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ON $N(k)$ MIXED QUASI EINSTEIN WARPED PRODUCTS

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ABSTRACT. In this paper we have studied $N(k)$-mixed quasi Einstein warped product manifolds for arbitrary dimension $n \geq 3$.

1. INTRODUCTION

The notion of quasi Einstein manifold was introduced in a paper [8] by M. C. Chaki and R. K. Maity. According to them a non-flat Riemannian manifold (M^n, g), $(n \geq 3)$ is defined to be a quasi Einstein manifold if its Ricci tensor S of type $(0, 2)$ satisfies the condition

$$S(X, Y) = \alpha g(X, Y) + \beta A(X)A(Y)$$

and is not identically zero, where α, β are scalars, $\beta \neq 0$ and A is a non-zero 1-form such that

$$g(X, \rho_1) = A(X), \quad \forall X \in TM,$$

where ρ_1 is a unit vector field.

In such a case α, β are called the associated scalars. A is called the associated 1-form and ρ_1 is called the generator of the manifold. Such an n-dimensional manifold is denoted by the symbol $(QE)_n$.

Again, in [14], U. C. De and G. C. Ghosh defined generalized quasi Einstein manifold. A non-flat Riemannian manifold is called a generalized quasi Einstein manifold if its Ricci-tensor S of type $(0, 2)$ is non-zero and satisfies the condition

$$S(X, Y) = \alpha g(X, Y) + \beta A(X)A(Y) + \gamma B(X)B(Y),$$

where α, β, γ are non-zero scalars and A, B are two 1-forms such that

$$g(X, \rho_1) = A(X) \quad \text{and} \quad g(X, \rho_2) = B(X), \tag{1}$$

where ρ_1, ρ_2 are unit vectors which are orthogonal, i.e,

$$g(\rho_1, \rho_2) = 0.$$

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The vector fields ρ_1 and ρ_2 are called the generators of the manifold. This type of manifold are denoted by $G(QE)_n$.

Again in [9], Chaki introduced super quasi Einstein manifold, denoted by $S(QE)_n$, where the Ricci-tensor S of type $(0,2)$ which is not identically zero satisfies the condition

$$S(X,Y) = \alpha g(X,Y) + \beta A(X)A(Y) + \gamma[A(X)B(Y) + A(Y)B(X)] + \delta D(X,Y),$$

where $\alpha, \beta, \gamma, \delta$ are scalars with $\beta \neq 0$, $\gamma \neq 0$, $\delta \neq 0$ and A, B are two non-zero 1-forms defined as (1) and ρ_1, ρ_2 being mutually orthogonal unit vector fields, D is a symmetric $(0,2)$ tensor with zero trace which satisfies the condition

$$D(X, \rho_1) = 0, \quad \forall X.$$

In such case $\alpha, \beta, \gamma, \delta$ are called the associated scalars, A, B are called the associated main and auxiliary 1-forms, ρ_1, ρ_2 are called the main and the auxiliary generators and D is called the associated tensor of the manifold.Such an n-dimensional manifold shall be denoted by the symbol $S(QE)_n$.

In the papers [2], [4] A. Bhattacharyya and T. De introduced the notion of mixed generalized quasi Einstein manifold. A non-flat Riemannian manifold is called a mixed generalized quasi-Einstein manifold if its Ricci tensor S of type $(0,2)$ is non-zero and satisfies the condition

$$S(X,Y) = \alpha g(X,Y) + \beta A(X)A(Y) + \gamma B(X)B(Y) + \delta[A(X)B(Y) + B(X)A(Y)],$$

where $\alpha, \beta, \gamma, \delta$ are non-zero scalars,

$$g(X, \rho_1) = A(X), \qquad g(X, \rho_2) = B(X),$$

and

$$g(\rho_1, \rho_2) = 0,$$

A, B are two non-zero 1-forms, ρ_1 and ρ_2 are unit vector fields corresponding to the 1-forms A and B, respectively.

If $\delta = 0$, then the manifold reduces to a $G(QE)_n$. This type of manifold is denoted by $MG(QE)_n$.

Again a Riemannian manifold is said to be a manifold of generalized quasi-constant curvature [3], [6], [13] if the curvature tensor $\acute{R}$ of type $(0,4)$ satisfies the condition

$$\begin{aligned} R(X,Y,Z,W) &= p[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] + q_1[g(X,W)A(Y)A(Z) \\ &\quad - g(X,Z)A(Y)A(W) + g(Y,Z)A(X)A(W) - g(Y,W)A(X)A(Z)] \\ &\quad + s[g(X,W)B(Y)B(Z) - g(X,Z)B(Y)B(W) \\ &\quad + g(Y,Z)B(X)B(W) - g(Y,W)B(X)B(Z)], \end{aligned}$$

where p, q_1, s are scalars, A and B are non-zero 1-forms, ρ_1 and ρ_2 are unit orthogonal vector fields, such that

$$g(X, \rho_1) = A(X) \quad \text{and} \quad g(X, \rho_2) = B(X) \tag{2}$$

and

$$g(\rho_1, \rho_2) = 0. \tag{3}$$

Again a Riemannian manifold is said to be a manifold of mixed generalized quasi-constant curvature [2], [4], [15] if the curvature tensor $\acute{R}$ of type $(0, 4)$ satisfies the condition

$$\begin{aligned}\acute{R}(X,Y,Z,W) = {}& p[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] + q_1[g(X,W)A(Y)A(Z) \\ & - g(Y,W)A(X)A(Z) + g(Y,Z)A(X)A(W) - g(X,Z)A(Y)A(W)] \\ & + s[g(X,W)B(Y)B(Z) - g(Y,W)B(X)B(Z) + g(Y,Z)B(X)B(W) \\ & - g(X,Z)B(Y)B(W) + t[\{A(Y)B(Z) + B(Y)A(Z)\}g(X,W) \\ & - \{A(X)B(Z) + B(X)A(Z)\}g(Y,W) + \{A(X)B(W) \\ & + B(X)A(W)\}g(Y,Z) - \{A(Y)B(W) + B(Y)A(W)\}g(X,Z)],\end{aligned}$$

where p, q_1, s, t are scalars, A, B are non-zero 1-forms, ρ_1 and ρ_2 are orthonormal unit vector fields corresponding to A and B which are defined as (2) and (3) and

$$g(R(X,Y)Z,W) = \acute{R}(X,Y,Z,W).$$

In [5] A. Bhattacharyya, M. Tarafdar, and D. Debnath introduced the notion of mixed super quasi Einstein manifold. A non-flat Riemannian manifold (M^n, g),$(n \geq 3)$ is called mixed super quasi Einstein manifold if its Ricci-tensor S of type $(0, 2)$ is not identically zero and satisfies the condition

$$\begin{aligned}S(X,Y) = {}& \alpha g(X,Y) + \beta A(X)A(Y) + \gamma B(X)B(Y) \\ & + \delta[A(X)B(Y) + B(X)A(Y)] + \epsilon D(X,Y),\end{aligned}$$

where $\alpha, \beta, \gamma, \delta, \epsilon$ are scalars with $\beta \neq 0$,$\gamma \neq 0$, $\delta \neq 0$, $\epsilon \neq 0$ and A, B are two non-zero 1-forms such that

$$g(X, \rho_1) = A(X) \quad \text{and} \quad g(X, \rho_2) = B(X), \quad \forall X, \tag{4}$$

ρ_1, ρ_2 are mutually orthogonal unit vector fields, D is a symmetric $(0, 2)$ tensor with zero trace which satisfies the condition

$$D(X, \rho_1) = 0, \quad \forall X. \tag{5}$$

In such case $\alpha, \beta, \gamma, \delta, \epsilon$ are called the associated scalars, A, B are called the associated main and auxiliary 1-forms, ρ_1, ρ_2 are called the main and the auxiliary generators and D is called the associated tensor of the manifold. Such an n-dimensional manifold shall be denoted by the symbol $MS(QE)_n$.

Again a Riemannian manifold is said to be a manifold of mixed super quasi-constant curvature [5] if the curvature tensor $\acute{R}$ of type $(0, 4)$ satisfies the condition

$$\begin{aligned}'R(X,Y,Z,W) = {} & p[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] + q_1[g(X,W)A(Y)A(Z) \\ & - g(Y,W)A(X)A(Z) + g(Y,Z)A(X)A(W) - g(X,Z)A(Y)A(W)] \\ & + s[g(X,W)B(Y)B(Z) - g(Y,W)B(X)B(Z) + g(Y,Z)B(X)B(W) \\ & - g(X,Z)B(Y)B(W) + t[\{A(Y)B(Z) + B(Y)A(Z)\}g(X,W) \\ & - \{A(X)B(Z) + B(X)A(Z)\}g(Y,W) + \{A(X)B(W) \\ & + B(X)A(W)\}g(Y,Z) - \{A(Y)B(W) + B(Y)A(W)\}g(X,Z)] \\ & + m_1[g(Y,Z)D(X,W) - g(X,Z)D(Y,W) \\ & + g(X,W)D(Y,Z) - g(Y,W)D(X,Z)],\end{aligned}$$

where p, q_1, s, t, m_1 are scalars, A, B are non-zero 1-forms defined as (4) and ρ_1, ρ_2 are mutually orthogonal unit vector fields, D is a symmetric $(0,2)$ tensor defined as (5).

The k-nullity distribution [12], [17], [22] of a Riemannian manifold M is defined by

$$N(k) : \zeta \to N_\zeta(k) = \{Z \in T_\zeta M \setminus R(X,Y)Z = k(g(Y,Z)X - g(X,Z)Y)\}$$

for all $X, Y \in TM$ and smooth function k. M. M. Tripathi and J. J. Kim [22] introduced the notion of $N(k)$-quasi Einstein manifold which defined as follows: if the generator ρ_1 belongs to the k-nullity distribution $N(k)$, then a quasi Einstein manifold (M^n, g) is called $N(k)$-quasi Einstein manifold.

In [17], H. G. Nagaraja introduced the concept of $N(k)$-mixed quasi Einstein manifold and mixed quasi constant curvature. A non-flat Riemannian manifold (M^n, g) is called an $N(k)$-mixed quasi Einstein manifold if its Ricci-tensor of type $(0,2)$ is non-zero and satisfies the condition

$$S(X,Y) = \alpha g(X,Y) + \beta A(X)B(Y) + \gamma B(X)A(Y), \tag{6}$$

where α, β, γ are smooth functions and A, B are non-zero 1-forms such that

$$g(X, \rho_1) = A(X) \quad \text{and} \quad g(X, \rho_2) = B(X), \quad \forall X,$$

where ρ_1, ρ_2 are the orthogonal unit vector fields called generators of the manifold belonging to $N(k)$. Such a manifold is denoted by the symbol $N(k) - (MQE)_n$.

Again a Riemannian manifold (M^n, g) is called of $N(k)$-mixed quasi constant curvature if it is conformally flat and its curvature tensor $'R$ of type $(0,4)$ satisfies the condition

$$\begin{aligned}'R(X,Y,Z,W) = {} & p[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\ & + q_1[g(X,W)A(Y)B(Z) - g(X,Z)A(Y)B(W) \\ & + g(X,W)A(Z)B(Y) - g(X,Z)A(W)B(Y)] \\ & + s[g(Y,Z)A(W)B(X) - g(Y,W)A(Z)B(X) \\ & + g(Y,Z)A(X)B(W) - g(Y,W)A(X)B(Z)],\end{aligned} \tag{7}$$

where p, q_1, s, are scalars, A, B are non-zero 1-forms defined as (17) and ρ_1, ρ_2 are mutually orthogonal unit vector fields.

Let M be an m-dimensional, $m \geq 3$, Riemannian manifold and $\zeta \in M$. Denote by $K(\omega)$ or $K(u \wedge v)$ the sectional curvature of M associated with a plane section $\omega \subset T_\zeta M$, where $\{u, v\}$ is an orthonormal basis of ω. For any n-dimensional subspace $L \subseteq T_\zeta M$, $2 \leq n \leq m$, its scalar curvature $\sigma(L)$ is denoted by $\sigma(L) = 2\sum_{1\leq i<j\leq n} K(e_i \wedge e_j)$, where $\{e_1, e_2, \ldots, e_n\}$ is an orthonormal basis of L. When $L = T_\zeta M$, the scalar curvature $\sigma(L)$ is just the scalar curvature $\sigma(\zeta)$ of M at ζ.

2. Warped product manifolds

The notion of warped product generalizes that of a surface of revolution. It was introduced in [19] for studying manifolds of negative curvature. Let (C, g_C) and (J, g_J) be two Riemannian manifolds and f is a positive, differentiable function on C. Consider the product manifold $C \times J$ with its projections $\omega : C \times J \longrightarrow C$ and $\theta : C \times J \longrightarrow J$. The warped product $C \times_f J$ is the manifold $C \times J$ with the Riemannian structure such that $\|X\|^2 = \|\omega^*(X)\|^2 + f^2(\omega(\zeta))\|\theta^*(X)\|^2$, for any vector field X on M. Thus

$$g = g_C + f^2 g_J \tag{8}$$

holds on M. The function f is called the warping function of the warped product [20].

Since $C \times_f J$ is a warped product, then we have $\nabla_X Z = \nabla_Z X = (X lnf)Z$ for unit vector fields X, Z on C and J, respectively. Hence, we find $K(X \wedge Z) = g(\nabla_Z\nabla_X X - \nabla_X\nabla_Z X, Z) = \frac{1}{f}\{(\nabla_X X_f - X^2 f\}$. If we chose a local orthonormal frame $\{e_1, e_2, \ldots, e_n\}$ such that $\{e_1, e_2, \ldots, e_{n_1}\}$ are tangent to C and $e_{n_1+1}, \ldots, e_n\}$ are tangent to J, then we have

$$\frac{\triangle f}{f} = \sum_{i=1}^{n} K(e_i \wedge e_j) \tag{9}$$

for each $s = n_1 + 1, \ldots, n$ [20].

We need the following two lemmas from [20], for later use:

Lemma 1. *Let $M = C \times_f J$ be a warped product with Riemannian curvature tensor R_M. Given fields X, Y, Z on C and U, V, W on J. Then*

(i) $R_M(X, Y)Z = R_C(X, Y)Z$,
(ii) $R_M(V, X)Y = -(H^f(X, Y)/f)V$, *where H^f is the Hessian of f,*
(iii) $R_M(X, Y)V = R_M(V, W)X = 0$,
(iv) $R_M(X, V)W = -(g(V, W)/f)\nabla_X(grad\ f)$,
(v) $R_M(V, W)U = R_J(V, W)U + (\|grad\ f\|^2/f^2)\{g(V, U)W - g(W, U)V\}$.

Lemma 2. *Let $M = C \times_f J$ be a warped product with Ricci-tensor S_M. Given fields X, Y on C and V, W on J. Then*

(i) $S_M(X, Y) = S_C(X, Y) - \frac{d}{f}H^f(X, Y)$, *where $d = \dim J$,*

(ii) $S_M(X,V)=0$,

(iii) $S_M(V,W)=S_J(V,W)-g(V,W)f^\star$, $f^\star=\frac{\triangle f}{f}+\frac{d-1}{f^2}\|grad\ f\|^2$, *where* $\triangle f$ *is the Laplacian of* f *on* C.

Moreover, the scalar curvature σ_M *of the manifold* M *satisfies the condition*

$$\sigma_M=\sigma_C+\frac{\sigma_J}{f^2}-2d\frac{\triangle f}{f}-d(d-1)\frac{|\nabla f|^2}{f^2},$$

where σ_C *and* σ_J *are the scalar curvatures of* C *and* J*, respectively.*

In [16], Gebarowski studied Einstein warped product manifolds and proved the following three theorems:

Theorem 1. *Let* (M,g) *be a warped product* $I\times_f J$, $\dim I=1$, $\dim J=(n-1),(n\geq 3)$. *Then* (M,g) *is an Einstein manifold if and only if* J *is Einstein with constant scalar curvature* σ_J *in the case* $n=3$ *and* f *is given by one of the following formulae, for any real number* b,

$$f^2(t)=\begin{cases}\frac{4}{a}K\sinh^2\frac{\sqrt{a}(t+b)}{2} & \text{for } a>0,\\ K(t+b)^2 & \text{for } a=0,\\ -\frac{4}{a}K\sin^2\frac{\sqrt{-a}(t+b)}{2} & \text{for } a<0\end{cases}$$

for $K>0$, $f^2(t)=b\exp(at),(a\neq 0)$ *for* $K=0$, $f^2(t)=-\frac{4}{a}K\cosh^2\frac{\sqrt{a}(t+b)}{2}$, $(a>0)$ *for* $K<0$, *where* a *is the constant appearing after the first integration of the equation* $q''\mathrm{e}^q+2K=0$ *and* $K=\frac{\sigma_J}{(n-1)(n-2)}$.

Theorem 2. *Let* (M,g) *be a warped product* $C\times_f J$ *of a complete connected* r*-dimensional* $(1<r<n)$ *Riemannian manifold* C *and* $(n-r)$*-dimensional Riemannian manifold* J. *If* (M,g) *is a space of constant sectional curvature* $K>0$, *then* C *is a sphere of radius* $\frac{1}{\sqrt{K}}$.

Theorem 3. *Let* (M,g) *be a warped product* $C\times_f J$ *of a complete connected* $(n-1)$*-dimensional Riemannian manifold* C *and one-dimensional Riemannian manifold* I. *If* (M,g) *is an Einstein manifold with scalar curvature* $\sigma_M>0$ *and the Hessian of* f *is proportional to the metric tensor* g_C, *then*

(i) (C,g_C) *is an* $(n-1)$*-dimensional sphere of radius* $\rho=\left(\frac{\sigma_C}{(n-1)(n-2)}\right)^{-\frac{1}{2}}$.

(ii) (M,g) *is a space of constant sectional curvature* $K=\frac{\sigma_M}{n(n-1)}$.

Motivated by the above study by Gebarowski and the paper by S. Sular and C. Ozgur [21], in the present paper my aim is to study the above theorems for $N(k)$-mixed quasi-Einstein manifolds.

3. $N(k)$-MIXED QUASI-EINSTEIN WARPED PRODUCTS

In this section, we consider $N(k)$-mixed quasi-Einstein warped product manifolds and prove some results concerning these type of manifolds.

Theorem 4. *Let (M,g) be a warped product manifold $I\times_f J$, where $\dim I=1$ and $\dim J=n-1, (n\geq 3)$. If (M,g) is an $N(k)$-mixed quasi-Einstein manifold with associated scalars α,β,γ, then J is also an $N(k)$-mixed quasi-Einstein manifold.*

Proof. Suppose that $(dt)^2$ is the metric on I. Taking $f=\exp\{\frac{q}{2}\}$ and making use of Lemma 2, we can write

$$S_M\left(\frac{\partial}{\partial t},\frac{\partial}{\partial t}\right)=-\frac{n-1}{4}[2q''+(q')^2] \tag{10}$$

and

$$S_M(V,W)=S_J(V,W)-\frac{1}{4}e^q[2q''+(n-1)(q')^2]g_J(V,W), \tag{11}$$

for all vector fields V,W on J. Since M is $N(k)$-mixed quasi-Einstein, from (6) we have

$$S_M\left(\frac{\partial}{\partial t},\frac{\partial}{\partial t}\right)=\alpha g\left(\frac{\partial}{\partial t},\frac{\partial}{\partial t}\right)+\beta A\left(\frac{\partial}{\partial t}\right)B\left(\frac{\partial}{\partial t}\right)+\gamma B\left(\frac{\partial}{\partial t}\right)A\left(\frac{\partial}{\partial t}\right) \tag{12}$$

and

$$S_M(V,W)=\alpha g(V,W)+\beta A(V)B(W)+\gamma B(V)A(W). \tag{13}$$

Now let $U,U'\in\chi(M)$. Decomposing the vector fields U and U' uniquely into its components U_I, U_J, and U'_I, U'_J on I and J, respectively, we can write $U=U_I+U_J$ and $U'=U'_I+U'_J$. Since $\dim\ I=1$, we can take $U_I=\xi_1\frac{\partial}{\partial t}$ which gives us $U=\xi_1\frac{\partial}{\partial t}+U_J$ and $U'_I=\xi_2\frac{\partial}{\partial t}$ which yields $U'=\xi_2\frac{\partial}{\partial t}+U'_J$, where ξ_1 and ξ_2 are functions on M. Then we can write

$$A\left(\frac{\partial}{\partial t}\right)=g\left(\frac{\partial}{\partial t},U\right)=\xi_1, B\left(\frac{\partial}{\partial t}\right)=g\left(\frac{\partial}{\partial t},U'\right)=\xi_2. \tag{14}$$

On the other hand, by the use of (8) and (14), the equations (12) and (13) reduce to

$$S_M\left(\frac{\partial}{\partial t},\frac{\partial}{\partial t}\right)=\alpha+\beta\xi_1\xi_2+\gamma\xi_1\xi_2 \tag{15}$$

and

$$S_M(V,W)=\alpha e^q g_J(V,W)+\beta A(V)B(W)+\gamma B(V)A(W). \tag{16}$$

Comparing the right hand side of the equations (10) and (15) we get

$$\alpha+\beta\xi_1\xi_2+\gamma\xi_1\xi_2=-\frac{n-1}{4}[2q''+(q')^2].$$

Similarly, comparing the right hand sides of (11) and (16) we obtain

$$S_J(V,W)=\frac{1}{4}e^q[2q''+(n-1)(q')^2+4\alpha]g_J(V,W)+\beta A(V)B(W)+\gamma B(V)A(W),$$

which implies that J is an $N(k)$-mixed quasi-Einstein manifold. This completes the proof of the theorem. □

Theorem 5. *Let (M, g) be a warped product $C \times_f J$ of a complete connected r-dimensional $(1 < r < n)$ Riemannian manifold C and an $(n-r)$-dimensional Riemannian manifold J.*

(i) If (M, g) is a space of $N(k)$-mixed quasi-constant sectional curvature, the Hessian of f is proportional to the metric tensor g_C and the associated vector fields E and E' are the general vector field on M or E, $E' \in \chi(C)$, then C is isometric to the sphere of radius $\frac{1}{\sqrt{k}}$ in the $(r+1)$ dimensional Euclidean space. For $r = 2$, C is a 2-dimensional Einstein manifold.

(ii) If (M, g) is a space of $N(k)$-mixed quasi-constant sectional curvature and the associated vector fields E, $E' \in \chi(J)$, then C is an Einstein manifold.

Proof. Assume that M is a space of $N(k)$-mixed quasi-constant sectional curvature. Then from equation (7), we can write

$$\begin{aligned} \acute{R}(X,Y,Z,W) &= p[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\ &+ q_1[g(X,W)A(Y)B(Z) - g(X,Z)A(Y)B(W) \\ &+ g(X,W)A(Z)B(Y) - g(X,Z)A(W)B(Y)] \\ &+ s[g(Y,Z)A(W)B(X) - g(Y,W)A(Z)B(X) \\ &+ g(Y,Z)A(X)B(W) - g(Y,W)A(X)B(Z)] \end{aligned} \tag{17}$$

for all vector fields X, Y, Z, W on C.

Decomposing the vector fields E and E' uniquely into its components E_C, E_J, and E'_C, E'_J on C and J, respectively, we can write $E = E_C + E_J$ and $E' = E'_C + E'_J$. Then we can write

$$\begin{aligned} A(X) &= g(X,E) = g(X,E_C) = g_C(X,E_C), \\ B(X) &= g(X,E') = g(X,E'_C) = g_C(X,E'_C). \end{aligned} \tag{18}$$

In view of Lemma 1 and by using (8) and (18) in equation (17) and then after a contraction over X and W (we put $X = W = e_i$), we get

$$S_C(Y,Z) = p(r-1)g_C(Y,Z) + [q_1(r-1) - s][A(Y)B(Z) + B(Y)A(Z)], \tag{19}$$

which shows us that C is a mixed quasi-Einstein manifold. Contracting from (19) over Y and Z, we can write

$$\sigma_C = p(r-1)r. \tag{20}$$

Since M is a space of constant sectional curvature, in view of (9) and (17) we get

$$\frac{\triangle f}{f} = \frac{pr}{2}. \tag{21}$$

On the other hand, since the Hessian of f is proportional to the metric tensor g_C, it can be written as follows

$$H^f(X,Y) = \frac{\triangle f}{r} g_C(X,Y). \tag{22}$$

Then by the use of (20) and (21) in (22) we obtain that

$$H^f(X,Y) + Kfg_C(X,Y) = 0$$

holds on C, where $K = -\frac{\sigma_C}{2r(r-1)}$.

So by Obata's theorem [18], C is isometric to the sphere of radius $\frac{1}{\sqrt{K}}$ in the $(r+1)$-dimensional Euclidean space. When $r = 2$ then since $\beta \neq 0$ and $\gamma \neq 0$, C becomes a 2-dimensional Einstein manifold.

Assume that the associated vector fields $E, E' \in \chi(C)$. Then in view of Lemma 1 and by making use of (8) and (17) and after a contraction over X and W we obtain

$$S_C(Y,Z) = p(r-1)g_C(Y,Z) + [q_1(r-1) - s][A(Y)B(Z) + B(Y)A(Z)],$$

which gives us that C is an $N(k)$-mixed quasi-Einstein manifold. By a contraction from the above equation over Y and Z, we get

$$\sigma_C = p(r-1)r.$$

Since M is a space of constant sectional curvature, in view of (9) and (17) (for the case of $E, E' \in \chi(C)$), we obtain

$$\frac{\triangle f}{f} = \frac{pr}{2}.$$

On the other hand, since the Hessian of f is proportional to the metric tensor g_C, it can be written as follows

$$H^f(X,Y) = \frac{\triangle f}{r} g_C(X,Y).$$

Then by the use of above three equations we get

$$H^f(X,Y) + Kfg_C(X,Y) = 0, \quad \text{where} \quad K = -\frac{\sigma_C}{2r(r-1)}$$

holds on C. So by Obata's theorem [18], C is isometric to the sphere of radius $\frac{1}{\sqrt{K}}$ in the $(r+1)$-dimensional Euclidean space. For $r = 2$ and as $\beta \neq 0$, $\gamma \neq 0$, C is a 2-dimensional Einstein manifold.

Assume that the associated vector fields $E, E' \in \chi(J)$, then equation (17) reduces to

$$R(X,Y,Z,W) = p[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)].$$

In view of Lemma 1 and by making use of (8), the above equation can be written as

$$R(X,Y,Z,W) = p[g_C(Y,Z)g_C(X,W) - g_C(X,Z)g_C(Y,W)].$$

Again we use a contraction of the above equation over X and W, we get

$$S_C(Y,Z) = p(r-1)g_C(Y,Z),$$

which implies that C ia an Einstein manifold with scalar curvature $\sigma_C = pr(r-1)$. Hence the proof of the theorem is completed. □

Theorem 6. *Let (M, g) be a warped product $C \times_f I$ of a complete connected $(n-1)$-dimensional Riemannian manifold C and a one-dimensional Riemannian manifold I. If (M, g) is an $N(k)$-mixed quasi-Einstein manifold with constant associated scalars α, β, and γ and the Hessian of f is proportional to the metric tensor g_C, then (C, g_C) is an $(n-1)$-dimensional sphere of radius $\frac{n-1}{\sqrt{\sigma_C+\alpha}}$.*

Proof. Assume that M is a warped product manifold. Then by using Lemma 2 we can write

$$S_C(X, Y) = S_M(X, Y) + \frac{1}{f} H^f(X, Y)$$

for any vector fields X, Y on C. On the other hand, since M is an $N(k)$-mixed quasi-Einstein manifold we have

$$S_M(X, Y) = \alpha g(X, Y) + \beta A(X)B(Y) + \gamma B(X)A(Y). \tag{23}$$

When $U, U' \in \chi(M)$, decomposing the vector fields U and U' uniquely into its components U_I, U_J, and U'_I, U'_J on B and I, respectively, we can write

$$U = U_B + U_I \quad \text{and} \quad U' = U'_B + U'_I.$$

In view of (8) and the above three equations,

$$\begin{aligned} S_C(X, Y) = \alpha g_C(X, Y) &+ \beta g_C(X, U_C) g_(Y, U'_C) \\ &+ \gamma g_C(X, U'_C) g_C(Y, U_C) + \frac{1}{f} H^f(X, Y). \end{aligned}$$

By contraction from the above equation over X, Y, we get

$$\sigma_C = \alpha(n-1) + \frac{\triangle f}{f}. \tag{24}$$

On the other hand, we know from equation (23) that

$$\sigma_M = \alpha n. \tag{25}$$

By using (25) in (24) we get $\sigma_C = \sigma_M - \alpha + \frac{\triangle f}{f}$.

In view of Lemma 2 we also know that

$$-\frac{\sigma_M}{n} = \frac{\triangle f}{f}. \tag{26}$$

The last two equations give us $\sigma_C = \frac{n-1}{n}\sigma_M - \alpha$. On the other hand, since the Hessian of f is proportional to the metric tensor g_C, it can be written as follows

$$H^f(X, Y) = \frac{\triangle f}{n-1} g_C(X, Y).$$

As the consequence of equation (26) we have $\frac{\triangle f}{n-1} = -\frac{1}{n(n-1)}\sigma_M f$, which implies that

$$H^f(X, Y) + \frac{\sigma_C + \alpha}{(n-1)^2} f g_C(X, Y) = 0.$$

So, by Obata's theorem C is isometric to the $(n-1)$-dimensional sphere of radius $\frac{n-1}{\sqrt{\sigma_C+\alpha}}$. □

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