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ON A MULTIPLIER OF THE PROGRESSIVE MEANS AND CONVEX MAPS OF THE UNIT DISC

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ABSTRACT. In this paper we are concerned with a multiplier $\varpi(n)$ of the Progressive means, and convex maps of the unit disc. With this concern we would have brought up in a rather unified approach the results of G. Pólya and I. J. Schoenberg in [7], T. Başgöze, J. L. Frank, and F. R. Keogh in [3], and Ziad S. Ali in [1]. More theorems on the properties of the multiplier $\varpi(n)$ are given, and a key lemma showing combinatorial trigonometric identities whose offsprings are: Several combinatorial, and combinatorial trigonometric identities, and a new method for generating the Chebyshev's polynomials. Finally we present a different form of $\varpi(n)$ as well as relating $\varpi(n)$ to the subordination principle.

1. Introduction

Let $\sum_{k=0}^{\infty} u_k$ be a given series, and let $\{S_n\}_0^\infty$ denote the sequence of its partial sums. Let $\{q_n\}_0^\infty$ be a sequence of real numbers with $q_0 > 0$, and $q_n \geq 0$ for all $n > 0$, and let $Q_n = \sum_{k=0}^{n} q_k$. By G. H. Hardy [6] the sequence-to-sequence transformation

$$T_n = \frac{1}{Q_n} \sum_{k=0}^{n} q_{n-k} S_k$$

is called the *Norlund means* of $\{S_n\}_0^\infty$, and is denoted by (N, q_n).

The (N, q_n) is regular if and only if $q_n = o(Q_n)$ as $n \to \infty$; furthermore, the sequence-to-sequence transformation

$$\overline{T_n} = \frac{1}{Q_n} \sum_{k=0}^{n} q_k S_k$$

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is called the *progressive means* of $\{S_n\}_0^\infty$, and is denoted by $(\overline{N}, q_n)$. The $(\overline{N}, q_n)$ is regular if and only if $Q_n \to \infty$ as $n \to \infty$. By Peter L. Duren [4] a function f analytic in a domain D is said to be simple, schlicht, or univalent if f is one-to-one mapping of D onto another domain. A domain E of the complex plane is said to be convex if and only if the line segment joining any two points of E lies entirely in E. A function f which is analytic, univalent in the unit disc $D = \{z : z < 1\}$, and is normalized by $f(0) = f'(0) - 1 = 0$ is said to belong to the class S. Now $f \in S$ is said to belong to the class K if and only if it is a conformal mapping of the unit disc $D = \{z : z < 1\}$ onto a convex domain. An analytic function g is said to be subordinate to an analytic function f (written $g \prec f$) if

$$g(z) = f\big(\omega(z)\big) \quad |z| < 1$$

for some analytic function ω with $|\omega(z)| \leq |z|$. It is known by the Koebe-One-Quarter theorem that the range of every function of the class S contains the disc $\{w : |w| < \frac{1}{4}\}$, i.e. $\frac{1}{4}z \prec f$. The strengthened version of the Koebe-One-Quarter theorem says that the range of every convex function$f \in K$ contains the disc $|w| < \frac{1}{2}$, i.e. $\frac{1}{2}z \prec f$. The Chebychev's polynomials of the first kind $T_n(x)$, and of the second kind $U_n(x)$ are respectively defined by:

$$T_n(x) = \cos n\theta, \qquad U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \qquad x = \cos\theta.$$

2. Means connected with power series

Suppose that $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is regular for $|z| < 1$. Let

- $S_n(z, f) = \sum_{k=0}^{n} a_k z^k$ be the sequence of partial sums of f,
- $\sigma_n(z, f) = \frac{1}{n+1} \sum_{k=0}^{n} S_k(z, f)$ be the Cesaro means or $(C, 1)$ means of f,
- $T_n(z, f) = \frac{1}{Q_n} \sum_{k=0}^{n} q_{n-k} S_k(z, f)$ be the Norlund means of f,
- $\overline{T_n}(z, f) = \frac{1}{Q_n} \sum_{k=0}^{n} q_k S_k(z, f)$ be the Progressive means of f,
- $V_n(z, f) = \frac{1}{\binom{2n}{n}} \sum_{k=1}^{n} \binom{2n}{n+k} a_k z^k$ be the de la Vallee Poussin means of f.

3. Known results

In [7] G. Pólya and I. J. Schoenberg proved the following theorem, and corollary:

Theorem 3.1. *For $f(z) \in K$, it is necessary and sufficient that $V_n(z, f) \in K$ for $n = 1, 2 \ldots$.*

Corollary 3.2. *For $f(z) \in K$, $V_n(z, f) \prec f$ for $n = 1, 2, \ldots$.*

In [3] T. Başgöze, J. L. Frank, and F. R. Keogh proved the following theorem:

Theorem 3.3. (i) *Suppose that the values taken by $f(z)$ for z in D lie in a convex domain D_w. Then the values taken by $\sigma_n(z,f)$ also lie in D_w for all n, and all z in D.*

(ii) *Conversely, suppose that the values taken by $\sigma_n(z,f)$ lie in a convex domain D_w; then the values taken by $f(z)$ lie in D_w for all z in D.*

In [1] Ziad S. Ali proved the following theorems:

Theorem 3.4. (i) *Let (N,q_n) be a regular Norlund transformation such that $\{q_n\}_0^\infty$ is a non-decreasing sequence of positive numbers. Suppose that the values taken by $f(z)$, for z in D, lie in a convex domain D_w,then the values taken by $T_n(z,f)$, also lie in D_w for all n, and all z in D.*

(ii) *Conversely, suppose that the values taken by $T_n(z,f)$ lie in a convex domain D_w; then the values taken by $f(z)$ lie in D_w for all z in D.*

Theorem 3.5. (i) *Let $(\overline{N},q_n)$ be a regular Progressive transformation such that $\{q_n\}_0^\infty$ is a non-increasing sequence of positive numbers. Suppose that the values taken by $f(z)$, for z in D, lie in a convex domain D_w,then the values taken by $\overline{T_n}(z,f)$, also lie in D_w for all n, and all z in D.*

(ii) *Conversely, suppose that the values taken by $\overline{T_n}(z,f)$ lie in a convex domain D_w; then the values taken by $f(z)$ lie in D_w for all z in D.*

In [2] Ziad S. Ali proved the following theorem:

Theorem 3.6. (i) *Let $f(z)=\sum\limits_{k=1}^{\infty} a_k z^k$, $(c_1=1)$ be regular in the unit disc $|z|<1$.*

(ii) *Let T_n be a transformation of the Norlund type. Let*

$$Q_k^n=\sum_{r=0}^{k} q_r^n=\sum_{r=0}^{k}\frac{(2n-2r+1)}{(2n-r+1)}\binom{2n}{r}q_0,$$

and

$$\omega(n)=\frac{-2}{Q_n^n}\sum_{k=1}^{n}(-1)^k Q_{n-k}^n,$$

then $\frac{1}{\omega(n)}T_n(z,f)\in K$ if and only if $f\in K$.

4. The Main Theorems

In this section we prove the following theorems:

Theorem 4.1. (i) *Let* $f(z) = \sum_{k=1}^{\infty} a_k z^k$, $(a_1 = 1)$ *be regular in the unit disc* $|z| < 1$, *and let* $\overline{T_n}(z,f) = \frac{1}{Q_n}\sum_{k=0}^{n} q_k S_k(z,f)$ *be a transformation of the progressive type.*

(ii) *Let*

$$Q_{n-k} = Q_{n-k}^n = \sum_{r=0}^{n-k} q_r^n, \text{ and } Q_n = Q_n^n = \sum_{r=0}^{n} q_r^n,$$

$$q_r^n = \begin{cases} \frac{(2n-2r+1)}{(2n-r+1)}\binom{2n}{r} q_0 & \text{if } r = 0,1,\dots,(n-k), \\ q_{n-r}^n & \text{if } r = (n-k)+1, (n-k)+2, \dots, n-1, n. \end{cases}$$

(iii) *Let*

$$\varpi(n) = \frac{-2}{Q_n^n}\sum_{k=1}^{n}(-1)^k\left(Q_n^n - Q_{k-1}^n\right),$$

then $\frac{1}{\varpi(n)}\overline{T_n}(z,f) \in K$ *if and only if* $f \in K$.

Proof. We begin first by noting that:

$$\frac{1}{\varpi(n)}\overline{T_n}(z,f) = \frac{1}{\frac{-2}{Q_n^n}\sum_{k=1}^{n}(-1)^k(Q_n^n - Q_{k-1}^n)}\frac{1}{Q_n^n}\sum_{k=1}^{n} q_k^n S_k(z,f)$$

expanding $\sum_{k=1}^{n} q_k^n S_k(z,f)$, we can easily see:

$$\frac{1}{\varpi(n)}\overline{T_n}(z,f) = \frac{1}{-2\sum_{k=1}^{n}(-1)^k(Q_n^n - Q_{k-1}^n)}\sum_{k=1}^{n}(Q_n^n - Q_{k-1}^n)a_k z^k.$$

Since

$$\begin{aligned} Q_n^n &= Q_{(n-k)}^n + (q_{(n-k)+1}^n + q_{(n-k)+2}^n + \dots + q_{n-1}^n + q_n^n), \quad \text{and} \\ Q_{k-1}^n &= q_{k-1}^n + q_{k-2}^n + \dots + q_1^n + q_0^n. \end{aligned}$$

Hence

$$\frac{1}{\varpi(n)}\overline{T_n}(z,f) =$$

$$\frac{\sum_{k=1}^{n}\Big((Q_{(n-k)}^n + (q_{(n-k)+1} + \dots + q_{n-1}^n + q_n^n)) - (q_{k-1}^n + q_{k-2}^n + \dots + q_1^n + q_0^n)\Big)a_k z^k}{-2\sum_{k=1}^{n}(-1)^k\Big((Q_{(n-k)}^n + (q_{(n-k)+1}^n + \dots + q_{n-1}^n + q_n^n)) - (q_{k-1}^n + q_{k-2}^n + \dots + q_1^n + q_0^n)\Big)}.$$

Equivalently we have:

$$\frac{1}{\varpi(n)}\overline{T_n}(z,f) =$$

$$\frac{\sum\limits_{k=1}^{n}\big(Q^n_{(n-k)}+(q^n_{n-(k-1)}+\cdots+q^n_{n-1}+q^n_n)-(q^n_{k-1}+q^n_{k-2}+\cdots+q^n_1+q^n_0)\big)a_kz^k}{-2\sum\limits_{k=1}^{n}(-1)^k\big(Q^n_{(n-k)}+(q^n_{n-(k-1)}+\cdots+q^n_{n-1}+q^n_n)-(q^n_{k-1}+q^n_{k-2}+\cdots+q^n_1+q^n_0)\big)}\,.$$

Since $q^n_r = q^n_{n-r}$ for $r = n-(k-1), n-(k-2), \dots (n-1), n$, it follows easily that:

$$(q^n_{n-(k-1)}+q^n_{n-(k-2)}+\cdots+q^n_{n-1}+q^n_n)-(q^n_{k-1}+q^n_{k-2}+\cdots+q^n_1+q^n_0)=0\,.$$

Hence

$$\frac{1}{\overline{\omega}(n)}\overline{T_n}(z,f)=\frac{1}{-2\sum\limits_{k=1}^{n}(-1)^kQ^n_{(n-k)}}\sum_{k=1}^{n}Q^n_{(n-k)}a_kz^k\,.$$

Now we can easily show that:

$$Q^n_{n-k}=\sum_{r=0}^{n-k}q^n_r=\sum_{r=0}^{n-k}\frac{(2n-2r+1)}{(2n-r+1)}\binom{2n}{r}q_0=\binom{2n}{n-k}q_0\,.$$

Hence

$$\frac{1}{\overline{\omega}(n)}\overline{T_n}(z,f)=\frac{1}{-2\sum\limits_{k=1}^{n}(-1)^k\binom{2n}{n-k}}\sum_{k=1}^{n}\binom{2n}{n-k}a_kz^k\,.$$

Finally we can show that for n odd we have:

$$-2\sum_{k=1}^{n}(-1)^k\binom{2n}{n-k}=\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}+\binom{2n}{n}\,,$$

and for n even we have:

$$-2\sum_{k=1}^{n}(-1)^k\binom{2n}{n-k}=-\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}+\binom{2n}{n}\,.$$

Now since:

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}=0.$$

It follows that

$$\frac{1}{\overline{\omega}(n)}\overline{T_n}(z,f)=\frac{1}{\binom{2n}{n}}\sum_{k=1}^{n}\binom{2n}{n+k}a_kz^k=V_n(z,f),$$

which are the de la Vallee Poussin means of f, and the theorem follows by G. Pólya and I. J. Schoenberg [7]. □

Theorem 4.2. (i) *Suppose that* $f(z)=\sum\limits_{k=0}^{\infty}a_kz^k$ *is regular for* $|z|<1$, *and suppose that* $\overline{T_n}$ *are the Progressive means.*

(ii) *Let* $Q_n^n = n+1$, *and let*

$$\overline{\omega}(n) = \begin{cases} \frac{-2}{Q_n^n} \sum_{k=1}^{n} (-1)^k (Q_n^n - Q_{k-1}^n) & n \text{ is odd} \\ \\ \frac{-2}{Q_n^n} \sum_{k=1}^{n} (-1)^k (Q_n^n - Q_{k-1}^n) + 1 & n \text{ is even,} \end{cases}$$

then

$$\frac{1}{\overline{\omega}(n)} \overline{T_n}(z,f) \in K \quad \text{if and only if } f \in K.$$

Proof. Clearly $Q_n^n - Q_{k-1}^n = n - k + 1$. Considering two separate cases for n even, and n odd we can easily see that

$$n+1 = \begin{cases} -2 \sum_{k=1}^{n} (-1)^k (n-k+1) & n \text{ is odd} \\ \\ -2 \sum_{k=1}^{n} (-1)^k (n-k+1) + 1 & n \text{ is even.} \end{cases}$$

Accordingly for any n we have:

$$\frac{1}{\overline{\omega}(n)} \overline{T_n}(z,f) = \frac{1}{n+1} \sum_{k=0}^{n} S_k(z,f) = \sigma_n(z,f),$$

which are the Cesaro means of f, and the result follows by T. Başgöze, J. L. Frank, and F. R. Keogh [3]. □

Theorem 4.3. (i) *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^k$ *be regular in the unit disc* $D = \{z : |z| < 1\}$.

(ii) *Let* $\overline{T_n}(z,f)$ *be a regular Progressive type transformation defined by a non-increasing sequence* $\{q_r^n\}_{r=1}$ *of positive real numbers such that* $\sum_{i \in odd} q_i^n = \sum_{i \in even} q_i^n$, *where* i *is a non-negative integer then:*

$$\frac{1}{\overline{\omega}(n)} \overline{T_n}(z,f) \in K \quad \text{if and only if} \quad f \in K.$$

Proof. For n odd integer, say $n = 2s+1$ we have:

$$-2 \sum_{i=1}^{n} (-1)^i (Q_n^n - Q_{i-1}^n) = -2 \sum_{i=1}^{2s+1} (-1)^i (Q_{2s+1}^{2s+1} - Q_{i-1}^{2s+1})$$

$$= -2\Big(- \sum_{t=0}^{s} q_{2t+1}^{2s+1} \Big) = 2 \sum_{i \in \text{odd}}^{n} q_i^n, \quad i = 1, 3, 5 \ldots.$$

Similarly for $n = 2s$ we have:

$$-2\sum_{i=1}^{n}(-1)^i(Q_n^n - Q_{i-1}^n) = -2\sum_{i=1}^{2s}(-1)^i(Q_{2s}^{2s} - Q_{i-1}^{2s})$$
$$= -2\big(-\sum_{t=0}^{s-1} q_{2t+1}^{2s}\big) = 2\sum_{i\in\text{odd}}^{n} q_i^n, \quad i = 1, 3, 5 \dots .$$

Therefore,

$$\overline{\omega}(n) = \frac{-2}{Q_n^n}\sum_{i=1}^{n}(-1)^i(Q_n^n - Q_{i-1}^n) = \frac{1}{Q_n^n}\big(\sum_{i\in\text{odd}}^{n} q_i^n + \sum_{i\in\text{even}}^{n} q_i^n\big) = 1.$$

Accordingly the result follows by Ziad S. Ali [1]. □

5. Theorems on $\overline{\omega}(n)$

In this section we see more of the properties of $\overline{\omega}(n)$ through the following theorems.

Theorem 5.1. *Let $\{q_r^n\}_{r=1}^n$ be a sequence of positive real numbers, then*

$$\overline{\omega}(n) = 1 \quad \textit{if and only if} \quad \sum_{r\in odd} q_r^n = \sum_{r\in even} q_r^n \,.$$

Proof. Let $\overline{\omega}(n) = 1$, then

$$-2\sum_{r=1}^{n}(-1)_r\,(Q_n^n - Q_r^n) = Q_n^n \qquad 2\sum_{r\in\text{odd}} q_r^n = \left(\sum_{r\in\text{odd}}^{n} q_r^n + \sum_{r\in\text{even}}^{n} q_r^n\right).$$

Now assume $\sum\limits_{r\in\text{odd}}^{n} q_r^n = \sum\limits_{r\in\text{even}}^{n} q_r^n$, then

$$\overline{\omega}(n) = \frac{1}{Q_n^n}\left(\sum_{r\in\text{even}}^{n} q_r^n + \sum_{r\in\text{odd}}^{n} q_r^n\right) = 1. \qquad \square$$

Theorem 5.1 can be used as a tool to generate or prove new Combinatorial identities as seen by the following theorems:

Theorem 5.2. *Let $q_r^n = \binom{n}{r}$, then*

$$\sum_{r\in odd} q_r^n = \sum_{r\in even} q_r^n \,, \quad \textit{and} \quad -\frac{1}{2^{n-1}}\sum_{r=1}^{n}(-1)^r\left(2^n - \sum_{j=0}^{r-1}\binom{n}{j}\right) = 1.$$

Proof. With $q_r^n = \binom{n}{r}$, we have:

$$\overline{\omega}(n) = -\frac{2}{Q_n^n}\left(\sum_{r=1}^{n}(-1)^r(Q_n^n - Q_{r-1}^n)\right) = \frac{1}{Q_n^n}\left(\sum_{r\in\text{odd}}\binom{n}{r} + \sum_{r\in\text{even}}^{n}\binom{n}{r}\right) = 1.$$

Accordingly by theorem 5.1 we have the following combinatorial identity:

$$-\frac{1}{2^{n-1}}\sum_{r=1}^{n}(-1)^r\left(2^n-\sum_{j=0}^{r-1}\binom{n}{j}\right)=1.$$

The above newly generated combinatorial identity is implicitly saying for example when n is even: The sum of all combinations of n elements taken r at a time with $r=1,3,5,\ldots$ is 2^{n-1}. $\square$

Theorem 5.3. *Let* $q_r^n=\frac{2n-2r+1}{2n-r+1}\binom{2n}{r}q_0$, *and* $Q_n^n=\sum_{r=0}^{n}q_r^n$. *Then we have:*

$$\sum_{r\in odd}^{n}q_r^n=\sum_{r\in even}^{n}q_r^n\,.$$

Proof. With $q_r^n=\frac{2n-2r+1}{2n-r+1}\binom{2n}{r}q_0$, we can show:

$$\begin{aligned}\overline{\omega}(n)&=\frac{-2}{Q_n^n}\left(\sum_{r=1}^{n}(-1)^r(Q_n^n-Q_{r-1}^n)\right)=\frac{2}{Q_n^n}\sum_{r\in\text{odd}}^{n}\frac{2n-2r+1}{2n-r+1}\binom{2n}{r}\\&=\frac{2}{Q_n^n}\sum_{r\in\text{even}}^{n}\frac{2n-2r+1}{2n-r+1}\binom{2n}{r}=1.\end{aligned}$$

Now we may apply theorem 5.1. Moreover for n even we have:

$$\sum_{r\in\text{even}}^{n}\frac{2n-2r+1}{2n-r+1}\binom{2n}{r}=\sum_{r=0}^{\frac{n}{2}}\binom{2n}{2r}-\sum_{r=0}^{\frac{n}{2}-1}\binom{2n}{2r+1}=\frac{1}{2}\binom{2n}{n}$$

$$\sum_{r\in\text{odd}}^{n}\frac{2n-2r+1}{2n-r+1}\binom{2n}{r}=\sum_{r=0}^{\frac{n}{2}-1}\binom{2n}{2r+1}-\sum_{r=0}^{\frac{n}{2}-1}\binom{2n}{2r}=\frac{1}{2}\binom{2n}{n}.$$

For $n\in$ odd we have:

$$\sum_{r\in\text{even}}\frac{2n-2r+1}{2n-r+1}\binom{2n}{r}=\sum_{r=0}^{\frac{n-1}{2}}\binom{2n}{2r}-\sum_{r=0}^{\frac{n-3}{2}}\binom{2n}{2r+1}=\frac{1}{2}\binom{2n}{n}$$

$$\sum_{r\in\text{odd}}\frac{2n-2r+1}{2-r+1}\binom{2n}{r}=\sum_{r=0}^{\frac{n-1}{2}}\binom{2n}{2r+1}-\sum_{r=0}^{\frac{n-1}{2}}\binom{2n}{2r}=\frac{1}{2}\binom{2n}{n}.$$

This completes the proof of theorem 5.3. $\square$

We note from theorem 5.3. above that for n even

$$\sum_{r=0}^{\frac{n}{2}}\binom{2n}{2r}=\sum_{r=0}^{\frac{n}{2}-1}\binom{2n}{2r+1}+\binom{2n-1}{n}\quad;\quad\sum_{r=0}^{\frac{n}{2}}\binom{2n}{2r}=2^{2n-2}+\binom{2n-1}{n}.$$

For n odd we can show

$$\sum_{r=0}^{\frac{n}{2}} \binom{2n}{2r} = 2^{2n-2}\,.$$

Accordingly for any n we have:

$$\sum_{r=0}^{[\frac{n}{2}]} \binom{2n}{2r} = 2^{2n-2} + \frac{1+(-1)^n}{2}\binom{2n-1}{n}, \qquad n \geq 1,$$

which is identity 1.92 of Henry W. Gould [5]. Similarly for n even we have from theorem 5.3:

$$\sum_{r=0}^{\frac{n}{2}-1} \binom{2n}{2r+1} = 2^{2n-2}\,.$$

Now for n odd we have:

$$\sum_{r=0}^{\frac{n-1}{2}} \binom{2n}{2r+1} = \sum_{r=0}^{\frac{n-1}{2}} \binom{2n}{2r} + \binom{2n-1}{n} = 2^{2n-2} + \binom{2n-1}{n}.$$

Accordingly for any n we have:

$$\sum_{r=0}^{[\frac{n-1}{2}]} \binom{2n}{2r+1} = 2^{2n-2} + \frac{1-(-1)^n}{2}\binom{2n-1}{n},$$

which is identity 1.98 of Henry W. Gould [5].

Theorem 5.4. *For $n > 1$ we have:*

$$-2\left(\sum_{k=1}^{n}(-1)^k \cdot \left(\sum_{r=k}^{n} r^2\binom{2n}{n-r}\right)\right) = \sum_{r=1}^{n} r^2\binom{2n}{n-r}.$$

Proof. Follows by theorem 5.1 and noting that for $n > 1$ we have:

$$\sum_{\substack{r\in\text{odd}\\ r\geq 1}}^{n} r^2\binom{2n}{n-r} = \sum_{\substack{r\in\text{even}\\ r\geq 2}}^{n} r^2\binom{2n}{n-r}.$$ □

6. A key lemma

In this section we have the following lemma:

Lemma 6.1. *For $1 \leq r \leq n$, and θ real we have:*

(i) $$\sum_{r=1}^{n} \binom{2n}{n-r}\left(\cos r\theta + (-1)^{r+1}\right) = 2^{n-1}(1+\cos\theta)^n.$$

(ii) $$\sum_{r=1}^{n}(-1)^r\binom{2n}{n-r}\cos r\theta + \frac{1}{2}\binom{2n}{n} = 2^{n-1}(1-\cos\theta)^n\,.$$

Proof. (i) Using induction on n, and by repeated application of the recurrence formula

$$\binom{n+1}{r+1} = \binom{n}{r} + \binom{n}{r+1},$$

the above lemma follows.Note now that the proof of the lemma also follows by noting that:

$$\cos^{2n}\frac{\theta}{2} = \frac{1}{2^n}\binom{2n}{n} + \frac{1}{2^{2n-1}}\sum_{r=0}^{n-1}\binom{2n}{r}\cos(n-r)\theta,$$

and where

$$\sum_{r=1}^{n}(-1)^{r+1}\binom{2n}{n-r} = \frac{1}{2}\binom{2n}{n}.$$

Furthermore note that the above lemma also follows from the following:

$$\text{Re.}\left(e^{in\theta}\left(1+e^{-i\theta}\right)^{2n}\right) = \sum_{r=0}^{2n}\binom{2n}{r}\cos(n-r)\theta = 2^{2n}\cos^{2n}\frac{\theta}{2}.$$

We can also see that

$$\sum_{r=0}^{n}\binom{2n}{r}\cos(n-r)\theta = \frac{1}{2}\sum_{r=0}^{2n}\binom{2n}{r}\cos(n-r)\theta + \frac{1}{2}\binom{2n}{n}.$$

Now with $k = n-r$ it follows that

$$\sum_{k=1}^{n}\binom{2n}{n-k}\cos k\theta + \frac{1}{2}\binom{2n}{n} = 2^{2n-1}\cos^{2n}\frac{\theta}{2}.$$

Accordingly

$$\sum_{r=1}^{n}\binom{2n}{n-r}\left(\cos r\theta + (-1)^{r+1}\right) = 2^{n-1}(\cos\theta+1)^n,$$

and the lemma follows again.

(ii) Follows since

$$\begin{aligned}\left(e^{-in\theta}(1-e^{i\theta})^{2n}\right) &= 2^{2n}\sin^{2n}\frac{\theta}{2}\cdot(-1)^n\\ &= (-1)^n\left(\binom{2n}{n} + 2\sum_{r=1}^{n}(-1)^r\binom{2n}{n-r}\cos r\theta\right),\\ &= \sum_{r=0}^{2n}(-1)^r\binom{2n}{r}\cos(n-r)\theta.\end{aligned}$$

This completes the proof of lemma 6.1. □

Remark 1. From above we have for $m = 2n$

$$\sum_{r=0}^{m} \binom{m}{r} \cos(\frac{m}{2} - r)\theta = 2^m \cos^m \frac{\theta}{2} \cdot 1\,.$$

Accordingly we have:

$$\sum_{r=0}^{m} \tbinom{m}{r} \cos r\theta = 2^m \cos \tfrac{m\theta}{2} \cos^m \tfrac{\theta}{2}$$

$$\sum_{r=0}^{m} \tbinom{m}{r} \sin r\theta = 2^m \sin \tfrac{m\theta}{2} \cos^m \tfrac{\theta}{2}\,.$$

For any m the above two combinatorial identities which are 1.26, and 1.27 in the list of identities of Henry W. Gould [5] follow by considering $(1 + e^{i\theta})^m$.

Remark 2. Similarly for $m = 2n$ we have:

$$\sum_{r=0}^{m} (-1)^r \binom{m}{r} \cos(\frac{m}{2} - r)\theta = (-1)^{\frac{m}{2}} 2^m \sin^m \frac{\theta}{2} \cdot 1$$

then we have:

$$\sum_{r=0}^{m} (-1)^r \binom{m}{r} \cos r\theta = (-1)^{\frac{m}{2}} 2^m \sin^m \frac{\theta}{2} \cos \frac{m\theta}{2}$$

$$\sum_{r=0}^{m} (-1)^r \binom{m}{r} \sin r\theta = (-1)^{\frac{m}{2}} 2^m \sin^m \frac{\theta}{2} \sin \frac{m\theta}{2}\,.$$

For any m the above two combinatorial identities which are 1.28, and 1.29 of the identities of Henry W. Gould [5] follow by considering $(1 - e^{i\theta})^m$. Now for $\theta = 0$ in lemma 6.1(i) we can show the following:

$$\sum_{r=0}^{n} \binom{2n}{r} = 2^{2n-1} + \binom{2n-1}{n}$$

$$\sum_{r=0}^{n} (-1)^r \binom{2n}{r} = (-1)^n \binom{2n-1}{n}$$

$$\sum_{r=0}^{n} \binom{2n+1}{r} = 4^n$$

which are 1.85, 1.86, and 1.83 of Henry W. Gould [5].

Corollary 6.2. *For $\theta \in$ real, and $r \leq n$ we have the following combinatorial trigonometric identities:*

$$\sum_{r \in even}^{n} \binom{2n}{n-r} \cos r\theta = 2^{2n-2} \left(\cos^{2n} \frac{\theta}{2} + \sin^{2n} \frac{\theta}{2} \right) - \frac{1}{2} \binom{2n}{n}$$

$$\sum_{r \in odd}^{n} \binom{2n}{n-r} \cos r\theta = 2^{2n-2} \left(\cos^{2n} \frac{\theta}{2} - \sin^{2n} \frac{\theta}{2} \right) .$$

Proof. Follows from lemma 6.1. □

Corollary 6.3. *For $r \leq n$ we have:*

$$\sum_{\substack{r \in odd \\ r \geq 1}}^{n} \binom{2n}{n-r} r \cdot \sin r\theta = n 2^{n-1} \sin \theta \cdot \left(\sum_{\substack{r \in even \\ r \geq 0}}^{n-1} \binom{n-1}{r} \cos^r \theta \right), \qquad n \geq 1$$

$$\sum_{\substack{r \in even \\ r > 1}}^{n} \binom{2n}{n-r} r \cdot \sin r\theta = n 2^{n-1} \sin \theta \cdot \left(\sum_{\substack{r \in odd \\ r \geq 1}}^{n-1} \binom{n-1}{r} \cos^r \theta \right), \qquad n \geq 1 .$$

Proof. Follows from lemma 6.1. □

Corollary 6.4. *For $n \geq 2$ we have:*

$$\sum_{\substack{r \in odd \\ r \geq 1}}^{n} \binom{2n}{n-r} r^2 \cdot \cos r\theta$$

$$= n \cdot 2^{n-1} \Big(\left(1 - n \sin^2 \theta\right) \cdot \sum_{\substack{r \in odd \\ r \geq 1}}^{n-2} \binom{n-2}{r} \cdot \cos^r \theta + \sum_{\substack{r \in even \\ r \geq 0}}^{n-2} \binom{n-2}{r} \cos^{r+1} \theta \Big)$$

$$\sum_{\substack{r \in even \\ r \geq 2}}^{n} \binom{2n}{n-r} r^2 \cdot \cos r\theta$$

$$= n \cdot 2^{n-1} \Big(\left(1 - n \sin^2 \theta\right) \cdot \sum_{\substack{r \in even \\ r \geq 0}}^{n-2} \binom{n-2}{r} \cdot \cos^r \theta + \sum_{\substack{r \in odd \\ r \geq 1}}^{n-2} \binom{n-2}{r} \cos^{r+1} \theta \Big).$$

Proof. Follows from lemma 6.1. □

Corollary 6.5. *For* $0 \leq r \leq n$ *we have:*

$$\sum_{r=0}^{n} r \binom{2n}{r} = n \cdot 2^{2n-1}$$

$$\sum_{r=0}^{n} r^2 \binom{2n}{r} = n \cdot 2^{2n-2} + n^2 2^{2n-1} - n^2 \binom{2n-1}{n}.$$

Proof. Since from lemma 6.1 we have:

$$\sum_{r=1}^{n} r^2 \binom{2n}{n-r} = n \cdot 2^{2n-2}.$$

Furthermore since we can also show that

$$\sum_{r=0}^{n} r \binom{2n}{n+r} = \frac{n}{2} \binom{2n}{n},$$

the corollary follows. □

7. Generating the Chebyshev's polynomials

Using lemma 6.1(i), then by the definition of the Chebyshev's polynomials of the first kind $T_n(x)$, we see that $T_n(x)$ satisfies the following formula:

$$\sum_{r=1}^{n} \binom{2n}{n-r} \left(T_r(x) + (-1)^{r+1}\right) = 2^{n-1}(x+1)^n, \quad x = \cos\theta.$$

Now by letting $r = 1, r = 2, r = 3, \ldots$ etc. we can respectively obtain

$$T_1(x) = x,\ T_2(x) = 2x^2 - 1,\ T_3(x) = 4x^3 - 3x, \ldots$$

hence generating the Chebyshev's polynomials of the first kind of degrees $1, 2, 3, \ldots$ etc. We can similarly see that $U_n(x)$, the Chebyshev's polynomials of the second kind satisfy:

$$\sum_{r=1}^{n} \binom{2n}{n-r} \cdot r \cdot U_{r-1}(x) = n \cdot 2^{n-1}(x+1)^{n-1}, \quad x = \cos\theta.$$

Again now for $r = 1, r = 2, r = 3, \ldots$ etc. we can respectively obtain

$$U_0(x) = 1, \quad U_1(x) = 2x, \quad U_2(x) = 4x^2 - 1, \ldots,$$

and hence generating the Chebyshev's polynomials of the second kind of degrees $0, 1, 2, \ldots$ etc.

8. An application on probabilities

Using lemma 6.1(i), we can show that the probability of n successes in $2n$ trials of a symmetric binomial distribution is given by:

$$\frac{\binom{2n}{n}}{2^{2n}} = \frac{1}{2^{2n-1}} \sum_{r=0}^{n} \binom{2n}{r} \cos\frac{(n-r)\pi}{2} - \frac{1}{2^n} \tag{1}$$

$$\frac{\binom{2n}{n}}{2^{2n}} = \frac{2\sum_{r=0}^{n}\binom{2n}{r}}{2^{2n}} - 1 \tag{2}$$

$$\frac{\binom{2n}{n}}{2^{2n}} = \frac{1}{2\pi}\int_0^{2\pi} \cos^{2n} t\, dt \tag{3}$$

$$\frac{\binom{2n}{n}}{2^{2n}} = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}. \tag{4}$$

9. A different form of $\overline{\omega}(n)$

A different form of $\overline{\omega}(n)$ is presented in this section, and this is seen by the following:

Theorem 9.1. (i) *Let* $f(z) = \sum_{k=1}^{\infty} c_k z^k$ $(c_1 = 1)$ *be regular in the unit disc* $|z| < 1$.

(ii) *Let*

$$Q_{n-k} = Q^n_{n-k} = \sum_{r=0}^{n-k} q^n_r, \text{ and } Q_n = Q^n_n = \sum_{r=0}^{n} q^n_r,$$

$$q^n_r = \begin{cases} \frac{(2n-2r+1)}{(2n-r+1)}\binom{2n}{r} q_0 & r = 0, 1, \ldots, (n-k), \\ q^n_{n-r} & r = (n-k)+1, (n-k)+2, \ldots, n-1, n. \end{cases}$$

(iii) *Let* $\overline{T_n}$ *be the Progressive means. With* $z = \rho e^{i\theta}$, *let*

$$\overline{\omega}_m(n,\theta) = \tfrac{-2}{Q^n_n} \min_{|z|\le 1} Re. \sum_{r=1}^{n} \left(Q^n_n - Q^n_{r-1}\right)\cdot z^r, \text{ then}$$

$$\tfrac{1}{\overline{\omega}_m(n,\theta)} \overline{T}_n(z,f) \in K \text{ if and only if } f \in K.$$

Proof. $u(\rho,\theta) = \sum_{k=1}^{n} \binom{2n}{n-k}\rho^k \cos k\theta$ is harmonic in

$$D = \{z : |z| < 1\} \quad \text{as } \nabla^2 u = \frac{\partial^2 u}{\partial\rho^2} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{1}{\rho}\frac{\partial u}{\partial\rho} = 0.$$

Furthermore u is continuous on $\overline{D} : \{z : |z| \le 1\}$. Accordingly by the minimum principle for harmonic functions u attains its minimum on the boundary of D. Now the proof of theorem 9.1 follows from lemma 6.1(i), and theorem 4.1.

Note that from lemma 6.1(i), or $-\sum_{k=1}^{n}\binom{2n}{n-k}k\sin k\theta$ guarantees a minimum at $\theta=\pi\in[0,2\pi]$. □

10. The subordination principle and $\overline{\omega}_m(n,\theta)$

In this section we relate $\overline{\omega}(n)$ to the subordination principle by the following theorem.

Theorem 10.1. (i) *Let K denote the class of "Schlicht" power series which map $|z|<1$ onto some convex domain, and let $f\in K$.*

(ii) *Let*

$$Q_{n-k}=Q_{n-k}^{n}=\sum_{r=0}^{n-k}q_r^n,\ \text{and}\ Q_n=Q_n^n=\sum_{r=0}^{n}q_r^n,$$

$$q_r^n=\begin{cases}\frac{(2n-2r+1)}{(2n-r+1)}\binom{2n}{r}q_0 & r=0,1,\ldots,(n-k),\\ q_{n-r}^n & r=(n-k)+1,(n-k)+2,\ldots,n-1,n.\end{cases}$$

(iii) *Let $\overline{T_n}$ be a transformation of the Progressive type. With $z=\rho e^{i\theta}$, let*

$$\overline{\omega}_m(n,\theta)=\frac{-2}{Q_n^n}\min_{|z|\le 1}\ Re.\sum_{r=1}^{n}\left(Q_n^n-Q_{r-1}^n\right)\cdot z^r,\ \text{then}$$

$$\frac{1}{\overline{\omega}_m(n,\theta)}\overline{T}_n(z,f)\prec f.$$

Proof. Follows from the proof of 9.1, and corollary 3.2 of G. Pólya and I. J. Schoenberg [7]. Note that

$$\frac{1}{\overline{\omega}_m(1,\theta)}\overline{T_1}(z,f)=\frac{1}{2}z\prec f,$$

which is the strengthened version of the Koebe-One-Quarter theorem, and

$$\frac{1}{\overline{\omega}_m(2,\theta)}\overline{T_2}(z,f)=\frac{2}{3}z+\frac{a_2}{6}z^2=V_2(z,f)\prec f.$$ □

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