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SOME NEW FUNCTIONAL EQUATIONS CONNECTED WITH CHARACTERIZATION PROBLEMS

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Abstract. Special cases of the functional equation

$$h_1\left(\frac{x}{c(y)}\right)\frac{1}{c(y)}f_Y(y)=h_2\left(\frac{y}{d(x)}\right)\frac{1}{d(x)}f_X(x),$$

supposed to hold for almost all $(x,y)\in\mathbb{R}_+^2$ and for the given functions c, d and the unknown functions h_1, h_2, f_X and f_Y, are investigated.

1. Introduction

Functional equations have many interesting applications in characterization problems of probability theory.

Probably Narumi was the first who studied some related questions in [12]. Later in [2] Arnold, Castillo and Sarabia showed how solutions of functional equations can be used in characterizing joint distributions from conditional distributions. They considered among others all possible distributions with given regression functions with conditionals in scale families.

They obtained those equations in the following way.

Let (X,Y) be an absolutely continuous bivariate random variable, whose joint, marginal and conditional density functions are denoted by $f_{(X,Y)}$, f_X, f_Y, $f_{X|Y}, f_{Y|X}$ respectively. One can write $f_{(X,Y)}$ in two different ways and obtain the functional equation

$$f_{(X,Y)}(x,y)=f_{X|Y}(x,y)\,f_Y(y)=f_{Y|X}(x,y)\,f_X(x) \tag{1}$$

for all $(x,y)\in\mathbb{R}^2$ (or for all $(x,y)\in\mathbb{R}_+^2$ if we restrict our investigations to the random variable (X,Y) with support in the positive quadrant).

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They studied joint densities whose conditional densities satisfy

$$f_{X|Y}\left(x,y\right)=h_{1}\left(\frac{x}{c\left(y\right)}\right)\frac{1}{c\left(y\right)} \tag{2}$$

and

$$f_{Y|X}\left(x,y\right)=h_{2}\left(\frac{y}{d\left(x\right)}\right)\frac{1}{d\left(x\right)} \tag{3}$$

for given positive functions c and d, where h_1, h_2 are positive unknown functions.

Then one can deduce from (1) the functional equation

$$h_{1}\left(\frac{x}{c\left(y\right)}\right)\frac{1}{c\left(y\right)}f_{Y}\left(y\right)=h_{2}\left(\frac{y}{d\left(x\right)}\right)\frac{1}{d\left(x\right)}f_{X}\left(x\right). \tag{4}$$

For a special choice of the given functions Arnold, Castillo and Sarabia solved (4) assuming the existence of derivatives of the unknown functions h_1, h_2, f_Y, f_X up to the second order. They restricted the search to random variable (X,Y) with support in the positive quadrant and thereby it was possible to determine the nature of the joint distribution.

In this paper, under special choices of the given functions, we assume only the measurability of the positive unknown functions h_1, h_2, f_Y, f_X and that the so obtained equations hold for almost all pairs (x,y) from $\mathbb{R}^2_+$.

We prove that the measurable solutions of (4) satisfied almost everywhere – in the different special cases – can uniquely be extended to continuous functions and when the measurable functions are replaced by the continuous functions, equation (4) is satisfied everywhere on $\mathbb{R}^2_+$.

Here we shall use the following result of A. Járai (see [5] and [6]).

Theorem 1 (Járai)**.** *Let Z be a regular topological space, Z_i $(i=1,2,\dots,n)$ be topological spaces and T be a first countable topological space. Let Y be an open subset of $\mathbb{R}^k$, X_i an open subset of $\mathbb{R}^{r_i}$, $r_i\in\mathbb{Z}$, $(i=1,2,\dots,n)$ and D an open subset of $T\times Y$. Let furthermore $T'\subset T$ be a dense subset, $f\colon T'\to Z$, $g_i\colon D\to X_i$ and $h\colon D\times Z_1\times\cdots\times Z_n\to Z$. Suppose that the function f_i is almost everywhere defined on X_i (with respect to the r_i-dimensional Lebesgue measure) with values in Z_i $(i=1,2,\dots n)$ and the following conditions are satisfied:*

(1) *for all $t\in T'$ and for almost all $y\in D_t=\{y\in Y\mid (t,y)\in D\}$*

$$f(t)=h(t,y,f_1(g_1(t,y)),\dots,f_n(g_n(t,y))); \tag{5}$$

(2) *for each fixed y in Y, the function h is continuous in the other variables;*
(3) *f_i is Lebesgue measurable on X_i $(i=1,2,\dots,n)$;*
(4) *g_i and the partial derivative $\frac{\partial g_i}{\partial y}$ are continuous on D $(i=1,2,\dots,n)$;*
(5) *for each $t\in T$ there exist a y such that $(t,y)\in D$ and the partial derivative $\frac{\partial g_i}{\partial y}$ has the rank r_i at $(t,y)\in D$ $(i=1,2,\dots,n)$.*

Then there exists a unique continuous function $\tilde{f}$ such that $f = \tilde{f}$ almost everywhere on T, and if f is replaced by $\tilde{f}$ then equation (5) *is satisfied almost everywhere on D.*

A different approach of this topic can be found in our paper [10] accepted to publication in Tatra Mountains Mathematical Publications.

2. First problem

Let us consider the case, when the functions c, d are of the form

$$c\,(y) = \frac{1}{\alpha + y}, \qquad d\,(x) = \frac{1}{\beta + x} \quad (x, y > 0)\,,$$

where α, β are non-negative constants.

From (4) we get the equation

$$h_1\left(\left(\alpha + y\right)x\right)\left(\alpha + y\right)f_Y\left(y\right) = h_2\left(\left(\beta + x\right)y\right)\left(\beta + x\right)f_X\left(x\right) \tag{6}$$

for almost all $(x, y) \in \mathbb{R}_+^2$, where h_1, h_2, f_X, $f_Y \colon \mathbb{R}_+ \to \mathbb{R}_+$ are measurable unknown functions, $\alpha, \beta \geq 0$ are arbitrary constants.

Easy calculation shows the validity of the following technical lemma.

Lemma 1. *The positive measurable functions h_1, h_2, f_X, f_Y satisfy equation* (6) *for almost all $(x, y) \in \mathbb{R}_+^2$ if and only if the measurable functions G_1, G_2, F_1, $F_2 \colon \mathbb{R}_+ \to \mathbb{R}$ defined by*

$$G_1\left(t\right) = \ln\left[h_1\left(t\right)\right], \quad G_2\left(t\right) = \ln\left[h_2\left(t\right)\right],$$

$$F_1\left(t\right) = \ln\left[\left(\alpha + t\right)f_Y\left(t\right)\right], \quad F_2\left(t\right) = \ln\left[\left(\beta + t\right)f_X\left(t\right)\right], \quad \left(t \in \mathbb{R}_+\right)$$

satisfy the functional equation

$$G_1\left(x\left(\alpha + y\right)\right) + F_1\left(y\right) = G_2\left(y\left(\beta + x\right)\right) + F_2\left(x\right), \tag{7}$$

for almost all $(x, y) \in \mathbb{R}_+^2$, where $\alpha, \beta \geq 0$ are arbitrary constants.

To get the measurable solution of equation (7) (and so (6)) satisfied almost everywhere, we distinguish 2 cases:

(1) $\alpha^2 + \beta^2 \neq 0$;
(2) $\alpha = \beta = 0$.

2.1. The $\alpha^2 + \beta^2 \neq 0$ case.

In this case, by the help of Theorem 1, we can prove the following

Theorem 2. *If the measurable functions $G_1, G_2, F_1, F_2 \colon \mathbb{R}_+ \to \mathbb{R}$ satisfy equation* (7) *for almost all $(x, y) \in \mathbb{R}_+^2$, then there exist unique continuous functions $\widetilde{G}_1, \widetilde{G}_2, \widetilde{F}_1, \widetilde{F}_2 \colon \mathbb{R}_+ \to \mathbb{R}$ such that $\widetilde{G}_1 = G_1$, $\widetilde{G}_2 = G_2$, $\widetilde{F}_1 = F_1$ and $\widetilde{F}_2 = F_2$ almost everywhere, and if G_1, G_2, F_1, F_2 are replaced by $\widetilde{G}_1$, $\widetilde{G}_2$, $\widetilde{F}_1$, $\widetilde{F}_2$ respectively, then equation* (7) *is satisfied everywhere on $\mathbb{R}_+^2$.*

Proof. First we prove that there exists unique continuous function $\widetilde{G}_1$ which is almost everywhere equal to G_1 on $\mathbb{R}_+$ and replacing G_1 by $\widetilde{G}_1$, equation (7) is satisfied almost everywhere.

With the substitution $t = x(\alpha + y)$ we get from (7) the equation

$$G_1(t) = G_2\left(y\left(\beta + \frac{t}{\alpha + y}\right)\right) + F_2\left(\frac{t}{\alpha + y}\right) - F_1(y) \tag{8}$$

which is satisfied for almost all $(t, y) \in \mathbb{R}_+^2$. By Fubini's Theorem it follows that there exists $T' \subseteq \mathbb{R}_+$ of full measure such that for all $t \in T'$ equation (8) is satisfied for almost every $y \in D_t$, where

$$D_t = \left\{y \in \mathbb{R}_+ \,\middle|\, (t, y) \in \mathbb{R}_+^2\right\} = \mathbb{R}_+.$$

Let us define the functions g_1, g_2, g_3, h in the following way:

$$g_1(t, y) = y\left(\beta + \frac{t}{\alpha + y}\right), \quad g_2(t, y) = \frac{t}{\alpha + y},$$
$$g_3(t, y) = y, \quad h(t, y, z_1, z_2, z_3) = z_1 + z_2 - z_3,$$

and let us now apply Theorem 1 of Járai to (8) with the following casting: $G_1 = f$, $G_2 = f_1$, $F_2 = f_2$, $F_1 = f_3$, $Z = Z_i = \mathbb{R}$, $T = Y = X_i = \mathbb{R}_+$, $(i = 1, 2, 3)$.

Hence the first assumption in Theorem 1 with respect to (8) holds. In the event of fixed y, the function h is continuous in the other variables, so the second assumption holds too. Because the functions in equation (8) are measurable, the third assumption is trivial.

The functions g_i are continuous, the partial derivatives

$$D_2 g_1(t, y) = \frac{t\alpha}{(y + \alpha)^2} + \beta, \quad D_2 g_2(t, y) = -\frac{t}{(y + \alpha)^2}, \quad D_2 g_3(t, y) = 1$$

are also continuous, so the fourth assumption holds too.

For each $t \in \mathbb{R}_+$ there exist a $y \in \mathbb{R}_+$ such that $(t, y) \in \mathbb{R}_+^2$ and the partial derivatives don't equal zero in (t, y), so they have rank 1. Thus the last assumption is satisfied in Theorem 1.

So we get from Theorem 1 that there exists unique continuous function $\widetilde{G}_1$ which is almost everywhere equal to G_1 on $\mathbb{R}_+$ and $\widetilde{G}_1, G_2, F_1, F_2$ satisfy equation (7) almost everywhere, which is equivalent to the equation

$$\widetilde{G}_1(x(\alpha + y)) + F_1(y) = G_2(y(\beta + x)) + F_2(x) \tag{9}$$

for almost all $(x, y) \in \mathbb{R}_+^2$.

By a similar argument we can prove the same for the function G_2.

From equation (9) with the substitution $t = y(\beta + x)$ we get the equation

$$G_2(t) = \widetilde{G}_1\left(\left(\frac{t}{y} - \beta\right)(\alpha + y)\right) + F_1(y) - F_2\left(\frac{t}{y} - \beta\right)$$

which with a suitable casting, by Fubini's Theorem, and the fact that the assumptions of Theorem 1 are fulfilled again, gives us that there exists unique continuous function $\widetilde{G}_2$ which is almost everywhere equal to G_2 on $\mathbb{R}_+$ and $\widetilde{G}_1$, $\widetilde{G}_2, F_1, F_2$ satisfy equation (7) almost everywhere, i.e.

$$\widetilde{G}_1\left(x\left(\alpha+y\right)\right)+F_1\left(y\right)=\widetilde{G}_2\left(y\left(\beta+x\right)\right)+F_2\left(x\right) \tag{10}$$

for almost all $(x,y)\in\mathbb{R}_+^2$.

There exist such x_0 and y_0 so that with the substitutions $x=x_0$ and $y=y_0$, respectively, we get from equation (10) that

$$F_1\left(y\right)=\widetilde{G}_2\left(y\left(\beta+x_0\right)\right)+F_2\left(x_0\right)-\widetilde{G}_1\left(x_0\left(\alpha+y\right)\right) \tag{11}$$

holds for almost all $y\in\mathbb{R}_+$, and

$$F_2\left(x\right)=\widetilde{G}_1\left(x\left(\alpha+y_0\right)\right)+F_1\left(y_0\right)-\widetilde{G}_2\left(y_0\left(\beta+x\right)\right) \tag{12}$$

holds for almost all $x\in\mathbb{R}_+$. Since $\widetilde{G}_1,\widetilde{G}_2\colon\mathbb{R}_+\to\mathbb{R}$ are continuous, therefore there exist unique continuous functions $\widetilde{F}_1$, $\widetilde{F}_2\colon\mathbb{R}_+\to\mathbb{R}$, defined by the right-hand side of the last two equality, which are almost everywhere equal to F_1 and F_2 on $\mathbb{R}_+$ respectively, and if we replace F_1 and F_2 by $\widetilde{F}_1$ and $\widetilde{F}_2$, respectively, the functional equation

$$\widetilde{G}_1\left(x\left(\alpha+y\right)\right)+\widetilde{F}_1\left(y\right)=\widetilde{G}_2\left(y\left(\beta+x\right)\right)+\widetilde{F}_2\left(x\right) \tag{13}$$

is satisfied almost everywhere on $\mathbb{R}_+^2$.

Both side of (13) define continuous functions on $\mathbb{R}_+^2$, which are equal to each other on a dense subset of $\mathbb{R}_+^2$, therefore we obtain that (13) is satisfied everywhere on $\mathbb{R}_+^2$.

Further $G_1=\widetilde{G}_1$, $G_2=\widetilde{G}_2$, $F_1=\widetilde{F}_1$ and $F_2=\widetilde{F}_2$ almost everywhere on $\mathbb{R}_+$. □

Therefore it is enough to determine the general continuous solutions $\widetilde{G}_1$, $\widetilde{G}_2$, $\widetilde{F}_1$, $\widetilde{F}_2\colon\mathbb{R}_+\to\mathbb{R}$ of equation

$$\widetilde{G}_1\left(x\left(\alpha+y\right)\right)+\widetilde{F}_1\left(y\right)=\widetilde{G}_2\left(y\left(\beta+x\right)\right)+\widetilde{F}_2\left(x\right),\quad (x,y)\in\mathbb{R}_+^2. \tag{14}$$

2.1.1. *The $\alpha>0$, $\beta>0$ case.* From (14) with the substitutions $x\to\beta x$, $y\to\alpha y$ and the notions

$$\overline{G}_1\left(t\right)=\widetilde{G}_1\left(\alpha\beta t\right),\quad \overline{G}_2\left(t\right)=\widetilde{G}_2\left(\alpha\beta t\right),$$

$$\overline{F}_1\left(t\right)=\widetilde{F}_1\left(\alpha t\right),\quad \overline{F}_2\left(t\right)=\widetilde{F}_2\left(\beta t\right)$$

the following equation arises

$$\overline{G}_1\left(x\left(1+y\right)\right)+\overline{F}_1\left(y\right)=\overline{G}_2\left(y\left(1+x\right)\right)+\overline{F}_2\left(x\right),\quad (x,y)\in\mathbb{R}_+^2. \tag{15}$$

Equation (15) was investigated in [9] by Lajkó, where the general measurable (continuous) solutions were given, and the following statement was proved.

Theorem 3. *[see [9]] If the continuous functions* $\overline{G}_1$, $\overline{G}_2$, $\overline{F}_1$, $\overline{F}_2\colon \mathbb{R}_+ \to \mathbb{R}$ *satisfy the functional equation* (15)*, then there exist constants* c_1, c_2, γ, δ_1, δ_2, δ_3, $\delta_4 \in \mathbb{R}$ *with* $\delta_1+\delta_3=\delta_2+\delta_4$ *such that*

$$\begin{aligned}
\overline{G}_1(x) &= c_1 \ln x + \gamma x + \delta_1,\\
\overline{G}_2(x) &= c_2 \ln x + \gamma x + \delta_2,\\
\overline{F}_1(x) &= c_2 \ln x - c_1 \ln(x+1) + \gamma x + \delta_3,\\
\overline{F}_2(x) &= c_1 \ln x - c_2 \ln(x+1) + \gamma x + \delta_4
\end{aligned}$$

for all $x \in \mathbb{R}_+$.

Now we can summarize the results of Lemma 1 and Theorems 2, 3 in the following theorem.

Theorem 4. *The measurable functions* h_1, h_2, f_X, $f_Y\colon \mathbb{R}_+ \to \mathbb{R}_+$ *satisfy functional equation* (6) *(in case* $\alpha,\beta>0$*) for almost all* $(x,y)\in\mathbb{R}_+^2$ *if and only if*

$$\begin{aligned}
h_1(x) &= \left(\frac{x}{\alpha\beta}\right)^{c_1} \exp\left(\frac{\gamma}{\alpha\beta}x+\delta_1\right) && a.e. && x\in\mathbb{R}_+,\\
h_2(x) &= \left(\frac{x}{\alpha\beta}\right)^{c_2} \exp\left(\frac{\gamma}{\alpha\beta}x+\delta_2\right) && a.e. && x\in\mathbb{R}_+,\\
f_Y(y) &= \alpha^{c_1-c_2}\frac{y^{c_2}}{(y+\alpha)^{c_1+1}} \exp\left(\frac{\gamma}{\alpha}y+\delta_3\right) && a.e. && y\in\mathbb{R}_+,\\
f_X(x) &= \beta^{c_2-c_1}\frac{x^{c_1}}{(x+\beta)^{c_2+1}} \exp\left(\frac{\gamma}{\beta}x+\delta_4\right) && a.e. && x\in\mathbb{R}_+,
\end{aligned}$$

where $c_1, c_2, \gamma, \delta_1, \delta_2, \delta_3, \delta_4 \in \mathbb{R}$ *are arbitrary constants with* $\delta_1+\delta_3=\delta_2+\delta_4$.

Proof. Theorems 2 and 3 imply that the measurable functions $\widetilde{G}_1$, $\widetilde{G}_2$, $\widetilde{F}_1$, $\widetilde{F}_2$ are almost everywhere equal to the functions $\overline{G}_1$, $\overline{G}_2$, $\overline{F}_1$, $\overline{F}_2$ given in Theorem 3, respectively. Then, by Lemma 1, we can easily get our result for the functions h_1, h_2, f_X, f_Y.

On the other hand, it is easy to see that functions h_1, h_2, f_X, f_Y, given in this theorem, satisfy the functional equation (6) for almost all $(x,y)\in\mathbb{R}_+^2$ if $\delta_1+\delta_3=\delta_2+\delta_4$ □

Remark 1. The previous theorem shows that h_1 and h_2 are gamma densities (with parameters $-\frac{\gamma}{\alpha\beta}$, c_1+1 and $-\frac{\gamma}{\alpha\beta}$, c_2+1, respectively). Thus (X,Y) has gamma conditionals in this case.

Remark 2. It is easy to see that in this special case the joint density function is of the form

$$f_{(X,Y)}(x,y) = \exp(\delta_1+\delta_3)\left(\frac{x}{\beta}\right)^{c_1}\left(\frac{y}{\alpha}\right)^{c_2}\exp\left(\gamma\left(\frac{x}{\beta}+\frac{xy}{\alpha\beta}+\frac{y}{\alpha}\right)\right)$$

for almost all $(x,y) \in \mathbb{R}_+^2$, i.e. the class of all solutions to (2) and (3) (in case $c(y) = \frac{1}{\alpha+y}$, $d(x) = \frac{1}{\beta+x}$) coincides with the MODEL II gamma conditional class (see [2]).

2.1.2. *The $\alpha = 0$, $\beta > 0$ case.* From (14) the following equation arises

$$\widetilde{G}_1(xy) + \widetilde{F}_1(y) = \widetilde{G}_2(y(\beta + x)) + \widetilde{F}_2(x), \quad (x,y) \in \mathbb{R}_+^2,$$

which with the substitutions

$$x \to \beta\frac{x}{y}, \quad y \to \frac{y}{\beta}$$

gives us the equation

$$\widetilde{G}_1(x) + \widetilde{F}_1\left(\frac{y}{\beta}\right) = \widetilde{G}_2(x+y) + \widetilde{F}_2\left(\beta\frac{x}{y}\right), \quad (x,y) \in \mathbb{R}_+^2.$$

With the definitions

$$\overline{F}_1(y) = \widetilde{F}_1\left(\frac{y}{\beta}\right), \quad \overline{F}_2(x) = \widetilde{F}_2(\beta x)$$

we get

$$\widetilde{G}_1(x) + \overline{F}_1(y) = \widetilde{G}_2(x+y) + \overline{F}_2\left(\frac{x}{y}\right), \quad (x,y) \in \mathbb{R}_+^2. \tag{16}$$

The following theorem can be derived from the main results of the papers [3], [8].

Theorem 5. *The continuous functions $\widetilde{G}_1$, $\widetilde{G}_2$, $\overline{F}_1$, $\overline{F}_2 \colon \mathbb{R}_+ \to \mathbb{R}$ satisfy the functional equation* (16) *if and only if*

$$\begin{aligned}
\widetilde{G}_1(x) &= \gamma x + c_1 \ln x + \delta_1,\\
\widetilde{G}_2(x) &= \gamma x + c_2 \ln x + \delta_2,\\
\overline{F}_1(x) &= \gamma x + (c_2 - c_1)\ln x + \delta_3,\\
\overline{F}_2(x) &= c_1 \ln x - c_2 \ln(x+1) + \delta_4
\end{aligned}$$

for all $x \in \mathbb{R}_+$, where γ, c_1, c_2, δ_1, δ_2, $\delta_3, \delta_4 \in \mathbb{R}$ are constants, such that $\delta_1 + \delta_3 = \delta_2 + \delta_4$.

Now we can summarize the results of Lemma 1 and Theorems 2, 5 in the following theorem.

Theorem 6. *The measurable functions h_1, h_2, f_X, $f_Y \colon \mathbb{R}_+ \to \mathbb{R}_+$ satisfy functional equation* (6) *(in case $\alpha = 0$, $\beta > 0$) for almost all $(x,y) \in \mathbb{R}_+^2$ if and only if*

$$\begin{aligned}
h_1(x) &= x^{c_1}\exp(\gamma x + \delta_1) && a.e. \quad x \in \mathbb{R}_+,\\
h_2(x) &= x^{c_2}\exp(\gamma x + \delta_2) && a.e. \quad x \in \mathbb{R}_+,\\
f_Y(y) &= \beta^{c_2-c_1}y^{c_2-c_1-1}\exp(\beta\gamma y + \delta_3) && a.e. \quad y \in \mathbb{R}_+,\\
f_X(x) &= \beta^{c_2-c_1}x^{c_1}(x+\beta)^{-c_2-1}\exp(\delta_4) && a.e. \quad x \in \mathbb{R}_+,
\end{aligned}$$

where $\gamma, c_1, c_2,\ \delta_1, \delta_2,\ \delta_3, \delta_4 \in \mathbb{R}$ *are arbitrary constants with* $\delta_1 + \delta_3 = \delta_2 + \delta_4$*.*

Proof. It goes similarly as in the case of Theorem 4. □

Remark 3. The previous theorem shows that h_1 and h_2 are gamma densities (with parameters $-\gamma$, $c_1 + 1$ and $-\gamma$, $c_2 + 1$, respectively). Thus (X, Y) has gamma conditionals in this case.

Remark 4. It is easy to see that in this special case the joint density function is of the form

$$f_{(X,Y)}(x, y) = \exp(\delta_1 + \delta_3)\, \beta^{c_2 - c_1} x^{c_1} y^{c_2} \exp(\gamma y (x + \beta))$$

for almost all $(x, y) \in \mathbb{R}_+^2$.

2.1.3. *The* $\alpha > 0$, $\beta = 0$ *case.* From (14) the following equation arises

$$\widetilde{G}_1(x(\alpha + y)) + \widetilde{F}_1(y) = \widetilde{G}_2(xy) + \widetilde{F}_2(x), \quad (x, y) \in \mathbb{R}_+^2,$$

which with the substitutions

$$x \to \frac{y}{\alpha}, \quad y \to \alpha \frac{x}{y}$$

gives us the equation

$$\widetilde{G}_1(x + y) + \widetilde{F}_1\left(\alpha \frac{x}{y}\right) = \widetilde{G}_2(x) + \widetilde{F}_2\left(\frac{y}{\alpha}\right), \quad (x, y) \in \mathbb{R}_+^2.$$

With the definitions

$$\overline{F}_1(x) = \widetilde{F}_1(\alpha x), \quad \overline{F}_2(y) = \widetilde{F}_2\left(\frac{y}{\alpha}\right)$$

we get

$$\widetilde{G}_2(x) + \overline{F}_2(y) = \widetilde{G}_1(x + y) + \overline{F}_1\left(\frac{x}{y}\right), \quad (x, y) \in \mathbb{R}_+^2. \tag{17}$$

Equation (17) is dual to (16) by simple changing $\left(\widetilde{G}_1, \overline{F}_1\right)$ into $\left(\widetilde{G}_2, \overline{F}_2\right)$, thus by the help of Theorem 5 we get the following result.

Theorem 7. *The continuous functions* $\widetilde{G}_1$, $\widetilde{G}_2$, $\overline{F}_1$, $\overline{F}_2 \colon \mathbb{R}_+ \to \mathbb{R}$ *satisfy the functional equation* (17) *if and only if*

$$\begin{aligned}
\widetilde{G}_1(x) &= \gamma x + c_2 \ln x + \delta_2, \\
\widetilde{G}_2(x) &= \gamma x + c_1 \ln x + \delta_1, \\
\overline{F}_1(x) &= c_1 \ln x - c_2 \ln(x + 1) + \delta_4, \\
\overline{F}_2(x) &= \gamma x + (c_2 - c_1) \ln x + \delta_3
\end{aligned}$$

for all $x \in \mathbb{R}_+$*, where* $\gamma, c_1, c_2,\ \delta_1, \delta_2,\ \delta_3, \delta_4 \in \mathbb{R}$ *are constants such that* $\delta_1 + \delta_3 = \delta_2 + \delta_4$*.*

Now we can summarize the results of Lemma 1 and Theorems 2, 7 in the following theorem.

Theorem 8. *The measurable functions* h_1, h_2, f_X, $f_Y : \mathbb{R}_+ \to \mathbb{R}_+$ *satisfy functional equation* (6) *(in case* $\alpha > 0$, $\beta = 0$*) for almost all* $(x,y) \in \mathbb{R}_+^2$ *if and only if*

$$\begin{aligned} h_1(x) &= x^{c_2} \exp(\gamma x + \delta_2) && a.e. && x \in \mathbb{R}_+, \\ h_2(x) &= x^{c_1} \exp(\gamma x + \delta_1) && a.e. && x \in \mathbb{R}_+, \\ f_Y(y) &= \alpha^{c_2 - c_1} y^{c_1} (y + \alpha)^{-c_2 - 1} \exp(\delta_4) && a.e. && y \in \mathbb{R}_+, \\ f_X(x) &= \alpha^{c_2 - c_1} x^{c_2 - c_1 - 1} \exp(\alpha \gamma x + \delta_3) && a.e. && x \in \mathbb{R}_+, \end{aligned}$$

where $\gamma, c_1, c_2, \delta_1, \delta_2, \delta_3, \delta_4 \in \mathbb{R}$ *are arbitrary constants with* $\delta_1 + \delta_3 = \delta_2 + \delta_4$.

Proof. It goes similarly as in the case of Theorem 4. □

Remark 5. The previous theorem shows that h_1 and h_2 are gamma densities (with parameters $-\gamma$, $c_2 + 1$ and $-\gamma$, $c_1 + 1$, respectively). Thus (X, Y) has gamma conditionals in this case.

Remark 6. It is easy to see that in this special case the joint density function is of the form

$$f_{(X,Y)}(x,y) = \exp(\delta_1 + \delta_3) \alpha^{c_2 - c_1} x^{c_2} y^{c_1} \exp(\gamma x (y + \alpha))$$

for almost all $(x,y) \in \mathbb{R}_+^2$.

2.2. **The** $\alpha = \beta = 0$ **case.** From (7) the following equation arises

$$G_1(xy) + F_1(y) = G_2(xy) + F_2(x)$$

and with the notations

$$H(t) = G_1(t) - G_2(t), \quad F(t) = F_2(t), \quad G(t) = -F_1(t)$$

we get the equation

$$H(xy) = F(x) + G(y) \tag{18}$$

for almost all $(x,y) \in \mathbb{R}_+^2$, where $F, G, H : \mathbb{R}_+ \to \mathbb{R}$ are measurable functions.

Similarly as in Theorem 2, by the help of Theorem 1, we can prove the following

Theorem 9. *If the measurable functions* $F, G, H : \mathbb{R}_+ \to \mathbb{R}$ *satisfy equation* (18) *for almost all* $(x,y) \in \mathbb{R}_+^2$*, then there exist unique continuous functions* $\widetilde{F}, \widetilde{G}, \widetilde{H} : \mathbb{R}_+ \to \mathbb{R}$ *such that* $\widetilde{F} = F$, $\widetilde{G} = G$ *and* $\widetilde{H} = H$ *almost everywhere, and if* F, G, H *are replaced by* $\widetilde{F}$, $\widetilde{G}$, $\widetilde{H}$ *respectively, then equation* (18) *is satisfied everywhere on* $\mathbb{R}_+^2$.

Therefore we only need the general continuous solutions $\widetilde{F}$, $\widetilde{G}$, $\widetilde{H} : \mathbb{R}_+ \to \mathbb{R}$ of the Pexider equation

$$\widetilde{H}(xy) = \widetilde{F}(x) + \widetilde{G}(y), \quad (x,y) \in \mathbb{R}_+^2 \tag{19}$$

which are the following:

$$\widetilde{H}(t) = c \ln t + \delta_1 + \delta_2, \quad (t \in \mathbb{R}_+),$$

$$\widetilde{F}(t) = c\ln t + \delta_1, \quad (t \in \mathbb{R}_+),$$
$$\widetilde{G}(t) = c\ln t + \delta_2, \quad (t \in \mathbb{R}_+),$$

where $c, \delta_1, \delta_2 \in \mathbb{R}$ are arbitrary constants (see e.g. [1], [7]). By the help of these solutions we can state the following

Theorem 10. *The measurable functions h_1, h_2, f_X, $f_Y\colon \mathbb{R}_+ \to \mathbb{R}_+$ satisfy functional equation* (6) *(in case $\alpha = \beta = 0$) for almost all $(x, y) \in \mathbb{R}_+^2$ if and only if*

$$\begin{aligned} h_1(x) &= e^{\delta_1+\delta_2}\exp(G_2(x))\,x^c \quad && a.e. \quad x \in \mathbb{R}_+,\\ h_2(x) &= \exp(G_2(x)) && a.e. \quad x \in \mathbb{R}_+,\\ f_X(x) &= e^{\delta_1}x^{c-1} && a.e. \quad x \in \mathbb{R}_+,\\ f_Y(x) &= e^{-\delta_2}x^{-c-1} && a.e. \quad x \in \mathbb{R}_+, \end{aligned}$$

where $G_2\colon \mathbb{R}_+ \to \mathbb{R}$ is an arbitrary measurable function and $c, \delta_1, \delta_2 \in \mathbb{R}$ are arbitrary constants.

Remark 7. The joint density function in this case has the form

$$f_{(X,Y)}(x, y) = x^c e^{G_2(xy)+\delta_1}$$

for almost all $(x, y) \in \mathbb{R}_+^2$.

3. Second problem

The second inquired case of the general equation (4) is the following. Let the functions c, d be linear, i.e.

$$c(y) = \lambda_1(\alpha + y), \qquad d(x) = \lambda_2(\beta + x) \quad (x, y > 0),$$

where λ_1, λ_2 are positive, α and β are non-negative constants.

Hence, from (4) we get the equation

$$h_1\left(\frac{x}{\lambda_1(\alpha + y)}\right)\frac{1}{\lambda_1(\alpha + y)}f_Y(y) = h_2\left(\frac{y}{\lambda_2(\beta + x)}\right)\frac{1}{\lambda_2(\beta + x)}f_X(x) \tag{20}$$

for almost all $(x, y) \in \mathbb{R}_+^2$, where h_1, h_2, f_X, $f_Y\colon \mathbb{R}_+ \to \mathbb{R}_+$ are measurable unknown functions, λ_1, $\lambda_2 \in \mathbb{R}_+$, α, $\beta \geq 0$ are arbitrary constants.

Easy calculation shows the validity of the following technical lemma.

Lemma 2. *The positive measurable functions h_1, h_2, f_X, f_Y satisfy equation* (20) *for almost all $(x, y) \in \mathbb{R}_+^2$ if and only if the measurable functions G_1, G_2, F_1, $F_2\colon \mathbb{R}_+ \to \mathbb{R}$ defined by*

$$G_1(t) = -\ln\left[h_2\left(\frac{1}{\lambda_2 t}\right)\right], \quad G_2(t) = -\ln\left[h_1\left(\frac{1}{\lambda_1 t}\right)\right],$$

$$F_1(t) = \ln\left[\frac{f_Y(t)}{\lambda_1(\alpha + t)}\right], \quad F_2(t) = \ln\left[\frac{f_X(t)}{\lambda_2(\beta + t)}\right], \quad (t \in \mathbb{R}_+)$$

satisfy the functional equation

$$G_1\left(\frac{x+\beta}{y}\right)+F_1(y)=G_2\left(\frac{y+\alpha}{x}\right)+F_2(x), \tag{21}$$

for almost all $(x,y)\in\mathbb{R}_+^2$*, where* $\alpha,\beta\geq 0$ *are arbitrary constants.*

To get the measurable solution of equation (21) (and so (20)) satisfied almost everywhere, we distinguish 2 cases:

(1) $\alpha^2+\beta^2\neq 0$;
(2) $\alpha=\beta=0$.

3.1. **The** $\alpha^2+\beta^2\neq 0$ **case.** Similarly as in Theorem 2, by the help of Theorem 1, we can prove the following

Theorem 11. *If the measurable functions* $G_1,G_2,F_1,F_2\colon\mathbb{R}_+\to\mathbb{R}$ *satisfy equation* (21) *for almost all* $(x,y)\in\mathbb{R}_+^2$*, then there exist unique continuous functions* $\widetilde{G}_1,\widetilde{G}_2,\widetilde{F}_1,\widetilde{F}_2\colon\mathbb{R}_+\to\mathbb{R}$ *such that* $\widetilde{G}_1=G_1$*,* $\widetilde{G}_2=G_2$*,* $\widetilde{F}_1=F_1$ *and* $\widetilde{F}_2=F_2$ *almost everywhere, and if* G_1,G_2,F_1,F_2 *are replaced by* $\widetilde{G}_1$*,* $\widetilde{G}_2$*,* $\widetilde{F}_1$*,* $\widetilde{F}_2$ *respectively, then equation* (21) *is satisfied everywhere on* $\mathbb{R}_+^2$*.*

Proof. First we prove that there exists unique continuous function $\widetilde{G}_1$ which is almost everywhere equal to G_1 on $\mathbb{R}_+$ and replacing G_1 by $\widetilde{G}_1$, equation (21) is satisfied almost everywhere.

With the substitution

$$t=\frac{x+\beta}{y}$$

we get from (21) the equation

$$G_1(t)=G_2\left(\frac{y+\alpha}{ty-\beta}\right)+F_2(ty-\beta)-F_1(y) \tag{22}$$

which is satisfied for almost all $(t,y)\in\mathbb{R}_+^2$. By Fubini's Theorem it follows that there exists $T'\subseteq\mathbb{R}_+$ of full measure such that for all $t\in T'$ equation (22) is satisfied for almost every $y\in D_t$, where

$$D_t=\left\{y\in\mathbb{R}_+\,\middle|\,(t,y)\in\mathbb{R}_+^2\right\}=\mathbb{R}_+.$$

Let us define the functions g_1,g_2,g_3,h in the following way:

$$g_1(t,y)=\frac{y+\alpha}{ty-\beta},\quad g_2(t,y)=ty-\beta,$$

$$g_3(t,y)=y,\quad h(t,y,z_1,z_2,z_3)=z_1+z_2-z_3,$$

and let us now apply Theorem 1 of Járai to (22) with the following casting: $G_1=f$, $G_2=f_1$, $F_2=f_2$, $F_1=f_3$, $Z=Z_i=\mathbb{R}$, $T=Y=X_i=\mathbb{R}_+$, $(i=1,2,3)$.

Hence the first assumption in Theorem 1 with respect to (22) holds. In the event of fixed y, the function h is continuous in the other variables, so the second assumption holds too. Because the functions in equation (22) are measurable, the third assumption is trivial.

The functions g_i are continuous, the partial derivatives

$$D_2 g_1(t,y) = -\frac{\beta + t\alpha}{(ty-\beta)^2}, \quad D_2 g_2(t,y) = t, \quad D_2 g_3(t,y) = 1$$

are also continuous, so the fourth assumption holds too.

For each $t \in \mathbb{R}_+$ there exist a $y \in \mathbb{R}_+$ such that $(t,y) \in \mathbb{R}_+^2$ and the partial derivatives don't equal zero in (t,y), so they have rank 1. Thus the last assumption is satisfied in Theorem 1.

So we get from Theorem 1 that there exists unique continuous function $\widetilde{G}_1$ which is almost everywhere equal to G_1 on $\mathbb{R}_+$ and $\widetilde{G}_1, G_2, F_1, F_2$ satisfy equation (21) almost everywhere, which is equivalent to the equation

$$\widetilde{G}_1\left(\frac{x+\beta}{y}\right) + F_1(y) = G_2\left(\frac{y+\alpha}{x}\right) + F_2(x) \tag{23}$$

for almost all $(x,y) \in \mathbb{R}_+^2$.

By a similar argument we can prove the same for the function G_2.

From equation (23) with the substitution

$$t = \frac{y+\alpha}{x}$$

we get the equation

$$G_2(t) = \widetilde{G}_1\left(\frac{y+\alpha+\beta t}{ty}\right) + F_1(y) - F_2\left(\frac{y+\alpha}{t}\right)$$

which with a suitable casting, by Fubini's Theorem, and the fact that the assumptions of Theorem 1 are fulfilled again, gives us that there exists unique continuous function $\widetilde{G}_2$ which is almost everywhere equal to G_2 on $\mathbb{R}_+$ and $\widetilde{G}_1$, $\widetilde{G}_2, F_1, F_2$ satisfy equation (21) almost everywhere, i.e.

$$\widetilde{G}_1\left(\frac{x+\beta}{y}\right) + F_1(y) = \widetilde{G}_2\left(\frac{y+\alpha}{x}\right) + F_2(x) \tag{24}$$

for almost all $(x,y) \in \mathbb{R}_+^2$.

There exist such x_0 and y_0 so that with the substitutions $x = x_0$ and $y = y_0$, respectively, we get from equation (24) that

$$F_1(y) = \widetilde{G}_2\left(\frac{y+\alpha}{x_0}\right) + F_2(x_0) - \widetilde{G}_1\left(\frac{x_0+\beta}{y}\right) \tag{25}$$

holds for almost all $y \in \mathbb{R}_+$, and

$$F_2(x) = \widetilde{G}_1\left(\frac{x+\beta}{y_0}\right) + F_1(y_0) - \widetilde{G}_2\left(\frac{y_0+\alpha}{x}\right) \tag{26}$$

holds for almost all $x \in \mathbb{R}_+$. Since $\widetilde{G}_1, \widetilde{G}_2 \colon \mathbb{R}_+ \to \mathbb{R}$ are continuous, therefore there exist unique continuous functions $\widetilde{F}_1$, $\widetilde{F}_2 \colon \mathbb{R}_+ \to \mathbb{R}$, defined by the right-hand side of the last two equality, which are almost everywhere equal to F_1 and F_2 on $\mathbb{R}_+$ respectively, and if we replace F_1 and F_2 by $\widetilde{F}_1$ and $\widetilde{F}_2$, respectively, the functional equation

$$\widetilde{G}_1\left(\frac{x+\beta}{y}\right) + \widetilde{F}_1(y) = \widetilde{G}_2\left(\frac{y+\alpha}{x}\right) + \widetilde{F}_2(x) \tag{27}$$

is satisfied almost everywhere on $\mathbb{R}_+^2$.

Both side of (27) define continuous functions on $\mathbb{R}_+^2$, which are equal to each other on a dense subset of $\mathbb{R}_+^2$, therefore we obtain that (27) is satisfied everywhere on $\mathbb{R}_+^2$.

Further $G_1 = \widetilde{G}_1$, $G_2 = \widetilde{G}_2$, $F_1 = \widetilde{F}_1$ and $F_2 = \widetilde{F}_2$ almost everywhere on $\mathbb{R}_+$. □

Hence it is enough to determine the general continuous solutions $\widetilde{G}_1$, $\widetilde{G}_2$, $\widetilde{F}_1$, $\widetilde{F}_2 \colon \mathbb{R}_+ \to \mathbb{R}$ of equation

$$\widetilde{G}_1\left(\frac{x+\beta}{y}\right) + \widetilde{F}_1(y) = \widetilde{G}_2\left(\frac{y+\alpha}{x}\right) + \widetilde{F}_2(x), \quad (x,y) \in \mathbb{R}_+^2. \tag{28}$$

3.1.1. *The $\alpha > 0$, $\beta > 0$ case.* From (28) with the substitutions $x \to \beta x$, $y \to \alpha y$ and the notions

$$\overline{G}_1(t) = \widetilde{G}_1\left(\frac{\beta}{\alpha}t\right), \quad \overline{G}_2(t) = \widetilde{G}_2\left(\frac{\alpha}{\beta}t\right),$$

$$\overline{F}_1(t) = \widetilde{F}_1(\alpha t), \quad \overline{F}_2(t) = \widetilde{F}_2(\beta t)$$

the following equation arises

$$\overline{G}_1\left(\frac{x+1}{y}\right) + \overline{F}_1(y) = \overline{G}_2\left(\frac{y+1}{x}\right) + \overline{F}_2(x), \quad (x,y) \in \mathbb{R}_+^2. \tag{29}$$

Equation (29) was investigated in [4] by Glavosits and Lajkó and the following result was proved.

Theorem 12. *[see [4]] If the continuous (or measurable) functions $\overline{G}_1$, $\overline{G}_2$, $\overline{F}_1$, $\overline{F}_2 \colon \mathbb{R}_+ \to \mathbb{R}$ satisfy the functional equation (29), then there exist constants $p_1, p_2, q, d_1, d_2, d_3, d_4 \in \mathbb{R}$ with $d_1 + d_3 = d_2 + d_4$ such that*

$$\begin{aligned}
\overline{G}_1(x) &= p_1 \ln x + q \ln(x+1) + d_1,\\
\overline{G}_2(x) &= p_2 \ln x + q \ln(x+1) + d_2,\\
\overline{F}_1(x) &= (p_1+q) \ln x + p_2 \ln(x+1) + d_3,\\
\overline{F}_2(x) &= (p_2+q) \ln x + p_1 \ln(x+1) + d_4
\end{aligned}$$

for all $x \in \mathbb{R}_+$

Finally, as an immediate consequence of Lemma 2, Theorems 11 and 12, we get the following result.

Theorem 13. *The measurable functions h_1, h_2, f_X, $f_Y : \mathbb{R}_+ \to \mathbb{R}_+$ satisfy the equation* (20) *(in case $\alpha, \beta > 0$) for almost all $(x, y) \in \mathbb{R}_+^2$ if and only if*

$$\begin{aligned} h_1(x) &= e^{-d_2}\left(\frac{\lambda_1\alpha}{\beta}\right)^{p_2} x^{p_2+q}\left(x+\frac{\beta}{\lambda_1\alpha}\right)^{-q} \quad a.e. \quad x \in \mathbb{R}_+, \\ h_2(x) &= e^{-d_1}\left(\frac{\lambda_2\beta}{\alpha}\right)^{p_1} x^{p_1+q}\left(x+\frac{\alpha}{\lambda_2\beta}\right)^{-q} \quad a.e. \quad x \in \mathbb{R}_+, \\ f_X(x) &= e^{d_4}\frac{\lambda_2}{\beta^{p_1+p_2+q}} x^{p_2+q}(x+\beta)^{p_1+1} \quad a.e. \quad x \in \mathbb{R}_+, \\ f_Y(x) &= e^{d_3}\frac{\lambda_1}{\alpha^{p_1+p_2+q}} x^{p_1+q}(x+\alpha)^{p_2+1} \quad a.e. \quad x \in \mathbb{R}_+, \end{aligned}$$

where $p_1, p_2, q, d_1, d_2, d_3, d_4 \in \mathbb{R}$ are arbitrary constants with $d_1 + d_3 = d_2 + d_4$.

Proof. Theorem 11 and 12 imply that the measurable functions G_1, G_2, F_1, F_2 are almost everywhere equal to the functions $\widetilde{G}_1$, $\widetilde{G}_2$, $\widetilde{F}_1$, $\widetilde{F}_2$ given in Theorem 12. Finally, by Lemma 2, we get the result of our theorem to the functions h_1, h_2, f_X, f_Y.

It is easy to see that functions h_1, h_2, f_X, f_Y, given in this theorem, indeed satisfy the functional equation (20) for almost all $(x, y) \in \mathbb{R}_+^2$ if $d_1 + d_3 = d_2 + d_4$. □

Remark 8. Theorem 13 shows that h_1 and h_2 are Pearson type VI distributions (with parameters $p_2 + q + 1$, $-p_2 - 1$ and $p_1 + q + 1$, $-p_1 - 1$, respectively), which are also called beta distributions of the second kind (see [2]). In this case the marginals f_X and f_Y has also the same Pearson type VI distribution.

Remark 9. One can get easily that in this case the joint density function is of the form

$$f_{(X,Y)}(x, y) = \exp(d_3 - d_2)\,\alpha^{-p_1}\beta^{-p_2}x^{p_2+q}y^{p_1+q}(\alpha x + \beta y + \alpha\beta)^{-q}$$

for almost all $(x, y) \in \mathbb{R}_+^2$, thus the class of all solutions to (2) and (3) coincides with an extension of the bivariate Pareto distribution introduced by Mardia (see [2], [11]).

3.1.2. *The $\alpha = 0$, $\beta > 0$ case.* From (28) the following equation arises

$$\widetilde{G}_1\left(\frac{x+\beta}{y}\right) + \widetilde{F}_1(y) = \widetilde{G}_2\left(\frac{y}{x}\right) + \widetilde{F}_2(x), \quad (x, y) \in \mathbb{R}_+^2,$$

which with the substitutions

$$x \to \beta\frac{x}{y}, \quad y \to \frac{1}{y}$$

gives us the equation

$$\widetilde{G}_1\left(\beta\left(x+y\right)\right)+\widetilde{F}_1\left(\frac{1}{y}\right)=\widetilde{G}_2\left(\frac{1}{\beta x}\right)+\widetilde{F}_2\left(\beta\frac{x}{y}\right),\quad (x,y)\in\mathbb{R}_+^2.$$

With the definitions

$$\overline{G}_1\left(x\right)=\widetilde{G}_1\left(\beta x\right),\quad \overline{G}_2\left(x\right)=\widetilde{G}_2\left(\frac{1}{\beta x}\right),$$

$$\overline{F}_1\left(y\right)=-\widetilde{F}_1\left(\frac{1}{y}\right),\quad \overline{F}_2\left(x\right)=-\widetilde{F}_2\left(\beta x\right)$$

we get

$$\overline{G}_2\left(x\right)+\overline{F}_1\left(y\right)=\overline{G}_1\left(x+y\right)+\overline{F}_2\left(\frac{x}{y}\right),\quad (x,y)\in\mathbb{R}_+^2. \tag{30}$$

Equation (30) is dual to equation (16) as well, by replacing $\left(\widetilde{G}_1,\widetilde{G}_2\right)$ by $\left(\overline{G}_2,\overline{G}_1\right)$, thus it comes easily from Theorem 5 the following statement.

Theorem 14. *The continuous functions $\overline{G}_1$, $\overline{G}_2$, $\overline{F}_1$, $\overline{F}_2\colon\mathbb{R}_+\to\mathbb{R}$ satisfy the functional equation* (30) *if and only if*

$$\begin{aligned}\overline{G}_1\left(x\right)&=\gamma x+c_2\ln x+\delta_2,\\ \overline{G}_2\left(x\right)&=\gamma x+c_1\ln x+\delta_1,\\ \overline{F}_1\left(x\right)&=\gamma x+\left(c_2-c_1\right)\ln x+\delta_3,\\ \overline{F}_2\left(x\right)&=c_1\ln x-c_2\ln\left(x+1\right)+\delta_4\end{aligned}$$

for all $x\in\mathbb{R}_+$, where $\gamma,c_1,c_2,\ \delta_1,\delta_2,\ \delta_3,\delta_4\in\mathbb{R}$ are constants, such that $\delta_1+\delta_3=\delta_2+\delta_4$.

Now we can summarize the results of Lemma 2 and Theorems 11, 14 in the following theorem.

Theorem 15. *The measurable functions h_1, h_2, f_X, $f_Y\colon\mathbb{R}_+\to\mathbb{R}_+$ satisfy functional equation* (20) *(in case $\alpha=0$, $\beta>0$) for almost all $(x,y)\in\mathbb{R}_+^2$ if and only if*

$$\begin{aligned}h_1\left(x\right)&=\left(\frac{\beta}{\lambda_1}\right)^{c_1}x^{-c_1}\exp\left(-\frac{\gamma\lambda_1}{\beta}x-\delta_1\right) && a.e. \quad x\in\mathbb{R}_+,\\ h_2\left(x\right)&=\left(\beta\lambda_2\right)^{c_2}x^{c_2}\exp\left(-\frac{\gamma}{\beta\lambda_2 x}-\delta_2\right) && a.e. \quad x\in\mathbb{R}_+,\\ f_Y\left(y\right)&=\lambda_1 y^{c_2-c_1+1}\exp\left(-\frac{\gamma}{y}-\delta_3\right) && a.e. \quad y\in\mathbb{R}_+,\\ f_X\left(x\right)&=\lambda_2\beta^{c_1-c_2}x^{-c_1}\left(x+\beta\right)^{c_2+1}\exp\left(-\delta_4\right) && a.e. \quad x\in\mathbb{R}_+,\end{aligned}$$

where $\gamma,c_1,c_2,\ \delta_1,\delta_2,\ \delta_3,\delta_4\in\mathbb{R}$ are arbitrary constants with $\delta_1+\delta_3=\delta_2+\delta_4$.

Remark 10. It is easy to see that in this special case the joint density function is of the form

$$f_{(X,Y)}(x,y)=\exp(-\delta_1-\delta_3)\,\beta^{c_1}x^{-c_1}y^{c_2}\exp\left(-\frac{\gamma}{\beta}\frac{x+\beta}{y}\right)$$

for almost all $(x,y)\in\mathbb{R}_+^2$.

3.1.3. *The $\alpha>0$, $\beta=0$ case.* From (28) the following equation arises

$$\widetilde{G}_1\left(\frac{x}{y}\right)+\widetilde{F}_1(y)=\widetilde{G}_2\left(\frac{y+\alpha}{x}\right)+\widetilde{F}_2(x),\quad (x,y)\in\mathbb{R}_+^2,$$

which with the substitutions

$$x\to\frac{1}{y},\quad y\to\alpha\frac{x}{y}$$

gives us the equation

$$\widetilde{G}_1\left(\frac{1}{\alpha x}\right)+\widetilde{F}_1\left(\alpha\frac{x}{y}\right)=\widetilde{G}_2(\alpha(x+y))+\widetilde{F}_2\left(\frac{1}{y}\right),\quad (x,y)\in\mathbb{R}_+^2.$$

With the definitions

$$\overline{G}_1(x)=\widetilde{G}_1\left(\frac{1}{\alpha x}\right),\quad \overline{G}_2(x)=\widetilde{G}_2(\alpha x),$$

$$\overline{F}_1(x)=-\widetilde{F}_1(\alpha x),\quad \overline{F}_2(y)=-\widetilde{F}_2\left(\frac{1}{y}\right)$$

we get

$$\overline{G}_1(x)+\overline{F}_2(y)=\overline{G}_2(x+y)+\overline{F}_1\left(\frac{x}{y}\right),\quad (x,y)\in\mathbb{R}_+^2. \tag{31}$$

Equation (31) is dual to (16) by replacing $\left(\widetilde{G}_1,\widetilde{G}_2\right)$ by $\left(\overline{G}_1,\overline{G}_2\right)$ and $\left(\overline{F}_1,\overline{F}_2\right)$ by $\left(\overline{F}_2,\overline{F}_1\right)$, thus, using again Theorem 5, we get

Theorem 16. *The continuous functions $\overline{G}_1$, $\overline{G}_2$, $\overline{F}_1$, $\overline{F}_2\colon\mathbb{R}_+\to\mathbb{R}$ satisfy the functional equation* (31) *if and only if*

$$\begin{aligned}\overline{G}_1(x)&=\gamma x+c_1\ln x+\delta_1,\\ \overline{G}_2(x)&=\gamma x+c_2\ln x+\delta_2,\\ \overline{F}_1(x)&=c_1\ln x-c_2\ln(x+1)+\delta_4,\\ \overline{F}_2(x)&=\gamma x+(c_2-c_1)\ln x+\delta_3\end{aligned}$$

for all $x\in\mathbb{R}_+$, where γ,c_1,c_2, δ_1,δ_2, $\delta_3,\delta_4\in\mathbb{R}$ are constants, such that $\delta_1+\delta_3=\delta_2+\delta_4$.

Now we can summarize the results of Lemma 2 and Theorems 11, 16 in the following theorem.

Theorem 17. *The measurable functions h_1, h_2, f_X, $f_Y\colon \mathbb{R}_+ \to \mathbb{R}_+$ satisfy functional equation* (20) *(in case $\alpha > 0$, $\beta = 0$) for almost all $(x, y) \in \mathbb{R}_+^2$ if and only if*

$$
\begin{aligned}
h_1(x) &= (\alpha\lambda_1)^{c_2} x^{c_2} \exp\left(-\frac{\gamma}{\alpha\lambda_1 x} - \delta_2\right) && a.e. && x \in \mathbb{R}_+,\\
h_2(x) &= \left(\frac{\alpha}{\lambda_2}\right)^{c_1} x^{-c_1} \exp\left(-\frac{\gamma\lambda_2}{\alpha} x - \delta_1\right) && a.e. && x \in \mathbb{R}_+,\\
f_Y(y) &= \lambda_1 \alpha^{c_1 - c_2} (y + \alpha)^{c_2 + 1} y^{-c_1} \exp(-\delta_4) && a.e. && y \in \mathbb{R}_+,\\
f_X(x) &= \lambda_2 x^{c_2 - c_1 + 1} \exp\left(-\frac{\gamma}{x} - \delta_3\right) && a.e. && x \in \mathbb{R}_+,
\end{aligned}
$$

where γ, c_1, c_2, δ_1, δ_2, $\delta_3, \delta_4 \in \mathbb{R}$ are arbitrary constants with $\delta_1 + \delta_3 = \delta_2 + \delta_4$.

Remark 11. It is easy to see that in this special case the joint density function is of the form

$$
f_{(X,Y)}(x, y) = \exp(-\delta_2 - \delta_4)\, \alpha^{c_2} x^{c_1} y^{-c_2} \exp\left(-\frac{\gamma}{\alpha} \frac{y + \alpha}{x}\right)
$$

for almost all $(x, y) \in \mathbb{R}_+^2$.

3.2. **The $\alpha = \beta = 0$ case.** From (21) the following equation arises

$$
G_1\left(\frac{x}{y}\right) + F_1(y) = G_2\left(\frac{y}{x}\right) + F_2(x)
$$

and with the substitution $x \to xy$ and the notations

$$
H(t) = F_2(t), \quad F(t) = G_1(t) - G_2\left(\frac{1}{t}\right), \quad G(t) = F_1(t)
$$

we get again the Pexider equation

$$
H(xy) = F(x) + G(y)
$$

for almost all $(x, y) \in \mathbb{R}_+^2$, where $F, G, H\colon \mathbb{R}_+ \to \mathbb{R}$ are measurable functions.

Let us use the result of Theorem 9, hence we only need the general continuous solutions $\widetilde{F}$, $\widetilde{G}$, $\widetilde{H}\colon \mathbb{R}_+ \to \mathbb{R}$ of the Pexider equation (15) for all $(x, y) \in \mathbb{R}_+^2$, (the measureable solutions of the almost everywhere satisfied Pexider equation are almost everywhere equal to these solutions), which are the following:

$$
\begin{aligned}
\widetilde{H}(t) &= c \ln t + \delta_1 + \delta_2, \quad (t \in \mathbb{R}_+),\\
\widetilde{F}(t) &= c \ln t + \delta_1, \quad (t \in \mathbb{R}_+),\\
\widetilde{G}(t) &= c \ln t + \delta_2, \quad (t \in \mathbb{R}_+),
\end{aligned}
$$

where $c, \delta_1, \delta_2 \in \mathbb{R}$ are arbitrary constants (see e.g. [1], [7]).

By the help of these solutions we can state the following

Theorem 18. *The measurable functions h_1, h_2, f_X, $f_Y\colon \mathbb{R}_+ \to \mathbb{R}_+$ satisfy functional equation* (20) *(in case $\alpha = \beta = 0$) for almost all $(x, y) \in \mathbb{R}_+^2$ if and only if*

$$\begin{aligned} h_1(x) &= \exp\left(-G_2\left(\frac{1}{\lambda_1 x}\right)\right) \qquad a.e. \quad x \in \mathbb{R}_+, \\ h_2(x) &= e^{\delta_1}\exp\left(-G_2(\lambda_2 x)\right)(\lambda_2 x)^c \qquad a.e. \quad x \in \mathbb{R}_+, \\ f_X(x) &= e^{\delta_1+\delta_2}\lambda_2 x^{c+1} \qquad a.e. \quad x \in \mathbb{R}_+, \\ f_Y(x) &= e^{\delta_2}\lambda_1 x^{c+1} \qquad a.e. \quad x \in \mathbb{R}_+, \end{aligned}$$

where $G_2\colon \mathbb{R}_+ \to \mathbb{R}$ is an arbitrary measurable function and $c, \delta_1, \delta_2 \in \mathbb{R}$ are arbitrary constants.

Remark 12. The joint density function in this case has the form

$$f_{(X,Y)}(x, y) = y^c e^{-G_2\left(\frac{y}{x}\right)+\delta_2}$$

for almost all $(x, y) \in \mathbb{R}_+^2$.

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