Acta Mathematica Academiae Paedagogicae Nyíregyháziensis
18 (2002), 13–18
www.emis.de/journals

THE FEKETE - SZEGŐ THEOREM FOR CLOSE-TO-CONVEX FUNCTIONS OF THE CLASS $K_{sh}(\alpha, \beta)$

MASLINA DARUS

Abstract. For $0 \leq \alpha < 1$ and $0 < \beta \leq 1$. Let $K_{sh}(\alpha, \beta)$ be the class of normalized close-to-convex functions defined in the open unit disc D by

$$\left|\arg\left(\frac{zf'(z)}{g(z)}\right)\right| \leq \frac{\pi\alpha}{2},$$

such that $g \in S^*(\beta)$, the class of analytic normalized starlike functions of order β, i.e. for $z \in D$,

$$\Re\left(\frac{zg'(z)}{g(z)}\right) > \beta.$$

For $f \in K_{sh}(\alpha, \beta)$ and given by $f(z) = z + a_2z^2 + a_3z^3 + \cdots$, some sharp bounds are obtained for the Fekete - Szegő functional $|a_3 - \mu a_2^2|$ when μ is real.

1. Introduction

Let S denote the class of normalized analytic univalent functions f defined by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

for $z \in D = \{z : |z| < 1\}$. A classical theorem of Fekete and Szegő [4] states that for $f \in S$ given by (1.1),

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu, & \text{if} \quad \mu \leq 0, \\ 1 + 2\exp(\dfrac{-2\mu}{1-\mu}), & \text{if} \quad 0 \leq \mu \leq 1, \\ 4\mu - 3, & \text{if} \quad \mu \geq 1, \end{cases}$$

and that this is sharp.

Later, several authors attempted the related problems for either μ is complex or μ is real. For the subclasses $C, S^\star$ and K of convex, starlike and close-to-convex functions respectively, sharp upper bounds for the functional $\left|a_3 - \mu a_2^2\right|$ have been obtained for all real μ [5], [7], [6]. In particular for $f \in K$ and given by (1.1), Keogh and Merkes [5] showed that

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} 3 - 4\mu, & \text{if} \quad \mu \leq 0, \\ \dfrac{1}{3} + \dfrac{4}{9\mu}, & \text{if} \quad \dfrac{1}{3} \leq \mu \leq \dfrac{2}{3}, \\ 1, & \text{if} \quad \dfrac{2}{3} \leq \mu \leq 1, \\ 4\mu - 3, & \text{if} \quad \mu \geq 1, \end{cases}$$

2000 *Mathematics Subject Classification.* 30C45.
Key words and phrases. Fekete - Szegő theorem, close-to-convex functions, starlike, convex.

and that for each μ there is a function in K such that equality holds. In [9], Fekete - Szegő functional is obtained for close-to-convex function defined as follows.

Definition 1. *Let $0 \le \alpha < 1$, $0 < \beta \le 1$ and let f be given by (1.1). Then $f \in K_{hs}(\alpha, \beta)$ if and only if, there exist $g \in S_s^\star(\beta)$ such that for $z \in D$,*

$$Re\left(\frac{zf'(z)}{g(z)}\right) > \alpha$$

where $S_s^\star(\beta)$ denotes the class of starlike functions of order β defined in a sector, i.e. $g \in S_s^\star(\beta)$ if and only if, g is analytic in D with $g(z) = z + b_2z^2 + b_3z^3 + \cdots$ and

$$\left|\arg\left(\frac{zg'(z)}{g(z)}\right)\right| \le \frac{\beta\pi}{2}$$

for $z \in D$.

The authors [9] prove the following:

Theorem 1. *Let $f \in K_{hs}(\alpha, \beta)$ and be given by (1.1), then for $0 \le \alpha < 1$, $0 < \beta \le 1$ and μ real,*

$$3|a_3 - \mu a_2^2| = \begin{cases} 3\beta^2(1-\mu) - 3\mu(1-\alpha)^2 - 2(1-\alpha)(\beta(3\mu-2)-1), \\ \qquad \textit{if} \quad \mu \le \dfrac{2\beta}{3(1+\beta-\alpha)}, \\ 3\beta^2(1-\mu) + 2(1-\alpha) + \dfrac{\beta^2}{3\mu}(2-3\mu)^2, \\ \qquad \textit{if} \quad \dfrac{2\beta}{3(1+\beta-\alpha)} \le \mu \le \dfrac{4\beta}{3(1+\beta)}, \\ 2 - 2\alpha + \beta, \\ \qquad \textit{if} \quad \dfrac{4\beta}{3(1+\beta)} \le \mu \le \dfrac{4[\beta(2+\alpha)+1]}{3[\beta(3+\alpha)+1-\alpha]}, \\ 3\beta^2(\mu-1) + 2(1-\alpha) + \dfrac{\beta^2(1-\alpha)(3\mu-2)^2}{4-3\mu(1-\alpha)}, \\ \qquad \textit{if} \quad \dfrac{4[\beta(2+\alpha)+1]}{3[\beta(3+\alpha)+1-\alpha]} \le \mu \le \dfrac{2(2+\beta)}{3(1+\beta-\alpha)}, \\ 3\beta^2(\mu-1) + 3\mu(1-\alpha)^2 + 2(1-\alpha)(\beta(3\mu-2)-1), \\ \qquad \textit{if} \quad \mu \ge \dfrac{2(2+\beta)}{3(1+\beta-\alpha)}. \end{cases}$$

For each μ, there is a function $f \in K_{hs}(\alpha, \beta)$ such that equality holds.

In this paper, we look into the class $K_{sh}(\alpha, \beta)$ defined as the following:

Definition 2. *Let $0 < \alpha \le 1$, $0 \le \beta < 1$ and let f be given by (1.1). Then $f \in K_{sh}(\alpha, \beta)$ if and only if, there exist $g \in S_h^\star(\beta)$ such that for $z \in D$,*

$$\left|\arg\left(\frac{zf'(z)}{g(z)}\right)\right| \le \frac{\pi\alpha}{2} \tag{1.2}$$

where $S_h^\star(\beta)$ denotes the class of starlike functions of order β defined in a half plane, i.e. $g \in S_h^\star(\beta)$ if and only if, g is analytic in D with $g(z) = z + b_2z^2 + b_3z^3 + \cdots$ and

$$\Re\left(\frac{zg'(z)}{g(z)}\right) > \beta \tag{1.3}$$

for $z \in D$.

2. Result

We prove the following:

Theorem 2. *Let* $f \in K_{sh}(\alpha,\beta)$ *and be given by (1.1), then for* $0 < \alpha \leq 1$, $0 \leq \beta < 1$ *and* μ *real,*

$$3|a_3 - \mu a_2^2| = \begin{cases} 1-\beta+(2-3\mu)(1+\alpha-\beta)^2, \\ \qquad \text{if} \quad \mu \leq \dfrac{2(\alpha-\beta)}{3(1+\alpha-\beta)}, \\ (1-\beta)(3-2\beta-3\mu(1-\beta))+2\alpha+\dfrac{\alpha(1-\beta)^2(2-3\mu)^2}{2-\alpha(2-3\mu)}, \\ \qquad \text{if} \quad \dfrac{2(\alpha-\beta)}{3(1+\alpha-\beta)} \leq \mu \leq \dfrac{2}{3}, \\ 1+2\alpha-\beta, \\ \qquad \text{if} \quad \dfrac{2}{3} \leq \mu \leq \dfrac{2(2+\alpha-\beta)}{3(1+\alpha-\beta)}, \\ \beta-1+(3\mu-2)(1+\alpha-\beta)^2, \\ \qquad \text{if} \quad \mu \geq \dfrac{2(2+\alpha-\beta)}{3(1+\alpha-\beta)}. \end{cases}$$

For each μ*, there is a function* $f \in K_{sh}(\alpha,\beta)$ *such that equality holds.*

We first state simple lemmas which we shall use throughout the paper.

Lemma 1. *(*[8, p. 166.]*) Let* $h \in P$ *i.e.* h *be analytic in* D *and be given by*

$$h(z) = 1 + c_1 z + c_2 z^2 + \cdot.$$

and $\Re h(z) > 0$ *for* $z \in D$*, then* $|c_n| \leq 2$ *and*

$$\left|c_2 - \frac{c_1^2}{2}\right| \leq \left(2 - \frac{|c_1|^2}{2}\right).$$

Lemma 2. *(*[2]*) For* $0 \leq \beta < 1$*, let* $g \in S_h^\star(\beta)$ *and*

$$g(z) = z + b_2 z^2 + b_3 z^3 + \cdots .$$

Then for μ *real,*

$$\left|b_3 - \frac{3}{4}\mu b_2^2\right| \leq (1-\beta)\max\{1, |3-2\beta-4\mu(1-\beta)|\}.$$

Lemma 3. *Let* $f \in K_{sh}(\alpha,\beta)$ *and be given by (1.1), then*

$$|a_2| \leq 1+\alpha-\beta,$$

and

$$3|a_3| \leq 2\alpha^2 + (4\alpha+3-2\beta)(1-\beta).$$

Proof. Since $g \in S_h^\star(\beta)$, it follows from (1.3) that

$$zg'(z) = g(z)[p(z)(1-\beta)+\beta] \tag{2.4}$$

for $z \in D$, with $p \in P$given by $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$. Equating coefficients, we obtain,

$$b_2 = (1-\beta)p_1, \tag{2.5}$$

and

$$2b_3 = (1-\beta)p_2 + (1-\beta)b_2 p_1. \tag{2.6}$$

Also it follows from (1.2) that

$$zf'(z) = g(z)h(z)^\alpha \tag{2.7}$$

where $h \in P$. Writing $h(z) = 1 + c_1 z + c_2 z^2 + \cdots$ and equating coefficients in (2.7) we have

$$2a_2 = b_2 + c_1\alpha, \tag{2.8}$$

and

$$3a_3 = b_3 + c_2\alpha + c_1 b_2 \alpha + \frac{\alpha}{2}(\alpha - 1)c_1^2. \tag{2.9}$$

The result now follows on using the classical inequalities $|c_1| = |c_2| \leq 2$, $|p_1| = |p_2| \leq 2$, and the inequalities $|b_2| \leq 2(1-\beta)$ and $|b_3| \leq (1-\beta)(3-2\beta)$ which follow from (2.5) and (2.6). □

Proof. It follows from (2.5),(2.7),(2.8) and (2.9) that

$$3\left(a_3 - \mu a_2^2\right) = \left(b_3 - \frac{3}{4}\mu b_2^2\right) + \alpha\left(c_2 - \frac{c_1^2}{2}\right) + \frac{\alpha^2}{4}(2-3\mu)c_1^2 + \frac{\alpha}{2}(2-3\mu)c_1 b_2. \tag{2.10}$$

And so equation (2.10) gives

$$3|a_3 - \mu a_2^2| \leq \left|b_3 - \frac{3}{4}\mu b_2^2\right| + \alpha\left|c_2 - \frac{c_1^2}{2}\right| + \frac{1}{4}\alpha|2-3\mu||c_1^2| + \frac{\alpha}{2}|2-3\mu||c_1||b_2|. \tag{2.11}$$

We first consider the case $\dfrac{2(\alpha-\beta)}{3(1+\alpha-\beta)} \leq \mu \leq \dfrac{2}{3}$. Equation (2.11) gives

$$\begin{aligned}
3|a_3 - \mu a_2^2| &\leq (1-\beta)(3-2\beta-3\mu(1-\beta)) + \alpha\left(2 - \frac{c_1^2}{2}\right) + \frac{1}{4}\alpha^2(2-3\mu)|c_1^2| \\
&\quad + \frac{\alpha}{2}(2-3\mu)|c_1||b_2| \\
&\leq (1-\beta)(3-2\beta-3\mu(1-\beta)) + \alpha\left(2 - \frac{c_1^2}{2}\right) + \frac{1}{4}\alpha^2(2-3\mu)c_1^2 \\
&\quad + \alpha(1-\beta)(2-3\mu)|c_1| \\
&= \Upsilon(x) \quad \text{say,} \quad \text{with } x = |c_1|,
\end{aligned}$$

where we have used Lemmas 1 and 2 and the fact that $|b_2| \leq 2(1-\beta)$ for $g \in S_h^*(\beta)$. An elementary argument shows that the function Υ attains a maximum at $x_0 = \dfrac{2(1-\beta)(2-3\mu)}{2-\alpha(2-3\mu)}$, and so $|a_3 - \mu a_2^2| \leq \Upsilon(x_0)$, which proves the theorem if $\mu \leq \dfrac{2}{3}$ and $\alpha \geq 0$. Choosing $c_1 = \dfrac{2(1-\beta)(2-3\mu)}{2-\alpha(2-3\mu)}$, $c_2 = 2$, $b_2 = 2(1-\beta)$ and $b_3 = (1-\beta)(3-2\beta)$ in (2.10) shows that the result is sharp. We note that $|c_1| \leq 2$, i.e. $\mu \geq \dfrac{2(\alpha-\beta)}{3(1+\alpha-\beta)}$.

Next consider the case $\mu \leq \dfrac{2(\alpha-\beta)}{3(1+\alpha-\beta)}$. Then

$$\begin{aligned}
|a_3 - \mu a_2^2| &\leq \left|a_3 - \frac{2(\alpha-\beta)}{3(1+\alpha-\beta)}a_2^2\right| + \left(\frac{2(\alpha-\beta)}{3(1+\alpha-\beta)} - \mu\right)|a_2|^2, \\
&\leq \frac{3+2\alpha-3\beta}{3} + \left(\frac{2(\alpha-\beta)}{3(1+\alpha-\beta)} - \mu\right)(1+\alpha-\beta)^2, \\
&= \frac{1-\beta}{3} + \frac{(2-3\mu)}{3}(1+\alpha-\beta)^2.
\end{aligned}$$

for $\alpha \geq 0$, where we have used the result already proved in the case $\mu = \dfrac{2(\alpha-\beta)}{3(1+\alpha-\beta)}$, and the fact that for $f \in K_{sh}(\alpha,\beta)$, the inequality $|a_2| \leq 1+\alpha-\beta$ holds. Equality is attained on choosing $c_1 = c_2 = 2$, $b_2 = 2(1-\beta)$ and $b_3 = (1-\beta)(3-2\beta)$ in (2.10).

Suppose now that $\frac{2}{3} \leq \mu \leq \frac{2(2+\alpha-\beta)}{3(1+\alpha-\beta)}$. Since $g \in S_h^*(\beta)$ we can write $zg'(z) = g(z)[\beta + (1-\beta)p(z)]$ for $p \in P$, with $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$, and so equating coefficients we have that $b_2 = p_1(1-\beta)$ and $2b_3 = (1-\beta)p_2 + (1-\beta)^2 p_1^2$.

We deal first with the case $\mu = \frac{2(2+\alpha-\beta)}{3(1+\alpha-\beta)}$. Thus (2.10) gives

$$a_3 - \frac{2(2+\alpha-\beta)}{3(1+\alpha-\beta)}a_2^2 = \frac{1}{6}(1-\beta)\left(p_2 - \frac{p_1^2}{2}\right) + \frac{\alpha}{3}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{(1-\beta)(\alpha-1+\beta)}{12(1+\alpha-\beta)}p_1^2$$
$$- \frac{\alpha(1-\beta)}{3(1+\alpha-\beta)}p_1 c_1 - \frac{\alpha^2}{6(1+\alpha-\beta)}c_1^2,$$

and so if $\alpha + \beta \leq 1$,

$$\left|a_3 - \frac{2(2+\alpha-\beta)}{3(1+\alpha-\beta)}a_2^2\right| \leq \frac{1}{6}(1-\beta)\left|p_2 - \frac{p_1^2}{2}\right| + \frac{\alpha}{3}\left|c_2 - \frac{c_1^2}{2}\right| + \frac{(1-\beta)(1-\alpha-\beta)}{12(1+\alpha-\beta)}|p_1|^2$$
$$+ \frac{\alpha(1-\beta)}{3(1+\alpha-\beta)}|p_1||c_1| + \frac{\alpha^2}{6(1+\alpha-\beta)}|c_1|^2,$$
$$\leq \frac{1}{6}(1-\beta)\left(2 - \frac{p_1^2}{2}\right) + \frac{\alpha}{3}\left(2 - \frac{c_1^2}{2}\right) + \frac{(1-\beta)(1-\alpha-\beta)}{12(1+\alpha-\beta)}|p_1|^2$$
$$+ \frac{\alpha(1-\beta)}{3(1+\alpha-\beta)}|p_1||c_1| + \frac{\alpha^2}{6(1+\alpha-\beta)}|c_1|^2,$$
$$= \frac{1+2\alpha-\beta}{3} - \frac{\alpha(1-\beta)}{6(1+\alpha-\beta)}(|p_1| - |c_1|)^2,$$
$$\leq \frac{1+2\alpha-\beta}{3},$$

where we have used Lemma 1.

Now write

$$a_3 - \mu a_2^2 = \frac{(1+\alpha-\beta)(3\mu-2)}{2}\left(a_3 - \frac{2(2+\alpha-\beta)}{3(1+\alpha-\beta)}a_2^2\right)$$
$$+ \frac{3(1+\alpha-\beta)}{2}\left(\frac{2(2+\alpha-\beta)}{3(1+\alpha-\beta)} - \mu\right)(a_3 - \frac{2}{3}a_2^2),$$

and the result follows at once on using the theorem already proved in the cases $\mu = \frac{2(2+\alpha-\beta)}{3(1+\alpha-\beta)}$ and $\mu = \frac{2}{3}$ for $\alpha + \beta \leq 1$. Equality is attained when f is given by

$$f'(z) = \frac{(1+z^2)^\alpha}{(1-z^2)^{1+\alpha-\beta}}.$$

We finally assume that $\mu \geq \frac{2(2+\alpha-\beta)}{3(1+\alpha-\beta)}$. Write

$$a_3 - \mu a_2^2 = \left(a_3 - \frac{2(2+\alpha-\beta)}{3(1+\alpha-\beta)}a_2^2\right) + \left(\frac{2(2+\alpha-\beta)}{3(1+\alpha-\beta)} - \mu\right)a_2^2,$$

and the result follows at once on choosing the theorem already proved for $\mu = \frac{2(2+\alpha-\beta)}{3(1+\alpha-\beta)}$ and the inequality $|a_2| \leq 1 + \alpha - \beta$, which was proved in Lemma 3. Equality is attained on choosing $c_1 = 2i$, $c_2 = -2$, $b_2 = 2i(1-\beta)$ and $b_3 = -(1-\beta)(3-2\beta)$ in (2.10).

We remark that whenever $\beta = 0$ the theorem reduces to [1]. We also note that [3] and [2] give a complete result for both $\frac{zf'(z)}{g(z)}$ and $\frac{zg'(z)}{g(z)}$ defined in a sector and both defined in a half plane respectively. □

References

[1] H.R. Abdel-Gawad and D.K. Thomas. On the Fekete - Szego theorem for strongly close-to-convex functions. *Proc. Amer. Math. Soc.*, 114(2):345–349, 1992.

[2] M. Darus and D.K. Thomas. On the Fekete - Szego theorem for close-to-convex functions. *Math. Japonica*, 44(3), 1996.

[3] M. Darus and D.K. Thomas. he Fekete - Szego theorem for strongly close-to-convex functions. *Scientiae Mathematicae*, 3:201–212, 2000.

[4] M. Fekete and G. Szego. Eine Bermerkung ber ungerade schlichte Funktionen. *J. Lond. Math. Soc.*, 8:85–89, 1933.

[5] F.R. Keogh and E.P. Merkes. A coefficient inequality for certain classes of analytic functions. *Proc. Amer. Math. Soc.*, 20:8–12, 1969.

[6] W. Koepf. On the Fekete - Szego problem for close-to-convex functions. *Proc. Amer. Math. Soc.*, 101:89–95, 1987.

[7] W. Koepf. On the Fekete - Szego problem for close-to-convex functions ii. *Arch. Math.*, 49:490–533, 1987.

[8] Ch. Pommerenke. *Univalent functions*. Vandenhoeck and Ruprecht, Göttingen, 1975.

[9] K.A. Rahman and M. Darus. The Fekete - Szego theorem for close-to-convex functions of the class $k_{hs}(\alpha, \beta)$. *Inst. Jour. Math & Comp Sci (Math Ser)*, 14(3):171–177, 2001.

Received November 05, 2001.

School of Mathematical Sciences,
Faculty of Science and Technology,
Universiti Kebangsaan Malaysia, Bangi 43600
Selangor Darul Ehsan, Malaysia
E-mail address: maslina@pkrisc.cc.ukm.my