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ON THE PERIOD OF SEQUENCES IN CL_n

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Abstract. In this paper we investigate the period of 2-step sequences and 3-step sequences in CL_n, the chain with n elements.

1. Introduction

The study of Fibonacci sequences in groups began with the earlier work of Wall [14] where the ordinary Fibonacci sequences in cyclic groups were investigated. In the mid eighties Wilcox extended the problem to Abelian groups [15]. Prolific cooperation of Campbell, Doostie and Robertson expanded the theory to some finite simple groups [4]. Aydın and Smith proved in [2] that the lengths of ordinary 2-step Fibonacci sequences are equal to the lengths of ordinary 2-step Fibonacci recurrences in finite nilpotent groups of nilpotency class 4 and a prime exponent. The theory has been generalized in [5,6,11] to the 3-step Fibonacci sequences in finite nilpotent groups of nilpotency class $2,3,n$ and exponent p, respectively. Then it is shown in [1] that the period of 2-step general Fibonacci sequence is equal to the length of fundamental period of the 2-step general recurrence constructed by two generating elements of the group of exponent p and nilpotency class 2. In the recent years, there has been much interest in applications of Fibonacci numbers and sequences. Karaduman and Aydın obtained 2-step General Fibonacci sequences in finite nilpotent groups of nilpotency class 4 and exponent p [8]. Karaduman and Yavuz proved that the periods of the 2-step Fibonacci recurrences in finite nilpotent groups of nilpotency class 5 and a prime exponent are $p \cdot k(p)$, for $2 < p \leq 2927$, where p is prime and $k(p)$ is the periods of ordinary 2-step Fibonacci sequences [9].

A k-nacci sequence in a finite group is a sequence of group elements

$$x_0, x_1, x_2, \dots, x_n, \dots$$

for which, given an initial (seed) set $x_0, x_1, x_2, \dots, x_{j-1}$, each element is defined by

$$x_n = \begin{cases} x_0 x_1 x_2 \cdots x_{n-1} & \text{for } j \leq n < k \\ x_{n-k} x_{n-k+1} \cdots x_{n-1} & \text{for } n \geq k \end{cases}. \tag{1}$$

We also require that the initial elements of the sequence,

$$x_0, x_1, x_2, \dots, x_{j-1},$$

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generate the group, thus forcing the k-nacci sequence to reflect the structure of the group. The k-nacci sequence of a group generated by $x_0, x_1, x_2, \ldots, x_{j-1}$ is denoted by $F_k(G; x_0, x_1, \ldots, x_{j-1})$.

2-step Fibonacci sequence in the integers modulo m can be written as

$$F_2(Z_m; 0, 1).$$

We call a 2-step Fibonacci sequence of a group elements a Fibonacci sequence of a finite group. A finite group G is k-nacci sequenceable if there exists a k-nacci sequence of G such that every element of the group appears in the sequence.

A sequence of group elements is periodic if, after a certain point, it consists only of repetitions of a fixed subsequence. The number of elements in the repeating subsequence is called period of the sequence. For example, the sequence $a, b, c, d, e, b, c, d, e, b, c, d, e, \cdots$ is periodic after the initial element a and has period 4. A sequence of group elements is simply periodic with period k if the first k elements in the sequence form a repeating subsequence. For example, the sequence $a, b, c, d, e, f, a, b, c, d, e, f, a, b, c, d, e, f, \cdots$ is simply periodic with period 6.

Semigroup presentations have been studied over a long period, usually as a means of providing examples of semigroups. In [10], B.H. Neumann introduced an enumeration method for finitely presented semigruops analogous to the Todd-Coxeter coset enumeration process for group [13]. For about semigroup presentations see [12].

Let p denote the period of sequences in CL_n, which is a commutative semigroup with n elements, where $n \in N$. In this paper we prove that the period of 2-step sequences in CL_n is

$$p = (n-2)n + 1$$

and the period of 3-step sequences in CL_n is

$$p = \begin{cases} \left[\left|\frac{n}{2} - 1\right|\right] n + 1, & \text{if } n \text{ is even} \\ \\ \left[\left|\frac{n}{2} - 1\right|\right] n + 2, & \text{if } n \text{ is odd} \end{cases}$$

where $\left[\left|\frac{n}{2} - 1\right|\right]$ is the integer part of $|\frac{n}{2} - 1|$. Let A be an alphabet. We denote by A^+ the free semigroup on A consisting of all non-empty words over A. A *semigroup presentation* is an ordered pair of $< A \mid R >$, where $R \subseteq A^+XA^+$ A semigroup S is said to be *defined* by the semi group presentation $< A \mid R >$ if S is isomorphic to A^+/ρ, where ρ is the congruence on A^+ generated by R. Let u and v be two words in A^+. We write $u \equiv v$ if u and v are identical words, and write $u = v$ if $(u, v) \in \rho$, that is v is obtained from u by applying relations from R, or equivalently there is a finite sequence

$$u \equiv \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n \equiv v$$

of words from A^+in which every α_i is obtained from α_{i-1} by applying a relation from R(see [7, Proposition 1.5.9]). If both A and R are finite sets then $< A \mid R >$ is said to be a *finite presentation.* If a semigroup S can be defined by a finite presentation then S is said to be *finitely presented.*

Let $Y_n = \{y_1, y_2, y_3, \ldots, y_n\}$and let $CL_n = \{Y_1, Y_2, Y_3, \ldots Y_n\}$. Consider the set-theorical union $\cup$ as a binary operation. With respect to this operation, CL_n is a commutative semigroup of idempotents, and Y_n is the zero element of CL_n. We call CL_n the *chain of order n.*

Now, we give the following Theorem giving information about the *presentation* of CL_n.

Theorem 1. *The presentation*

$$P_n =< a_1, a_2, a_3, \dots, a_n | a_1^2 = a_1, a_i a_{i+1}^2 a_i = a_{i+1} (1 \leq i \leq n-1) >$$

defines the chain CL_n of order n and we have $a_i a_j = a_j$ and $a_j a_i = a_j$, for $1 \leq i < j \leq n$.

Proof. Let ϕ be the homomorphism from H_n, the semigroup defined by P_n, into CL_n defined by $a_i \to Y_i$. It is clear that ϕ is onto, and so CL_n is homomorphic image of H_n. Now we show that the order of H_n is n.

From the relations $a_1 a_2^2 a_1 = a_2$ and $a_1^2 = a_1$, we have

$$a_1 a_2 = a_1(a_1 a_2^2 a_1) = a_1 a_2^2 a_1 = a_2$$

and

$$a_2 a_1 = (a_1 a_2^2 a_1) a_1 = a_1 a_2^2 a_1 = a_2.$$

It follows that $a_2^2 = (a_1 a_2)(a_2 a_1) = a_2$ If we continue inductively, we obtain the followings:

$$a_i a_{i+1} = a_{i+1}, a_{i+1} a_i = a_{i+1}$$

and

$$a_{i+1}^2 = a_{i+1} (1 \leq i \leq n-1).$$

For $1 \leq i < j \leq n$, we show that $a_i a_j = a_j$ and $a_j a_i = a_j$. For this end, we use induction on $j - i$. For $j - i = 1$, we have just shown. Assume that, for $j - i = k$, we have $a_i a_{i+k} = a_{i+k}$. For $j - i = k + 1$, it follows from

$$a_{i+k} a_{(i+k)+1} = a_{(i+k)+1}$$

that

$$a_i a_j \equiv a_i a_{i+k+1} = a_i (a_{i+k} a_{i+k+1}) \equiv (a_i a_{i+k}) a_{i+k+1} = a_{i+k} a_{i+k+1} = a_{i+k+1} \equiv a_j$$

as required. Similarly, we show that $a_j a_i = a_j$, for $1 \leq i < j \leq n$, as follow.

For this again, we use induction on $j - i$. For $j - i = 1$, we have just shown. Assume that, for $j - i = k$, we have $a_j a_i = a_{i+k} a_i = a_{i+k}$. For $j - i = k + 1$, it follows from $a_{(i+k)+1} a_{i+k} = a_{(i+k)+1}$ that

$$a_j a_i \equiv a_{i+k+1} a_i = (a_{i+k} a_{i+k+1}) a_i \equiv a_{i+k+1} (a_{i+k} a_i) = a_{i+k+1} a_{i+k} = a_{i+k+1} \equiv a_j$$

as required.

Therefore, for every word $w \in A^+$ where $A = \{a_1, a_2, a_3, \dots a_n\}$, there exists a generator $a_i \in A$ such that the relation $w = a_i$ holds in the semigroup H_n defined by P_n, and hence the order of H_n is n. Therefore P_n defines CL_n.

The same proof of this Theorem has been given in [3]. □

If we define the sequences in CL_n as in formula (1), it is clear that the sequences is periodic and $p = n$. Now we define 2-step sequences in CL_n as $x_i = x_{i-n} x_{i-(n-1)}$ and 3-step sequences in CL_n as $x_i = x_{i-n} x_{i-(n-1)} x_{i-(n-2)}$, for $i > n$.

Theorem 2. *Let*

$$P_n =< a_1, a_2, a_3, \dots, a_n \mid a_1^2 = a_1, a_i a_{i+1}^2 a_i = a_{i+1} (1 \leq i \leq n-1) >$$

be presentation of CL_n.

i. *2-step sequences in CL_n is periodic and the period of the sequence is equal to*

$$p = (n-2)n + 1,$$

ii. *3-step sequences in CL_n is periodic and the period of the sequence is equal to*

$$p = \begin{cases} \left[\left|\frac{n}{2} - 1\right|\right] n + 1, & \text{if } n \text{ is even} \\ \left[\left|\frac{n}{2} - 1\right|\right] n + 2, & \text{if } n \text{ is odd} \end{cases}$$

Proof. **i.** The first n terms of sequence are $a_1, a_2, a_3, \ldots, a_n$. For simplicity, we use indices instead of generating elements of CL_n in our process. Since $x_i = x_{i-n}x_{i-(n-1)}$, for $i > n$, we have

$$\begin{aligned} x_{n+1} &= x_2 = 2, \\ x_{n+2} &= x_3 = 3, \\ x_{n+3} &= x_4 = 4, \\ &\vdots \\ x_{n+n-1} = x_n &= x_{2n-1} = n, \\ x_{2n} &= x_n = n, \\ x_{2n+1} &= x_3 = 3, \\ &\vdots \\ x_{3n+1} &= x_4 = 4, \\ &\vdots \\ x_{(n-2)n+1} &= x_{n-1} = n-1, \\ x_{(n-2)n+2} &= x_n = n, \end{aligned}$$

from defining relations in CL_n. It follows that $x_j = n$ for $l.n - (l-1) \leq j \leq l.n$, where $1 \leq l \leq (n-2)$. We also have $x_j = n$ and $x_{l.n+1} = x_{(l-1)n+2}$ Since the elements succeeding

$$x_{(n-2)n+1}, x_{(n-2)n+2},$$

depend on $n-1, n$ for their values, we have

$$x_{(n-2)n+m} = x_n = n$$

for $m > 1$. So, 2-step sequences in CL_n is periodic and the period of the sequence is equal to

$$p = (n-2)n + 1.$$

ii. The first n terms of sequence are $a_1, a_2, a_3, \ldots, a_n$. For simplicity, we use indices instead of generating elements of CL_n in our process. It is clear that the period of the sequence is 2 when $n = 2$. Firstly, we consider the case of n is even, $n > 2$. Since $x_i = x_{i-n}x_{i-(n-1)}x_{i-(n-2)}$, for $i > n$, we have

$$x_{n+1} = \prod_{j=n+1-n}^{n+1-(n-2)} x_j = x_3 = 3,$$

$$x_{n+2} = \prod_{j=n+1-(n-1)}^{n+1-(n-3)} x_j = x_4 = 4,$$

$$\vdots$$

$$x_{2n-2} = \prod_{j=n-2}^{n} x_j = x_n = n,$$

$$x_{2n-1} = \prod_{j=n-1}^{n+1} x_j = x_n = n,$$

$$x_{2n} = \prod_{j=n}^{n+2} x_j = x_n = n,$$

$$x_{2n+1} = \prod_{j=n+1}^{n+3} x_j = x_{n+3} = 5,$$

$$x_{2n+2} = \prod_{j=n+2}^{n+4} x_j = x_{n+4} = 6,$$

$$x_{2n+3} = \prod_{j=n+3}^{n+5} x_j = x_{n+5} = 7,$$

$$\vdots$$

$$x_{3n} = \prod_{j=2n}^{2n+2} x_j = x_{2n+2} = n,$$

$$x_{3n+1} = \prod_{j=2n+1}^{2n+3} x_j = x_{2n+3} = 7,$$

$$\vdots$$

$$x_{[|\frac{n}{2}-1|]n} = \prod_{j=[|\frac{n}{2}-1|]n-n}^{[|\frac{n}{2}-1|]n-(n-2)} x_j = x_n = n,$$

$$x_{[|\frac{n}{2}-1|]n+1} = \prod_{j=[|\frac{n}{2}-1|]n-(n-1)}^{[|\frac{n}{2}-1|]n-(n-3)} x_j = x_{n-1} = n-1,$$

$$x_{[|\frac{n}{2}-1|]n+2} = \prod_{j=[|\frac{n}{2}-1|]n-(n-2)}^{[|\frac{n}{2}-1|]n-(n-4)} x_j = x_n = n,$$

from defining relations in CL_n. Since the elements succeeding

$$x_{[|\frac{n}{2}-1|]n}, x_{[|\frac{n}{2}-1|]n+1}, x_{[|\frac{n}{2}-1|]n+2}, \dots,$$

depend on $n, n-1$, and n for their values, we have

$$x_{[|\frac{n}{2}-1|]n+m} = x_n = n$$

for $m > 1$. So, 3-step sequences in CL_n is periodic and the period of the sequence is equal to

$$p = \left[\left|\frac{n}{2} - 1\right|\right] n + 1$$

when n is even. Now we consider the case of n is odd. Since

$$x_i = x_{i-n} x_{i-(n-1)} x_{i-(n-2)}$$

for $i > n$, we have

$$x_{n+1} = \prod_{j=n+1-n}^{n+1-(n-2)} x_j = x_3 = 3,$$

$$x_{n+2} = \prod_{j=n+1-(n-1)}^{n+1-(n-3)} x_j = x_4 = 4,$$

$$\vdots$$

$$x_{2n-2} = \prod_{j=n-2}^{n} x_j = x_n = n,$$

$$x_{2n-1} = \prod_{j=n-1}^{n+1} x_j = x_n = n,$$

$$x_{2n} = \prod_{j=n}^{n+2} x_j = x_n = n,$$

$$x_{2n+1} = \prod_{j=n+1}^{n+3} x_j = x_{n+3} = 5,$$

$$x_{2n+2} = \prod_{j=n+2}^{n+4} x_j = x_{n+4} = 6,$$

$$x_{2n+3} = \prod_{j=n+2}^{n+5} x_j = x_{n+5} = 7,$$

$$\vdots$$

$$x_{3n} = \prod_{j=2n}^{2n+2} x_j = x_{2n+2} = n,$$

$$x_{3n+1} = \prod_{j=2n+1}^{2n+3} x_j = x_{2n+3} = 7,$$

$$\vdots$$

$$x_{\left[\left|\frac{n}{2}-1\right|\right]n} = \prod_{j=\left[\left|\frac{n}{2}-1\right|\right]n-(n-3)}^{\left[\left|\frac{n}{2}-1\right|\right]n-(n-5)} x_j = x_n = n$$

$$x_{\left[\left|\frac{n}{2}-1\right|\right]n+1} = \prod_{j=\left[\left|\frac{n}{2}-1\right|\right]n-(n-4)}^{\left[\left|\frac{n}{2}-1\right|\right]n-(n-6)} x_j = n-2,$$

$$x_{\left[\left|\frac{n}{2}-1\right|\right]n+2} = \prod_{j=\left[\left|\frac{n}{2}-1\right|\right]n-(n-5)}^{\left[\left|\frac{n}{2}-1\right|\right]n-(n-7)} x_j = n-1,$$

$$x_{\left[\left|\frac{n}{2}-1\right|\right]n+3} = \prod_{j=\left[\left|\frac{n}{2}-1\right|\right]n-(n-6)}^{\left[\left|\frac{n}{2}-1\right|\right]n-(n-8)} x_j = n,$$

from defining relations in CL_n. Since the elements succeeding

$$x_{\left[\left|\frac{n}{2}-1\right|\right]n+1}, x_{\left[\left|\frac{n}{2}-1\right|\right]n+2}, x_{\left[\left|\frac{n}{2}-1\right|\right]n+3}, \ldots,$$

depend on $n-2, n-1$, and n for their values, we have

$$x_{\left[\left|\frac{n}{2}-1\right|\right]n+m} = n$$

for $m > 2$. So, 3-step sequences in CL_n is periodic and the period of the sequence is equal to

$$p = \left[\left|\frac{n}{2}-1\right|\right]n+2.$$

when n is odd. □

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