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A NOTE ON ZERO DIVISOR GRAPH WITH RESPECT TO ANNIHILATOR IDEALS OF A RING

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Abstract. The zero divisor graph has been investigated in general for a commutative ring R. We consider a (not necessarily commutative) ring as a right module over itself. We consider annihilators of right ideals of R and define a graph related to these annihilators. Let R be a ring and I be an ideal of R. We denote the annihilator of I (viewed as a right R-module) by $\mathrm{Ann}(I)$. We define a graph with respect to $\mathrm{Ann}(I)$ as follows and denote it by $\Gamma_{A(I)}(R)$:

$$\Gamma_{A(I)}(R) = \{E = (a, b) \mid a \in I, b \in \mathrm{Ann}(I)\}.$$

With this we prove that for a right ideal of a ring R if $I^* \cap \mathrm{Ann}(I)^* = \phi$, then $\Gamma_{A(I)}(R)$ is bipartite, where $K^* = K \setminus \{0\}$ for any subset $K \subseteq R$.

1. Introduction

The concept of zero divisor graph has been an active area of research since the notion was introduced by Beck [5]. Different aspects of zero divisor graph have been studied and investigations are on.

In most of the cases zero divisor graph of a commutative ring has been investigated.

Let R be a commutative ring with identity $1 \neq 0$. Let $Z(R)$ be the set of zero divisors of R and $Z(R)^* = Z(R) \setminus \{0\}$. Two elements $a, b \in Z(R)^*$ are adjacent if and only if $ab = 0 = ba$. The zero divisor graph of R is denoted by $\Gamma(R)$.

For any graph $G = (V, E)$, the set of vertices shall be denoted by $V(G)$ and the set of edges shall be denoted by $E(G)$.

The zero divisor graph has been studied for the ring of continuous functions by Azarpanah and Motamedi [4], for a semiprime Gifand ring by Samei [10]. A relation of $\Gamma(R)$ for a reduced ring has been given with respect to the prime radical of R (Samei [10, Theorem 3.1]).

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For more details and results the reader is referred to Anderson et al. [1, 2, 3]. Some more treatment could be found in Levy and Shepiro [6], Akbari et al. [1], Maimani et al. [7] and Samei [10].

2. Zero divisor graph with respect to annihilators

We continue the above investigation. We relate zero divisor graph of a (not necessarily commutative) ring to annihilator primes. Let R be a ring and I be a right ideal of R. Let $A = \{a \in R \mid a \text{ is regular}\}$, $B = \{b \in R \mid b \text{ is a zero divisor}\}$, and $B^* = B \setminus \{0\}$. Furthermore, $\mathrm{Ann}(I) = \{r \in R \mid ar = 0, \text{ for all } a \in I\}$. We know that $\mathrm{Ann}(I)$ is an ideal of R. Let $\mathrm{Ann}(I)^* = \mathrm{Ann}(I) \setminus \{0\}$ and $I^* = I \setminus \{0\}$. We introduce zero divisor graph with respect to I in the following way. Let $a \in I^*$, $b \in B^*$. Then a, b are adjacent if and only if $ab = 0$. We denote the graph by $\Gamma_{A(I)}(R)$. We note that $\Gamma_{A(I)}(R)$ is a directed graph and $V(\Gamma_{A(I)}(R)) \subseteq B^*$ and $\mathrm{Ann}(I)^* \subseteq B^*$. If $b \in \mathrm{Ann}(I)^*$, then obviously $ab = 0$ for all $a \in I^*$. With this we prove the following:

Theorem 1. *Let R be a ring and I be a right ideal of R. If $I^* \cap \mathrm{Ann}(I)^* = \phi$ and we ignore isolated vertices, then $\Gamma_{A(I)}(R)$ is bipartite.*

Proof. $I^* \cap \mathrm{Ann}(I)^* = \phi$ implies that if $E = (a, b) \in \Gamma_{A(I)}(R)$, then $a \in I^*$, $a \notin \mathrm{Ann}(I)^*$ and $b \in \mathrm{Ann}(I)^*$, $b \notin I^*$. □

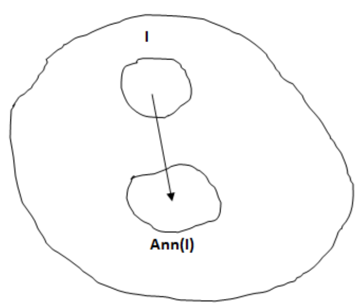

Figure 1

Example 2. *Let $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{C} \right\}$ and $I = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix} \mid c \in \mathbb{C} \right\}$. Then I is a right ideal of R and $\mathrm{Ann}(I) = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in \mathbb{C} \right\}$. Now clearly $I^* \cap \mathrm{Ann}(I)^* = \phi$, therefore, $\Gamma_{A(I)}(R)$ is bipartite.*

Example 3. *Let $R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z} \right\}$ and $J = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in \mathbb{C} \right\}$. Then J is a right ideal of R and $\mathrm{Ann}(J) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$. We note that J is* *faithful as a right R-module. (Recall that a right module M over a ring R is called faithful if $\mathrm{Ann}(M) = \{0\}$). We note that here $\Gamma_{A(I)}(R)$ is the null graph.*

Recall that $\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$ (where n is a positive integer) is a ring under addition modulo n and multiplication modulo n.

Example 4. *Let* $R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in \mathbb{Z}_n \text{ where } n \text{ is some positive integer} \right\}$ *and*

$$J = \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \mid a \in \mathbb{Z}_n \right\}.$$

Then J is a right ideal of R and we see that $\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, *therefore, $J^* \cap \mathrm{Ann}(J)^* \neq \phi$ which implies that $\Gamma_{A(J)}(R)$ is not bipartite. We also note that the zero divisor graph of J as a ring (i.e. $\Gamma(J)$) is complete. Here for all $a, b \in J$ we have $ab = 0 = ba$.*

Example 5. *Let $R = M_2(\mathbb{Z}_2)$ and*

$$I = \left\{ A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, C = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \right\}.$$

Then I is a right ideal of R and

$$\mathrm{Ann}(I)^* = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}.$$

Therefore, I is faithful as a right R-module and $\Gamma_{A(I)}(R)$ is the null graph.

Example 6. *Let*

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{Z}_2 \right\},$$

$$I = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix} \mid c \in \mathbb{Z}_2 \right\},$$

$$I^* = \left\{ A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\},$$

$$\mathrm{Ann}(I)^* = \left\{ B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, D = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\},$$

$$AB = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$AC = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$AD = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

We see that $\Gamma_{A(I)}(R) = K_{1,3}$ is complete bipartite.

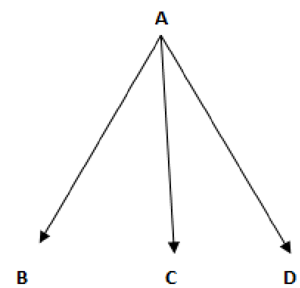

FIGURE 2

Example 7. *Consider* $\mathbb{Z}_3 = \{0, 1, 2\}$. *Let* $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{Z}_3 \right\}$ *and* $I = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} \mid a \in \mathbb{Z}_3 \right\}$. *Then* I *is a right ideal of* R. *Now*

$$I^* = \left\{ A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \right\}$$

and

$$\begin{aligned} P &= \operatorname{Ann}(I)^* \\ &= \left\{ C = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, D = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, F = \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix}, \right. \\ &\quad \left. G = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, H = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, J = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}, K = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} \right\}. \end{aligned}$$

Now $P \cap I^* = \phi$. *Therefore,* $\Gamma_{A(I)}(R)$ *is bipartite.*

Proposition 8. *Let* P *be defined as in Example 7. Then* $P \cup \{0\}$ *is a prime ideal of* R.

Proof. Since

$$ARB \in P = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in \mathbb{Z}_3 \right\}$$

is valid,

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} R \begin{pmatrix} u & v \\ 0 & u \end{pmatrix} \in \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \right\}.$$

Therefore,

$$\begin{pmatrix} aRu & arv + bRu \\ 0 & aRu \end{pmatrix} \in \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \right\}.$$

Thus,

$$aRu = 0$$

and hence

$$a = 0 \text{ or } u = 0.$$

If $a = 0$, we have $A = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \in P$.

If $u = 0$, we have $B = \begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix} \in P$. Hence P is a prime ideal of R. □

3. Zero divisor graph with respect to ideals

The concept of zero divisor graph with respect to an ideal was introduced by Redmond [9]. Let R be a commutative ring with identity $1 \neq 0$ and I an ideal of R. The zero divisor graph with identity respect to I is denoted by $\Gamma_I(R)$ and

$$\Gamma_I(R) = \{a \in R \setminus I \text{ such that } ab \in I \text{ for some } b \in R \setminus I\}$$

with distinct vertices a and b adjacent if and only if $ab \in I$. Thus if $I = \{0\}$, then $\Gamma_I(R) = \Gamma(R)$. Redmond [9] found a relationship between $\Gamma_I(R)$ and $\Gamma(R/I)$, and proved that for a finite ideal of a commutative ring R, $\Gamma_I(R)$ contains $\mid I \mid$ distinct subgraphs isomorphic to $\Gamma(R/I)$, where $|I|$ denotes the order of I.

Maimani et al. [7, Theorem 2.2] have proved the following concerning isomorphisms of zero divisor graphs. Let R and S be two rings. Let I be a finite ideal of R and J be a finite ideal of S such that $\sqrt{I} = I$ and $\sqrt{J} = J$. Then the following hold.

(1) If $|I| = |J|$ and $\Gamma(R/I) \cong \Gamma(S/J)$, then $\Gamma_I(R) \cong \Gamma_J(S)$.
(2) If $\Gamma_I(R) \cong \Gamma_J(S)$, then $\Gamma(R/I) \cong \Gamma(S/J)$.

Remark 9. Let P be a prime ideal of a commutative ring R. Then $\Gamma_P(R)$ is the null graph.

We take the notion of zero divisor graph with respect to an ideal of a commutative ring further in noncommutative set up in the following way.

Definition 10. Let R be a (not necessarily commutative) ring with identity $1 \neq 0$ and I be a right ideal of R. The zero divisor graph with respect to I is denoted by $\Gamma_I(I_R)$ and

$$\Gamma_I(I_R) = \{a \in R \setminus I \text{ such that } ab \in I, ba \in I \text{ for some } b \in R \setminus I\}$$

with distinct vertices a and b adjacent if and only if $ab \in I$ and $ba \in I$.

Example 11. *Let $A = \mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ and $R = M_2(A)$. Now $I = \{0, 3\}$ is an ideal of A and $K = M_2(I)$ is an ideal of R. Let $U = \begin{pmatrix} 3 & 2 \\ 0 & 4 \end{pmatrix} \in R$, $V = \begin{pmatrix} 1 & 3 \\ 0 & 0 \end{pmatrix} \in R$, and $W = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \in R$. Then $U \notin K$, $V \notin K$ and*

$W \notin K$. Now we see that $UV = \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \in K$, $VU = \begin{pmatrix} 3 & 2 \\ 0 & 0 \end{pmatrix} \notin K$, $UW = \begin{pmatrix} 3 & 3 \\ 0 & 0 \end{pmatrix} \in K$, $WU = \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \in K$. *Therefore,* $U, W \in \Gamma_K(K_R)$.

Example 12. *Let $S = \mathbb{Z}_2 = \{0, 1\}$ and $R = M_2(S)$. Now*

$$I = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$$

is a right ideal of R. Also

$$R \setminus I = \left\{ A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, D = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \right.$$
$$E = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, F = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, G = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, H = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},$$
$$\left. I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, J = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, K = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, L = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}.$$

We see that $\Gamma_I(I_R) = \{F, D\}$ as $FD \in I$ and $DF \in I$.

Remark 13. Let P be a completely prime ideal of a ring R. Then $\Gamma_P(P_R)$ is the null graph.

Recall that an ideal P of a ring R is completely prime if R/P is a domain, i.e., $ab \in P$ implies $a \in P$ or $b \in P$ for a, $b \in R$ (McCoy [8]).

Remark 14. Let R be a (not necessarily commutative) ring with identity $1 \neq 0$ and I be a right ideal of R. Then $\Gamma_I(I_R) \cup \{0\}$ need not be a right ideal of R. We saw in Example 12 that $D + F = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \notin \Gamma_I(I_R)$.

We note that in Example 11 there were vertices $U \notin I$, $V \notin I$ such that $UV \in I$ but $VU \notin I$. Similarly in Example 12 we had $AB \in I$, $BA \notin I$, $AD \in I$, $DA \notin I$, $CA \in I$, $AC \notin I, \dots, LE \in I$, $EL \notin I$, $LF \in I$, $FL \notin I$.

This motivates one to define the graph (directed) with respect to a right ideal in the following way. Here we use the notation as in Redmond [9].

Definition 15. Let R be a (not necessarily commutative) ring with identity $1 \neq 0$ and I be a right ideal of R. The zero divisor graph (directed) with respect to I is denoted by $\Gamma_I(R)$ and is defined as

$$\Gamma_I(R) = \{a \in R \setminus I \text{ such that } ab \in I \text{ for some } b \in R \setminus I\}$$

with distinct vertices a and b adjacent if and only if $ab \in I$.

Remark 16. Let R and I be as in Example 12. Then

$$\Gamma_I(R) = \{A, C, D, F, H, J, K, L\}.$$

We note that $B \notin \Gamma_I(R)$ as there does not exist any element $T \in R \setminus I$ such that $BT \in I$.

Remark 17. Let R be a (not necessarily commutative) ring with identity $1 \neq 0$ and I be a right ideal of R. Then $\Gamma_I(R) \cup \{0\}$ need not be a right ideal of R. We saw in Example 12 that $A + C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \notin \Gamma_I(R)$.

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