörös is the following

Theorem 1.1. *If $(U_n)_{n\geq 1}$ is a sequence of random variables on the field of probability $(\Omega, \mathcal{K}, P)$, with $U_n \sim Po(n)$, $\forall n \in \mathbf{N}$, then*

$$\lim_{n\to\infty} P(U_n \leq n) = \lim_{n\to\infty} \frac{\sum\limits_{i=0}^{n} \frac{n^i}{i!}}{e^n} = \frac{1}{2}. \tag{2}$$

Using the fact that

$$\frac{\sum\limits_{i=0}^{n} \frac{n^i}{i!}}{e^n} = 1 - \frac{1}{n!} \int\limits_0^n e^{-x} x^n dx, \tag{3}$$

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which is easy to obtain on integrating by parts, we see that the Gamma distribution has a similar property. Using this observation, we will give in the next section a result for the Gamma distribution which generalizes the above theorem. Finally, we extend this result to a larger class of distributions.

2. Main results

For $b \in \overline{\mathbf{R}} = \mathbf{R} \cup \{-\infty, \infty\}$ and $\beta > 0$, let us define the sequence $(c_n(b))_{n\geq 1}$ as follows:

- if $b \in \mathbf{R}$, then

$$c_n(b) = \frac{2(n-\beta) + b^2\beta - b\sqrt{\beta(b^2\beta + 4n)}}{2\beta},\ \forall n \in \mathbf{N}^*; \tag{4}$$

- if $b = \infty$, we take $c_n(\infty) = C_1 n + D_1$, $\forall n \geq 1$, where $C_1, D_1 \in \mathbf{R}$ with $0 < C_1 < \frac{1}{\beta}$;
- if $b = -\infty$, we take $c_n(-\infty) = C_2 n + D_2$, $\forall n \geq 1$, where $C_2, D_2 \in \mathbf{R}$ with $C_2 > \frac{1}{\beta}$;

We define now $(d_n(b))_{n\geq 1}$ to be the sequence given by

$$d_n(b) = [c_n(b)],\ \forall n \in \mathbf{N}^*, \tag{5}$$

where by [.] we denote the integer part of a real number. Obviously, $\lim\limits_{n\to\infty} c_n(b) = \infty$, for all $b \in \overline{\mathbf{R}}$, so there is an $n_0(b) \in \mathbf{N}^*$ such that for $n \geq n_0(b)$, $d_n(b) \in \mathbf{N}^*$. We will show that in fact we have

Theorem 2.1. *If $(U_n)_{n\geq n_0(b)}$ is a sequence of random variables on the field of probability $(\Omega, \mathcal{K}, P)$, with $U_n \sim Gamma(1 + d_n(b), \beta)$, $\forall n \geq n_0(b)$, then*

$$\lim_{n\to\infty} P(U_n \leq n) = \lim_{n\to\infty} \frac{1}{\beta^{1+d_n(b)} d_n(b)!} \int_0^n e^{-\frac{x}{\beta}} x^{d_n(b)} dx = \Phi(b), \tag{6}$$

for all $b \in \overline{\mathbf{R}}$, where $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{x} e^{-\frac{t^2}{2}} dt$, $\forall x \in \mathbf{R}$ is the Laplace function.

For $\beta = 1$ and $b = 0$ we reobtain the Theorem 1.1.

This result can be generalized to the case of other distributions $G(n)$, $n \in \mathbf{N}^*$, which have finite moments up to order 2 and for every $n \in \mathbf{N}^*$ the characteristic function of the distribution $G(n)$ is absolutely integrable on $\mathbf{R}$ and has the form $(\varphi(t))^n$, where $\varphi(t)$ is the characteristic function of $G(1)$ and we assume that it has the property that $\mu \stackrel{not}{=} \frac{\varphi'(0)}{i} \in (0, \sqrt{-\varphi''(0)})$.

In such cases, for all $b \in \overline{\mathbf{R}}$ we may consider the sequences defined in a similar way with $(c_n(b))_{n\geq 1}$ and $(d_n(b))_{n\geq 1}$:

- if $b \in \mathbf{R}$ then

$$\tilde{c}_n(b) = \frac{2n\mu + b^2\sigma^2 - b\sigma\sqrt{b^2\sigma^2 + 4n\mu}}{2\mu^2},\ \forall n \in \mathbf{N}^* \tag{7}$$

- if $b = \infty$, we take $\tilde{c}_n(\infty) = \tilde{C}_1 n + \tilde{D}_1$, $\forall n \geq 1$, where $\tilde{C}_1, \tilde{D}_1 \in \mathbf{R}$ with $0 < \tilde{C}_1 < \frac{1}{\mu}$;
- if $b = -\infty$, we take $\tilde{c}_n(-\infty) = \tilde{C}_2 n + \tilde{D}_2$, $\forall n \geq 1$, where $\tilde{C}_2, \tilde{D}_2 \in \mathbf{R}$ with $\tilde{C}_2 > \frac{1}{\mu}$;

and

$$\tilde{d}_n(b) = [\tilde{c}_n(b)],\ \forall n \in \mathbf{N}^*, \tag{8}$$

where $\sigma = \sqrt{(\varphi'(0))^2 - \varphi''(0)} > 0$. Obviously, $\lim_{n\to\infty} \tilde{c}_n(b) = \infty$, $\forall b \in \overline{\mathbf{R}}$, so there is an $\tilde{n}_0(b) \in \mathbf{N}^*$ such that for $n \geq \tilde{n}_0(b)$, $\tilde{d}_n(b) \geq 2$.

With these notations we have

Theorem 2.2. *If $(U_n)_{n\geq 1}$ is a sequence of random variables on the field of probability $(\Omega, \mathcal{K}, P)$, with $U_n \sim G(n)$, $\forall n \in \mathbf{N}^*$, then*

$$(9) \qquad \lim_{n\to\infty} P(U_{\tilde{d}_n(b)} \leq n) = \lim_{n\to\infty} \frac{1}{2\pi} \int_{-\infty}^{n} \int_{-\infty}^{\infty} e^{-itx} \varphi(t)^{\tilde{d}_n(b)} dt \; dx = \Phi(b),$$

for all $b \in \overline{\mathbf{R}}$.

Remark 2.3. The repartitions appearing in the Theorems 2.1 and 2.2 above are of continuous type. Of course, for discrete repartitions with properties similar to the above ones for $G(n)$, there also can be stated an analogous result to the Theorem 2.2., which generalizes in a natural way the Theorem 1.1 and in the final part of our paper we shall give another illustration of this result in the particular case of the binomial distribution.

3. Proofs

For $b \in \overline{\mathbf{R}}$ let us now define the sequence $(x_n(b))_{n\geq n_0(b)}$ by

$$(10) \qquad x_n(b) = \frac{n - \beta(1 + d_n(b))}{\beta\sqrt{1 + d_n(b)}}, \; \forall n \geq n_0(b).$$

We will need in the sequel the following

Lemma 3.1. *For the sequence $(x_n(b))_{n\geq 1}$ defined by (10) we have*

$$(11) \qquad b \leq x_n(b), \;\; \forall n \geq n_0(b) \; \textit{and} \; \lim_{n\to\infty} x_n(b) = b \; \textit{if} \;\; b \in \mathbf{R}$$

and

$$(12) \qquad \lim_{n\to\infty} x_n(b) = b \; \textit{if} \;\; b \in \{-\infty, \infty\}.$$

Proof. Using the fact that $x - 1 < [x] \leq x$, $\forall x \in \mathbf{R}$, we have that

$$(13) \qquad \frac{n - \beta(1 + c_n(b))}{\beta\sqrt{1 + c_n(b)}} \leq x_n(b) \leq \frac{n - \beta c_n(b)}{\beta\sqrt{c_n(b)}}, \; \forall n \geq n_0(b),$$

for all $b \in \overline{\mathbf{R}}$. If $b \in \{-\infty, \infty\}$ it is easy to see that

$$(14) \qquad \lim_{n\to\infty} \frac{n - \beta(1 + c_n(b))}{\beta\sqrt{1 + c_n(b)}} = \lim_{n\to\infty} \frac{n - \beta c_n(b)}{\beta\sqrt{c_n(b)}} = b$$

and if $b \in \mathbf{R}$ we have

$$(15) \qquad \frac{n - \beta(1 + c_n(b))}{\beta\sqrt{1 + c_n(b)}} = b, \; \forall n \geq n_0(b).$$

and

$$(16) \qquad \frac{n - \beta c_n(b)}{\beta\sqrt{c_n(b)}} = b\sqrt{1 + \frac{1}{c_n(b)}} + \frac{1}{\sqrt{c_n(b)}}, \; \forall n \geq n_0(b).$$

Thus we obtain the result by passing to the limit. □

Now we give the proof of the Theorem 2.1:

Proof. Let now $(X_n)_{n\geq 1}$ be a sequence of independent random variables with the distribution $X_n \sim Gamma(1,\beta)$, $\forall n \in \mathbf{N}^*$. We will denote by $S_n = \sum\limits_{i=1}^{n} X_i$, $\forall n \in \mathbf{N}^*$ and it is well known that $S_n \sim Gamma(n,\beta)$, $\forall n \in \mathbf{N}^*$.
Thus

$$P(\sum_{i=1}^{1+d_n(b)} X_i \leq n) = \frac{1}{\beta^{d_n(b)+1}\Gamma(1+d_n(b))}\int_0^n e^{-\frac{x}{\beta}} x^{d_n(b)} dx, \tag{17}$$

for all $n \geq n_0(b)$. From $X_n \sim Gamma(1,\beta)$ we know that $E(X_n) = \beta$ and $Var(X_n) = \beta^2$. It follows that $E(S_n) = n\beta$ and $Var(S_n) = n\beta^2$ and by the Central Limit Theorem we have that

$$\lim_{n\to\infty} P\Big(\frac{S_n - n\beta}{\beta\sqrt{n}} \leq x\Big) = \Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt, \tag{18}$$

for all real x.

We treat only the case where b is real. The cases $b = \pm\infty$ are analogous. Let now $\epsilon > 0$. Then from Lemma 3.1 it follows that there is an $n_1(b) \in \mathbf{N}$, $n_1(b) \geq n_0(b)$, such that $b \leq x_n(b) \leq b+\epsilon$, $\forall n \geq n_1(b)$, thus

$$\begin{aligned} &P\Big(\frac{S_{1+d_n(b)} - \beta(1+d_n(b))}{\beta\sqrt{1+d_n(b)}} \leq b\Big) \\ &\leq P\Big(\frac{S_{1+d_n(b)} - \beta(1+d_n(b))}{\beta\sqrt{1+d_n(b)}} \leq x_n(b)\Big) \\ &\leq P\Big(\frac{S_{1+d_n(b)} - \beta(1+d_n(b))}{\beta\sqrt{1+d_n(b)}} \leq b+\epsilon\Big), \quad \forall\, n \geq n_1(b) \end{aligned} \tag{19}$$

It follows that

$$\begin{aligned} \Phi(b) &= \lim_{n\to\infty} P\Big(\frac{S_{1+d_n(b)} - \beta(1+d_n(b))}{\beta\sqrt{1+d_n(b)}} \leq b\Big) \\ &\leq \liminf_{n\to\infty} P\Big(\frac{S_{1+d_n(b)} - \beta(1+d_n(b))}{\beta\sqrt{1+d_n(b)}} \leq x_n(b)\Big) \\ &\leq \limsup_{n\to\infty} P\Big(\frac{S_{1+d_n(b)} - \beta(1+d_n(b))}{\beta\sqrt{1+d_n(b)}} \leq x_n(b)\Big) \\ &\leq \lim_{n\to\infty} P\Big(\frac{S_{1+d_n(b)} - \beta(1+d_n(b))}{\beta\sqrt{1+d_n(b)}} \leq b+\epsilon\Big) = \Phi(b+\epsilon) \end{aligned} \tag{20}$$

and taking $\epsilon \to 0$ we have

$$\lim_{n\to\infty} P\Big(\frac{S_{1+d_n(b)} - \beta(1+d_n(b))}{\beta\sqrt{1+d_n(b)}} \leq x_n(b)\Big) = \Phi(b). \tag{21}$$

Observing now that

$$P\Big(\frac{S_{1+d_n(b)} - \beta(1+d_n(b))}{\beta\sqrt{1+d_n(b)}} \leq x_n(b)\Big) = P\Big(S_{1+d_n(b)} \leq n\Big), \tag{22}$$

we get the desired conclusion. □

The Theorem 2.2 can be proved in a way similar to the previous theorem:

Proof. We may consider in this case, too a sequence $(\tilde{x}_n(b))_{n\geq\tilde{n}_0(b)}$ defined by

$$\tilde{x}_n(b) = \frac{n - \mu\tilde{d}_n(b)}{\sigma\sqrt{\tilde{d}_n(b)}},\ \forall n \geq \tilde{n}_0(b),\ \forall b \in \overline{\mathbf{R}}, \tag{23}$$

which has similar properties with the sequence defined in (10), namely:

- $\frac{n-\mu\tilde{c}_n(b)}{\sigma\sqrt{\tilde{c}_n(b)}} \le \tilde{x}_n(b) \le \frac{n+\mu-\mu\tilde{c}_n(b)}{\sigma\sqrt{\tilde{c}_n(b)-1}}$, $\forall n \ge \tilde{n}_0(b)$, $\forall b \in \overline{\mathbf{R}}$.
- for $b = \pm\infty$ we have
$$\lim_{n\to\infty} \frac{n-\mu\tilde{c}_n(b)}{\sigma\sqrt{\tilde{c}_n(b)}} = \lim_{n\to\infty} \frac{n+\mu-\mu\tilde{c}_n(b)}{\sigma\sqrt{\tilde{c}_n(b)-1}} = b$$
- for $b \in \mathbf{R}$ we have
$$\frac{n-\mu\tilde{c}_n(b)}{\sigma\sqrt{\tilde{c}_n(b)}} = b, \forall n \ge \tilde{n}_0(b)$$
$$\frac{n+\mu-\mu\tilde{c}_n(b)}{\sigma\sqrt{\tilde{c}_n(b)-1}} = b\frac{1}{\sqrt{1-\frac{1}{\tilde{c}_n(b)}}} + \frac{\mu}{\sigma\sqrt{\tilde{c}_n(b)-1}}, \forall n \ge \tilde{n}_0(b).$$

Thus in this case we also get $\lim_{n\to\infty} \tilde{x}_n(b) = b$, $\forall b \in \overline{\mathbf{R}}$. With the same arguments as in the previous proof we obtain the conclusion. □

4. Concluding Remarks

If we choose $b = -\infty,\ C_2 = 2,\ D_2 = 0,\ \beta = 1$ in the Theorem 2.1, we obtain that

$$\lim_{n\to\infty} \frac{\sum_{i=0}^{2n} \frac{n^i}{i!}}{e^n} = 1 - \lim_{n\to\infty} \frac{1}{(2n)!} \int_0^n e^{-x} x^{2n} dx = 1, \tag{24}$$

and if we choose $b = \infty,\ C_1 = \frac{1}{2},\ D_1 = 0,\ \beta = 1$ we have

$$\lim_{n\to\infty} \frac{\sum_{i=0}^{\left[\frac{n}{2}\right]} \frac{n^i}{i!}}{e^n} = 1 - \lim_{n\to\infty} \frac{1}{\left[\frac{n}{2}\right]!} \int_0^n e^{-x} x^{\left[\frac{n}{2}\right]} dx = 0. \tag{25}$$

In fact, Theorem 2.1 expresses the fact that, by appropriately modifying the summation limit in the first sum above, one can obtain as limit any value in $[0, 1]$. This remark was first made in [2].

Let us now give an illustration of how this method works in the case of the binomial distribution.

Proposition 4.1. *For all $b \in \overline{\mathbf{R}}$ and $p \in (0, 1)$,*

$$\lim_{n\to\infty} \sum_{k=0}^{\min\{n,\tilde{d}_n(b)\}} \binom{\tilde{d}_n(b)}{k} p^k q^{\tilde{d}_n(b)-k} = \Phi(b), \tag{26}$$

where $q = 1 - p$ and $\tilde{d}_n(b)$ is given by

$$\tilde{d}_n(b) = [\tilde{c}_n(b)], \forall n \in \mathbf{N}^*$$

with $\tilde{c}_n(b)$ having in this case the following characterization

- *for $b \in \mathbf{R}$,*
$$\tilde{c}_n(b) = \frac{2n + b^2 q - b\sqrt{q(b^2q^2 + 4n)}}{2p}, \forall n \in \mathbf{N}^* \tag{27}$$
- *if $b = \infty$, we may choose $\tilde{c}_n(\infty) = \tilde{C}_1 n + \tilde{D}_1$, $\forall n \ge 1$, where $\tilde{C}_1, \tilde{D}_1 \in \mathbf{R}$ with $0 < \tilde{C}_1 < \frac{1}{p}$;*
- *if $b = -\infty$, we may choose $\tilde{c}_n(-\infty) = \tilde{C}_2 n + \tilde{D}_2$, $\forall n \ge 1$, where $\tilde{C}_2, \tilde{D}_2 \in \mathbf{R}$ with $\tilde{C}_2 > \frac{1}{p}$.*

In particular, from the above proposition we have

$$\lim_{n\to\infty} \sum_{k=0}^{\min\{n,\left[\frac{n}{2p}\right]\}} \binom{\left[\frac{n}{2p}\right]}{k} p^k q^{\left[\frac{n}{2p}\right]-k} = 1, \tag{28}$$

$$\lim_{n\to\infty} \sum_{k=0}^{n} \binom{\left[\frac{n}{p}\right]}{k} p^k q^{\left[\frac{n}{p}\right]-k} = \frac{1}{2}, \tag{29}$$

$$\lim_{n\to\infty} \sum_{k=0}^{n} \binom{\left[\frac{2n}{p}\right]}{k} p^k q^{\left[\frac{2n}{p}\right]-k} = 0, \tag{30}$$

for all $p \in (0,1)$.

Finally, such properties may be used in statistics, in order to find some estimates for the quantiles of the corresponding distributions.

References

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