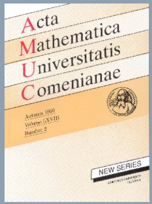

HARDY'S AND RELATED INEQUALITIES IN QUOTIENTS

S. IQBAL, K. KRULIĆ HIMMELREICH AND J. PEČARIĆ

Abstract. The main purpose of this paper is to give the well-known Hardy, Pólya-Knopp, Hardy-Hilbert, Hardy-Littlewood-Pólya and Hilbert-Hardy-type inequalities in quotients. We apply our result on multidimensional setting to obtain new results.

1. Introduction

We recall some well-known integral inequalities. First inequality is classical Hardy's inequality

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)\mathrm{d}t\right)^p \mathrm{d}x \le \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)\,\mathrm{d}x, \tag{1.1}$$

where $1 < p < \infty$, $\mathbb{R}_+ = (0, \infty)$, and $f \in L^p(\mathbb{R}_+)$ is a non-negative function. By rewriting (1.1) with the function $f^{\frac{1}{p}}$ instead of f and then by letting limit $p \to \infty$, we get the limiting case of Hardy's inequality known as Pólya-Knopp's inequality, that is.

$$\int_0^\infty \exp\left(\frac{1}{x}\int_0^x \ln f(t)\mathrm{d}t\right)\mathrm{d}x \le \mathrm{e}\int_0^\infty f(x)\,\mathrm{d}x$$

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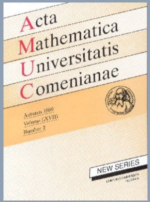


which holds for all positive functions $f \in L^1(\mathbb{R}_+)$. Two important inequalities related to (1.1) are Hardy-Hilbert's inequality

$$\int_0^\infty \Big(\int_0^\infty \frac{f(x)}{(x+y)}\,\mathrm{d}x\Big)^p \mathrm{d}y \le \Big(\frac{\pi}{\sin\frac{\pi}{p}}\Big)^p \int_0^\infty f^p(x)\,\mathrm{d}x$$

and the Hardy-Littlewood-Pólya inequality

$$\int_0^\infty \Big(\int_0^\infty \frac{f(y)}{\max\{x,y\}}\,\mathrm{d}x\Big)^p \mathrm{d}y \le \Big(pp'\Big)^p \int_0^\infty f^p(y)\,\mathrm{d}y,$$

which holds for $1 < p < \infty$, p' is the conjugate exponent of p, that is, $p' = \frac{p}{p-1}$ and non-negative $f \in L^p(\mathbb{R}_+)$. The constants $\left(\frac{p}{p-1}\right)^p$, e, $\left(\frac{\pi}{\sin\frac{\pi}{p}}\right)^p$, $(pp')^p$ in the above inequalities are the best possible constants. For further details we refer [1]–[5], [11], [13] and the references therein.

Godunova in [7] (see also [14]) proved the following inequality

$$\int_{\mathbb{R}_+^n} \Phi\Big(\frac{1}{x_1,\dots x_n}\int_{\mathbb{R}_+^n} l\Big(\frac{y_1}{x_1},\dots,\frac{y_n}{x_n}\Big) f(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\Big)\frac{\mathrm{d}\boldsymbol{x}}{x_1,\dots x_n} \le \int_{\mathbb{R}_+^n} \frac{\Phi(f(y))}{x_1,\dots,x_n}\,\mathrm{d}\boldsymbol{x} \tag{1.2}$$

which holds for all non-negative measurable functions $l\colon \mathbb{R}_+^n \to \mathbb{R}_+$ such that $\int_{\mathbb{R}_+^n} l(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = 1$, convex function $\Phi\colon [0,\infty) \to [0,\infty)$, and a non-negative function f on $\mathbb{R}_+^n$, such that the function $\boldsymbol{x} \to \int_{\mathbb{R}_+^n} \frac{\Phi(f(\boldsymbol{x}))}{x_1,\dots,x_n}$ is integrable on $\mathbb{R}_+^n$.

Let $(\Omega_1,\Sigma_1,\mu_1)$ and $(\Omega_2,\Sigma_2,\mu_2)$ be measure spaces with positive σ-finite measures, $k\colon \Omega_1\times\Omega_2\to\mathbb{R}$ be a measurable and non-negative kernel, and

$$0<K(x)=\int\limits_{\Omega_2} k(x,y)\,\mathrm{d}\mu_2(y)<\infty,\qquad x\in\Omega_1. \tag{1.3}$$

Let $U(k)$ denote the class of measurable functions $g\colon \Omega_1\to\mathbb{R}$ with the representation

$$g(x)=\int\limits_{\Omega_2} k(x,y)f(y)\mathrm{d}\mu_2(y),\qquad x\in\Omega_1, \tag{1.4}$$

where $f\colon \Omega_2\to\mathbb{R}$ is a measurable function.

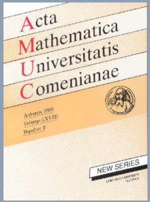
In [12] (see also [6]) K. Krulić et al. studied some new weighted Hardy-type inequalities on $(\Omega_1,\Sigma_1,\mu_1)$, $(\Omega_2,\Sigma_2,\mu_2)$, measure spaces with σ-finite measures by taking an integral operator A_k defined by

$$A_kf(x):=\frac{1}{K(x)}\int\limits_{\Omega_2} k(x,y)f(y)\,\mathrm{d}\mu_2(y), \tag{1.5}$$

where $f\colon \Omega_2\to\mathbb{R}$ is a measurable function, K is defined by (1.3). They proved the following theorem.

Theorem 1.1. *Let $(\Omega_1,\Sigma_1,\mu_1)$ and $(\Omega_2,\Sigma_2,\mu_2)$ be measure spaces with σ-finite measures, u be a weight function on Ω_1, k be a non-negative measurable function on $\Omega_1\times\Omega_2$ and K be defined on Ω_1 by (1.3). Suppose that the function $x\mapsto u(x)\frac{k(x,y)}{K(x)}$ is integrable on Ω_1 for each fixed $y\in\Omega_2$, and v is defined on Ω_2 by*

$$v(y):=\int\limits_{\Omega_1}\frac{u(x)k(x,y)}{K(x)}\mathrm{d}\mu_1(x)<\infty.$$



If Φ is convex on the interval $I \subseteq \mathbb{R}$, then the inequality

$$\int\limits_{\Omega_1} u(x)\Phi\Big(A_k f(x)\Big)\mathrm{d}\mu_1(x) \le \int\limits_{\Omega_2} v(y)\Phi\Big(f(y)\Big)\mathrm{d}\mu_2(y)$$

holds for all measurable functions $f\colon \Omega_2 \to \mathbb{R}$ such that $\operatorname{Im} f \subseteq I$, where A_k is defined by (1.5).

From Theorem 1.1, we can easily obtain Hardy's inequality, Hardy-Hilbert's inequality and Godunova's inequality and it also covers general situation that is a multidimensional case.

Before presenting the results for multidimensional setting, it is necessary to introduce some further notations. For $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n_+$, $\boldsymbol{u} = (u_1, u_2, \ldots, u_n)$, $\boldsymbol{v} = (v_1, v_2, \ldots, v_n)$, let

$$\frac{\boldsymbol{u}}{\boldsymbol{v}} = \left(\frac{u_1}{v_1}, \frac{u_2}{v_2}, \ldots, \frac{u_n}{v_n}\right) \qquad \text{and } \boldsymbol{u}^{\boldsymbol{v}} = u_1^{v_1} u_2^{v_2} \ldots u_n^{v_n}.$$

In particular, $\boldsymbol{u}^{\mathbf{1}} = \prod_{i=1}^n u_i$, $\boldsymbol{u}^{\mathbf{2}} = \left(\prod_{i=1}^n u_i\right)^2$ and $\boldsymbol{u}^{-\mathbf{1}} = \left(\prod_{i=1}^n u_i\right)^{-1}$, where $\boldsymbol{n} = (n, n, \ldots, n)$. We write $\boldsymbol{u} < \boldsymbol{v}$ if componentwise $u_i < v_i$, $i = 1, \ldots, n$. Relations $\le$, $>$, and $\ge$ are defined analogously.

Applying Theorem 1.1 with $\Omega_1 = \Omega_2 = \mathbb{R}^n_+$, the Lebesgue measure $\mathrm{d}\mu_1(\boldsymbol{x}) = \mathrm{d}\boldsymbol{x}$ and $\mathrm{d}\mu_2(\boldsymbol{y}) = \mathrm{d}\boldsymbol{y}$, and the kernel $k\colon \mathbb{R}^n_+ \times \mathbb{R}^n_+ \to \mathbb{R}$ of the form $k(\boldsymbol{x}, \boldsymbol{y}) = l\left(\frac{\boldsymbol{y}}{\boldsymbol{x}}\right)$, where $l\colon \mathbb{R}^n_+ \to \mathbb{R}$ is a non-negative measurable function, the following corollary is obtained in [12].

Corollary 1.2. *Let l and u be non-negative measurable functions on $\mathbb{R}^n_+$ such that $0 < L(\boldsymbol{x}) = \boldsymbol{x}^{\mathbf{1}} \int_{\mathbb{R}^n_+} l(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} < \infty$ for all $\boldsymbol{x} \in \mathbb{R}^n_+$ and the function $\boldsymbol{x} \mapsto u(\boldsymbol{x})\frac{l\left(\frac{\boldsymbol{y}}{\boldsymbol{x}}\right)}{L(\boldsymbol{x})}$ is integrable on $\mathbb{R}^n_+$ for each* *fixed* $\boldsymbol{y} \in \mathbb{R}_+^n$. *Let the function* v *be defined on* $\mathbb{R}_+^n$ *by*

$$v(\boldsymbol{y}) = \int_{\mathbb{R}_+^n} u(\boldsymbol{x}) \frac{l\left(\frac{\boldsymbol{y}}{\boldsymbol{x}}\right)}{L(\boldsymbol{x})} \,\mathrm{d}\boldsymbol{x}.$$

If Φ *is a convex function on an interval* $I \subseteq \mathbb{R}$, *then the inequality*

$$\int_{\mathbb{R}_+^n} u(\boldsymbol{x}) \Phi\Big(\frac{1}{L(\boldsymbol{x})} \int_{\mathbb{R}_+^n} l\Big(\frac{\boldsymbol{y}}{\boldsymbol{x}}\Big) f(\boldsymbol{y}) d\boldsymbol{y}\Big) \mathrm{d}\boldsymbol{x} \leq \int_{\mathbb{R}_+^n} v(\boldsymbol{y}) \Phi(f(\boldsymbol{y})) \mathrm{d}\boldsymbol{y} \tag{1.6}$$

holds for all measurable functions $f\colon \mathbb{R}_+^n \to \mathbb{R}$ *such that* $\operatorname{Im} f \subseteq I$.

Example 1.3. Especially, for $\int_{\mathbb{R}_+^n} l(\boldsymbol{t}) \mathrm{d}\boldsymbol{t} = 1$ and $u(\boldsymbol{x}) = \boldsymbol{x}^{-1}$, Corollary 1.2 reduces to Godunova's inequality (1.2). This shows that Corollary 1.2 is a genuine generalization of the Godunova inequality (1.2).

Next theorem is the generalized form of the Theorem 1.1 given in [12].

Theorem 1.4. *Let* $(\Omega_1, \Sigma_1, \mu_1)$ *and* $(\Omega_2, \Sigma_2, \mu_2)$ *be measure spaces with* σ*-finite measures,* u *be a weight function on* Ω_1, k *be a non-negative measurable function on* $\Omega_1 \times \Omega_2$ *and* K *be defined on* Ω_1 *by* (1.3). *Let* $0 < p \leq q < \infty$, *the function* $x \mapsto u(x)\left(\frac{k(x,y)}{K(x)}\right)^{\frac{q}{p}}$ *be integrable on* Ω_1 *for each fixed* $y \in \Omega_2$ *and* v *be defined on* Ω_2 *by*

$$v(y) := \Big(\int_{\Omega_1} u(x) \Big(\frac{k(x,y)}{K(x)}\Big)^{\frac{q}{p}} \mathrm{d}\mu_1(x)\Big)^{\frac{p}{q}} < \infty. \tag{1.7}$$

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If Φ is a positive convex function on the interval $I \subseteq \mathbb{R}$, then the inequality

$$\left(\int_{\Omega_1} u(x)[\Phi\Big(A_k f(x)\Big)]^{\frac{q}{p}} \mathrm{d}\mu_1(x)\right)^{\frac{1}{q}} \leq \left(\int_{\Omega_2} v(y)\Phi\Big(f(y)\Big) d\mu_2(y)\right)^{\frac{1}{p}} \tag{1.8}$$

holds for all measurable functions $f\colon \Omega_2 \to \mathbb{R}$ such that $\operatorname{Im} f \subseteq I$, *where A_k is defined by* (1.5).

For the case $p = q$, we obtain Theorem 1.1 and as expected by applying Theorem 1.4 we obtain the following further generalization of the Godunova result.

Corollary 1.5. *Let $0 < p \leq q < \infty$ and the assumptions in the Corollary 1.2 be satisfied with v defined by*

$$v(\boldsymbol{y}) = \left(\int_{\mathbb{R}^n_+} u(\boldsymbol{x}) \left(\frac{l\left(\frac{\boldsymbol{y}}{\boldsymbol{x}}\right)}{L(\boldsymbol{x})}\right)^{\frac{q}{p}} \mathrm{d}\boldsymbol{x}\right)^{\frac{p}{q}}.$$

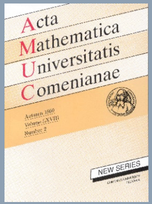
If Φ is a positive convex function on an interval $I \subseteq \mathbb{R}$, then the inequality

$$\left(\int_{\mathbb{R}^n_+} u(\boldsymbol{x})\left[\Phi\Big(\frac{1}{L(\boldsymbol{x})}\int_{\mathbb{R}^n_+} l\Big(\frac{\boldsymbol{y}}{\boldsymbol{x}}\Big) f(\boldsymbol{y}) d\boldsymbol{y}\Big)\right]^{\frac{q}{p}} \mathrm{d}\boldsymbol{x}\right)^{\frac{1}{q}} \leq \left(\int_{\mathbb{R}^n_+} v(\boldsymbol{y})\Phi(f(\boldsymbol{y})) d\boldsymbol{y}\right)^{\frac{1}{p}} \tag{1.9}$$

holds for all measurable functions $f\colon \mathbb{R}^n_+ \to \mathbb{R}$ such that $\operatorname{Im} f \subseteq I$.

S. Iqbal et al. in their recent paper [9] proved an inequality for an arbitrary convex and increasing function with some applications for different kinds of fractional integrals and fractional derivatives. The main purpose of this paper is to give the Hardy's and related inequalities in quotients.

Throughout this paper, all measures are assumed to be positive, all functions are assumed to be positive and measurable, and expressions of the form $0 \cdot \infty$, $\frac{\infty}{\infty}$ and $\frac{0}{0}$ are taken to be equal to



zero. By a weight function (shortly: a weight) we mean a non-negative measurable function on the actual set. $B(\,\cdot\,,\,\cdot\,)$ denotes the standard Beta function defined by

$$B(a,b)=\int_0^1 t^{a-1}(1-t)^{b-1}\,\mathrm{d}t,\qquad a,b>0.$$

The rest of the paper is organized in the following way. In Section 2, we give the well-known Hardy, Pólya-Knopp, Hardy-Hilbert, Hardy-Littlewood-Pólya and Hilbert-Hardy-type inequalities in quotients. We consider some particular weight functions to give the related examples. We conclude this paper by providing the new results for multidimensional setting.

2. Results

First we obtain our central result using a particular substitution, that is, if we substitute $k(x,y)$ by $k(x,y)f_2(y)$ and f by f_1/f_2, where $f_i\colon \Omega_2\to\mathbb{R}$, $(i=1,2)$ are measurable functions in Theorem 1.4, we obtain the following result.

Theorem 2.1. *Let $(\Omega_1,\Sigma_1,\mu_1)$ and $(\Omega_2,\Sigma_2,\mu_2)$ be measure spaces with σ-finite measures, u be a weight function on Ω_1 and k be a non-negative measurable function on $\Omega_1\times\Omega_2$. Let $0<p\le q<\infty$, the function $x\mapsto u(x)\Big(\frac{k(x,y)f_2(y)}{g_2(x)}\Big)^{\frac{q}{p}}$ be integrable on Ω_1 for each fixed $y\in\Omega_2$ and v be defined on Ω_2 by*

$$v(y):=f_2(y)\Bigg(\int_{\Omega_1} u(x)\Big(\frac{k(x,y)}{g_2(x)}\Big)^{\frac{q}{p}}\mathrm{d}\mu_1(x)\Bigg)^{\frac{p}{q}}<\infty,\qquad g_2(x)\neq 0. \tag{2.1}$$

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If Φ is a positive convex function on the interval $I \subseteq \mathbb{R}$, then the following inequality

$$\left(\int_{\Omega_1} u(x)\left[\Phi\left(\frac{g_1(x)}{g_2(x)}\right)\right]^{\frac{q}{p}} \mathrm{d}\mu_1(x)\right)^{\frac{1}{q}} \le \left(\int_{\Omega_2} v(y)\Phi\left(\frac{f_1(y)}{f_2(y)}\right) \mathrm{d}\mu_2(y)\right)^{\frac{1}{p}} \tag{2.2}$$

holds for all measurable functions $f_i\colon \Omega_2 \to \mathbb{R}$, $(i = 1, 2)$, such that $\frac{f_1(y)}{f_2(y)} \in I$ and

$$g_i(x) = \int_{\Omega_2} k(x, y) f_i(y)\, \mathrm{d}y, \qquad (i = 1, 2). \tag{2.3}$$

As a special case of Theorem 2.1 for $p = q$, we obtain the upcoming corollary. Also note that the function Φ need not to be positive.

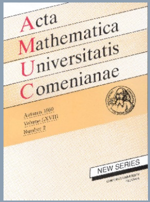
Corollary 2.2. *Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces with σ-finite measures, u be a weight function on Ω_1 and k be a non-negative measurable function on $\Omega_1 \times \Omega_2$. Suppose that the function $x \mapsto u(x)\frac{k(x,y)f_2(y)}{g_2(x)}$ is integrable on Ω_1 for each fixed $y \in \Omega_2$ and v is defined on Ω_2 by*

$$v(y) := f_2(y) \int_{\Omega_1} u(x) \frac{k(x, y)}{g_2(x)} \mathrm{d}\mu_1(x) < \infty, \qquad g_2(x) \neq 0. \tag{2.4}$$

If Φ is a convex function on the interval $I \subseteq \mathbb{R}$, then the inequality

$$\int_{\Omega_1} u(x)\Phi\left(\frac{g_1(x)}{g_2(x)}\right) \mathrm{d}\mu_1(x) \le \int_{\Omega_2} v(y)\Phi\left(\frac{f_1(y)}{f_2(y)}\right) \mathrm{d}\mu_2(y) \tag{2.5}$$

holds for all measurable functions $f_i\colon \Omega_2 \to \mathbb{R}$, $(i = 1, 2)$, such that $\frac{f_1(y)}{f_2(y)} \in I$ and g_i is defined by (2.3)



Remark 2.3. If we take $p = q$, $\Omega_1 = \Omega_2 = (a, b)$, $\mathrm{d}\mu_1(x) = \mathrm{d}x$ and $\mathrm{d}\mu_2(y) = \mathrm{d}y$ in Theorem 2.1, we obtain the result given in [8, Theorem 2.1]. So Theorem 2.1 is the generalized version of [8, Theorem 2.1].

Although the inequality (2.2) holds for all positive convex functions, some choices of Φ are of our particular interest. Let the function $\Phi\colon \mathbb{R}_+ \to \mathbb{R}$ be defined by $\Phi(x) = x^p$, so Φ is convex for $p \in \mathbb{R} \setminus [0, 1)$, concave for $p \in (0, 1]$, and affine, that is, both convex and concave for $p = 1$. In upcoming results we apply our results to power functions.

In next theorem, we give a general result for Hardy's inequality in quotient.

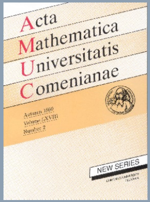
Theorem 2.4. *Let $0 < p \leq q < \infty$ and u be a weight function defined on $(0, \infty)$. Define v on $(0, \infty)$ by*

$$v(y) = f_2(y)\Big(\int_y^\infty \Big(\int_0^x f_2(y)\,\mathrm{d}y\Big)^{-\frac{q}{p}} u(x)\,\mathrm{d}x\Big)^{\frac{p}{q}} < \infty. \tag{2.6}$$

If Φ is a positive convex function on the interval $I \subseteq \mathbb{R}$, then the following inequality

$$\left(\int_0^\infty u(x)\left[\Phi\left(\frac{\int_0^x f_1(y)\,\mathrm{d}y}{\int_0^x f_2(y)\,\mathrm{d}y}\right)\right]^{\frac{q}{p}} \mathrm{d}x\right)^{\frac{1}{q}} \leq \left(\int_0^\infty v(y)\Phi\left(\frac{f_1(y)}{f_2(y)}\right)\mathrm{d}y\right)^{\frac{1}{p}} \tag{2.7}$$

holds for all measurable functions $f_i\colon (0, \infty) \to \mathbb{R}$, $(i = 1, 2)$, such that $\frac{f_1(y)}{f_2(y)} \in I$.



Proof. Rewrite the inequality (2.2) with $\Omega_1 = \Omega_2 = \mathbb{R}_+$, $\mathrm{d}\mu_1(x) = \mathrm{d}x$, $\mathrm{d}\mu_2(y) = \mathrm{d}y$. Let us define the kernel $k\colon \mathbb{R}_+^2 \to \mathbb{R}$ by

$$k(x,y) = \begin{cases} 1, & 0 < y \le x; \\ 0, & y > x, \end{cases} \tag{2.8}$$

then g_i defined in (2.3) takes the form

$$g_i(x) = \int_0^x f_i(y)\,\mathrm{d}y. \tag{2.9}$$

Substituting $g_i(x)$, $(i = 1, 2)$, in (2.2), we get (2.7). $\square$

Example 2.5. If we take $\Phi(x) = x^p$, $p \ge 1$ and a particular weight function $u(x) = \frac{1}{x^2}\Big(\int_0^x f_2(y)\,\mathrm{d}y\Big)^{\frac{q}{p}}$, $x \in (0,\infty)$ in (2.6), we obtain $v(y) = y^{-\frac{p}{q}} f_2(y)$ and the inequality (2.7) becomes

$$\begin{aligned} &\Big(\int_0^\infty \Big(\int_0^x f_1(y)\,\mathrm{d}y\Big)^q \Big(\int_0^x f_2(y)\,\mathrm{d}y\Big)^{q(\frac{1}{p}-1)} \frac{\mathrm{d}x}{x^2}\Big)^{\frac{1}{q}} \\ &\qquad \le \Big(\int_0^\infty y^{-\frac{p}{q}} f_1^p(y) f_2^{1-p}(y)\,\mathrm{d}y\Big)^{\frac{1}{p}}. \end{aligned} \tag{2.10}$$

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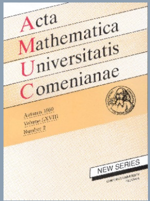

For $q = p$, the inequality (2.10) reduces to

$$\int_0^\infty \Big(\int_0^x f_1(y)\,\mathrm{d}y\Big)^p \Big(\int_0^x f_2(y)\,\mathrm{d}y\Big)^{1-p} \frac{\mathrm{d}x}{x^2} \le \int_0^\infty f_1^p(y) f_2^{1-p}(y) \frac{\mathrm{d}y}{y}. \tag{2.11}$$

If we take $f_2(y) = 1$ in (2.11), we obtain the following inequality (for details see [10] and [12])

$$\int_0^\infty \Big(\frac{1}{x}\int_0^x f_1(y)\,\mathrm{d}y\Big)^p \frac{\mathrm{d}x}{x} \le \int_0^\infty f_1^p(y) \frac{\mathrm{d}y}{y}. \tag{2.12}$$

On the other hand, for the convex function $\Phi\colon \mathbb{R} \to \mathbb{R}$ defined by $\Phi(x) = \mathrm{e}^x$, we can give the general form of Pólya-Knopp's inequality in quotients.

Corollary 2.6. *Let $0 < p \le q < \infty$ and u be a weight function defined on $(0,\infty)$. Defining v on $(0,\infty)$ by*

$$v(y) = \ln f_2(y) \Big(\int_y^\infty \Big(\int_0^x \ln f_2(y)\,\mathrm{d}y\Big)^{-\frac{q}{p}} u(x)\,\mathrm{d}x\Big)^{\frac{p}{q}} < \infty,$$

the following inequality

$$\left(\int_0^\infty u(x)\left[\exp\left(\frac{\int_0^x \ln f_1(y)\,\mathrm{d}y}{\int_0^x \ln f_2(y)\,\mathrm{d}y}\right)\right]^{\frac{q}{p}} \mathrm{d}x\right)^{\frac{1}{q}} \le \left(\int_0^\infty v(y)\exp\left(\frac{\ln f_1(y)}{\ln f_2(y)}\right)\mathrm{d}y\right)^{\frac{1}{p}} \tag{2.13}$$

holds for all positive measurable functions $f_i\colon (0,\infty) \to \mathbb{R}$, $(i = 1, 2)$.

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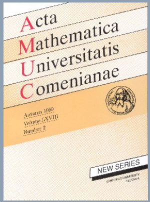

Proof. Rewrite the inequality (2.2) with $\Omega_1 = \Omega_2 = \mathbb{R}_+$, $\mathrm{d}\mu_1(x) = \mathrm{d}x$, $\mathrm{d}\mu_2(y) = \mathrm{d}y$ and $\Phi\colon \mathbb{R}_+ \to \mathbb{R}$ defined by $\Phi(x) = \mathrm{e}^x$. We obtain

$$\left(\int_0^\infty u(x)\left[\exp\left(\frac{g_1(x)}{g_2(x)}\right)\right]^{\frac{q}{p}}\mathrm{d}x\right)^{\frac{1}{q}} \le \left(\int_0^\infty v(y)\exp\left(\frac{f_1(y)}{f_2(y)}\right)\mathrm{d}y\right)^{\frac{1}{p}}. \tag{2.14}$$

Define $k(x,y)$ as in the proof of Theorem 2.4. Substituting $g_i(x)$, $(i = 1,2)$, defined by (2.9) in (2.14), we get

$$\left(\int_0^\infty u(x)\left[\exp\left(\frac{\int_0^x f_1(y)\,\mathrm{d}y}{\int_0^x f_2(y)\,\mathrm{d}y}\right)\right]^{\frac{q}{p}}\mathrm{d}x\right)^{\frac{1}{q}} \le \left(\int_0^\infty v(y)\exp\left(\frac{f_1(y)}{f_2(y)}\right)\mathrm{d}y\right)^{\frac{1}{p}}. \tag{2.15}$$

Replacing f_i by $\ln f_i$, $(i = 1,2)$ in (2.15), we get (2.13). $\square$

Remark 2.7. In particular, for the weight function $u(x) = \frac{1}{x^2}\left(\int_0^x \ln f_2(y)\,\mathrm{d}y\right)^{\frac{q}{p}}$, $x \in (0,\infty)$ in Corollary 2.6, we obtain $v(y) = y^{-\frac{p}{q}}\ln f_2(y)$ and the inequality (2.13) becomes

$$\begin{aligned}\left(\int_0^\infty \frac{1}{x^2}\left(\int_0^x \ln f_2(y)\,\mathrm{d}y\right)^{\frac{q}{p}}\left[\exp\left(\frac{\int_0^x \ln f_1(y)\,\mathrm{d}y}{\int_0^x \ln f_2(y)\,\mathrm{d}y}\right)\right]^{\frac{q}{p}}\mathrm{d}x\right)^{\frac{1}{q}}&\\ \le \left(\int_0^\infty y^{-\frac{p}{q}}\ln f_2(y)\exp\left(\frac{\ln f_1(y)}{\ln f_2(y)}\right)\mathrm{d}y\right)^{\frac{1}{p}}&.\end{aligned} \tag{2.16}$$

If we put $p = q$ and $f_2(y) = e$ in (2.16), we obtain the following inequality (for details see [10] and [12])

$$\int_0^\infty \exp\Big(\frac{1}{x}\int_0^x \ln f_1(y)\,\mathrm{d}y\Big)\frac{\mathrm{d}x}{x} \le \int_0^\infty f_1(y)\frac{\mathrm{d}y}{y}.$$

Our next general result is for Hardy-Hilbert's inequality.

Theorem 2.8. *Let $0 < p \le q < \infty$, $s \in \mathbb{R}$ and u be a weight function defined on $(0,\infty)$. Define v on $(0,\infty)$ by*

$$v(y) = f_2(y)\Big(\int_0^\infty \frac{u(x)}{(x+y)^{\frac{sq}{p}}}\Big(\int_0^\infty \frac{f_2(y)}{(x+y)^s}\,\mathrm{d}y\Big)^{-\frac{q}{p}}\mathrm{d}x\Big)^{\frac{p}{q}} < \infty. \tag{2.17}$$

If Φ is a positive convex function on the interval $I \subseteq \mathbb{R}$, then the following inequality

$$\left(\int_0^\infty u(x)\left[\Phi\left(\frac{\int_0^\infty \frac{f_1(y)}{(x+y)^s}\,\mathrm{d}y}{\int_0^\infty \frac{f_2(y)}{(x+y)^s}\,\mathrm{d}y}\right)\right]^{\frac{q}{p}}\mathrm{d}x\right)^{\frac{1}{q}} \le \left(\int_0^\infty v(y)\Phi\left(\frac{f_1(y)}{f_2(y)}\right)\mathrm{d}y\right)^{\frac{1}{p}} \tag{2.18}$$

holds for all measurable functions $f_i\colon (0,\infty) \to \mathbb{R}$, $(i = 1,2)$, such that $\frac{f_1(y)}{f_2(y)} \in I$.

Proof. Rewrite the inequality (2.2) with $\Omega_1 = \Omega_2 = \mathbb{R}_+$, $\mathrm{d}\mu_1(x) = \mathrm{d}x$, $\mathrm{d}\mu_2(y) = \mathrm{d}y$. Let us define the kernel $k\colon \mathbb{R}_+^2 \to \mathbb{R}$ by

$$k(x,y) = \Big(\frac{y}{x}\Big)^{\frac{s-2}{p}}(x+y)^{-s}, \qquad p > 1.$$

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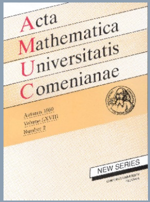


Then g_i defined in (2.3) becomes

$$g_i(x) = \int_0^\infty \Big(\frac{y}{x}\Big)^{\frac{s-2}{p}} (x+y)^{-s} f_i(y)\,\mathrm{d}y = x^{\frac{2-s}{p}} \int_0^\infty y^{\frac{s-2}{p}} \frac{f_i(y)}{(x+y)^s}\,\mathrm{d}y.$$

Substituting $g_i(x)$, $(i = 1, 2)$, in (2.2), we get

$$\Bigg(\int_0^\infty u(x)\Bigg[\Phi\Bigg(\frac{\int_0^\infty y^{\frac{s-2}{p}} \frac{f_1(y)}{(x+y)^s}\,\mathrm{d}y}{\int_0^\infty y^{\frac{s-2}{p}} \frac{f_2(y)}{(x+y)^s}\,\mathrm{d}y}\Bigg)\Bigg]^{\frac{q}{p}} \mathrm{d}x\Bigg)^{\frac{1}{q}} \le \Bigg(\int_0^\infty v(y)\Phi\Big(\frac{f_1(y)}{f_2(y)}\Big)\,\mathrm{d}y\Bigg)^{\frac{1}{p}}. \tag{2.19}$$

Writing $f_i(y)$ instead of $f_i(y)y^{\frac{s-2}{p}}$ in (2.19), we obtain (2.18). $\square$

Example 2.9. For $0 < \alpha < \frac{sq}{p}$, taking the particular weight function $u(x) = x^{\alpha-1} \cdot \Big(\int_0^\infty (x+y)^{-s} f_2(y)\,\mathrm{d}y\Big)^{\frac{q}{p}}$, $x \in (0,\infty)$, in (2.17), we obtain $v(y) = y^{\frac{\alpha p}{q}-s} f_2(y)\Big(B\Big(\alpha, \frac{sq}{p} - \alpha\Big)\Big)^{\frac{p}{q}}$, where B is the usual beta function. Let $p \ge 1$ and the function $\Phi\colon \mathbb{R}_+ \to \mathbb{R}$ be defined by $\Phi(x) = x^p$, then the inequality (2.18) becomes

$$\begin{aligned}\Big(\int_0^\infty x^{\alpha-1}\Big(\int_0^\infty \frac{f_1(y)}{(x+y)^s}\,\mathrm{d}y\Big)^q\Big(\int_0^\infty \frac{f_2(y)}{(x+y)^s}\,\mathrm{d}y\Big)^{q(\frac{1}{p}-1)}\,\mathrm{d}x\Big)^{\frac{1}{q}}&\\ \le \Big(B\Big(\alpha, \frac{sq}{p} - \alpha\Big)\Big)^{\frac{1}{q}}\Big(\int_0^\infty y^{\frac{\alpha p}{q}-s} f_1^p(y) f_2^{1-p}(y)\,\mathrm{d}y\Big)^{\frac{1}{p}}&.\end{aligned} \tag{2.20}$$

In the upcoming theorem, we give the Hardy-Littlewood-Pólya inequality in quotient.

Theorem 2.10. *Let $0 < p \leq q < \infty$, $s \in \mathbb{R}$ and u be a weight function defined on $(0,\infty)$. Define v on $(0,\infty)$ by*

$$v(y) = f_2(y)\Big(\int_0^\infty \frac{u(x)}{\max\{x,y\}^{\frac{sq}{p}}}\Big(\int_0^\infty \frac{f_2(y)}{\max\{x,y\}^s}\,\mathrm{d}y\Big)^{-\frac{q}{p}}\mathrm{d}x\Big)^{\frac{p}{q}} < \infty. \tag{2.21}$$

If Φ is a positive convex function on the interval $I \subseteq \mathbb{R}$, then the following inequality

$$\left(\int_0^\infty u(x)\left[\Phi\left(\frac{\int_0^\infty \frac{f_1(y)}{\max\{x,y\}^s}\,\mathrm{d}y}{\int_0^\infty \frac{f_2(y)}{\max\{x,y\}^s}\,\mathrm{d}y}\right)\right]^{\frac{q}{p}}\mathrm{d}x\right)^{\frac{1}{q}} \leq \left(\int_0^\infty v(y)\Phi\left(\frac{f_1(y)}{f_2(y)}\right)\mathrm{d}y\right)^{\frac{1}{p}} \tag{2.22}$$

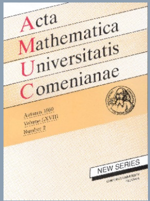
holds for all measurable functions $f_i\colon (0,\infty) \to \mathbb{R}$, $(i = 1,2)$, such that $\frac{f_1(y)}{f_2(y)} \in I$.

Proof. Rewrite the inequality (2.2) with $\Omega_1 = \Omega_2 = \mathbb{R}_+$, $\mathrm{d}\mu_1(x) = \mathrm{d}x$, $\mathrm{d}\mu_2(y) = \mathrm{d}y$. Let us define the kernel $k\colon \mathbb{R}_+^2 \to \mathbb{R}$ by

$$k(x,y) = \left(\frac{y}{x}\right)^{\frac{s-2}{p}} \max\{x,y\}^{-s}.$$

Then g_i defined in (2.3) takes the form

$$g_i(x) = \int_0^\infty \left(\frac{y}{x}\right)^{\frac{s-2}{p}} \max\{x,y\}^{-s} f_i(y)\,\mathrm{d}y = x^{\frac{2-s}{p}} \int_0^\infty y^{\frac{s-2}{p}} \frac{f_i(y)}{\max\{x,y\}^s}\,\mathrm{d}y.$$



Substituting $g_i(x)$, $(i = 1, 2)$ in (2.2), we get

$$\left(\int_0^\infty u(x)\left[\Phi\left(\frac{\int_0^\infty y^{\frac{s-2}{p}}\frac{f_1(y)}{\max\{x,y\}^s}\,\mathrm{d}y}{\int_0^\infty y^{\frac{s-2}{p}}\frac{f_2(y)}{\max\{x,y\}^s}\,\mathrm{d}y}\right)\right]^{\frac{q}{p}}\mathrm{d}x\right)^{\frac{1}{q}} \le \left(\int_0^\infty v(y)\Phi\left(\frac{f_1(y)}{f_2(y)}\right)\mathrm{d}y\right)^{\frac{1}{p}}. \tag{2.23}$$

Writing $f_i(y)$ instead of $f_i(y)y^{\frac{s-2}{p}}$ in (2.23), we obtain (2.22). $\square$

Example 2.11. For $0 < \alpha < \frac{sq}{p}$, taking the particular weight function $u(x) = x^{\alpha-1}\left(\int_0^\infty \frac{f_2(y)}{\max\{x,y\}^s}\,\mathrm{d}y\right)^{\frac{q}{p}}$, $x \in (0,\infty)$ in (2.21), we obtain $v(y) = y^{\frac{\alpha p}{q}-s}f_2(y) \times \left(\frac{sq}{\alpha(sq-\alpha p)}\right)^{\frac{p}{q}}$. Let $p \ge 1$ and the function $\Phi\colon \mathbb{R}_+ \to \mathbb{R}$ be defined by $\Phi(x) = x^p$. Then the inequality (2.22) becomes

$$\begin{aligned}\left(\int_0^\infty x^{\alpha-1}\left(\int_0^\infty \frac{f_1(y)}{\max\{x,y\}^s}\,\mathrm{d}y\right)^q\left(\int_0^\infty \frac{f_2(y)}{\max\{x,y\}^s}\,\mathrm{d}y\right)^{q\left(\frac{1}{p}-1\right)}\mathrm{d}x\right)^{\frac{1}{q}}&\\ \le \left(\frac{sq}{\alpha(sq-\alpha p)}\right)^{\frac{1}{q}}\left(\int_0^\infty y^{\frac{\alpha p}{q}-s}f_1^p(y)f_2^{1-p}(y)\,\mathrm{d}y\right)^{\frac{1}{p}}.&\end{aligned} \tag{2.24}$$

Now we give the result for Hardy-Hilbert-type inequality.

Theorem 2.12. *Let $0 < p \le q < \infty$ and u be a weight function defined on $(0,\infty)$. Define v on $(0,\infty)$ by*

$$v(y)=f_2(y)\Big(\int_0^\infty u(x)\Big(\frac{\ln y-\ln x}{y-x}\Big)^{\frac{q}{p}}\Big(\int_0^\infty\frac{\ln y-\ln x}{y-x}f_2(y)\,\mathrm{d}y\Big)^{-\frac{q}{p}}\mathrm{d}x\Big)^{\frac{p}{q}}<\infty. \tag{2.25}$$

If Φ is a positive convex function on the interval $I \subseteq \mathbb{R}$, then the following inequality

$$\left(\int_0^\infty u(x)\left[\Phi\left(\frac{\int_0^\infty \frac{\ln y-\ln x}{y-x}f_1(y)\,\mathrm{d}y}{\int_0^\infty \frac{\ln y-\ln x}{y-x}f_2(y)\,\mathrm{d}y}\right)\right]^{\frac{q}{p}}\mathrm{d}x\right)^{\frac{1}{q}} \le \left(\int_0^\infty v(y)\Phi\left(\frac{f_1(y)}{f_2(y)}\right)\mathrm{d}y\right)^{\frac{1}{p}} \tag{2.26}$$

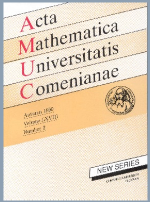
holds for all measurable functions $f_i\colon (0,\infty)\to\mathbb{R}$, $(i=1,2)$, such that $\frac{f_1(y)}{f_2(y)} \in I$.

Proof. Rewrite the inequality (2.2) with $\Omega_1 = \Omega_2 = \mathbb{R}_+$, $\mathrm{d}\mu_1(x) = \mathrm{d}x$, $\mathrm{d}\mu_2(y) = \mathrm{d}y$. For $\alpha \in (0,1)$, we define the kernel $k\colon \mathbb{R}_+^2 \to \mathbb{R}$ by

$$k(x,y)=\frac{\ln y-\ln x}{y-x}\Big(\frac{x}{y}\Big)^{\alpha}.$$

Then g_i defined in (2.3) takes the form

$$g_i(x)=\int_0^\infty \frac{\ln y-\ln x}{y-x}\Big(\frac{x}{y}\Big)^{\alpha}f_i(y)\,\mathrm{d}y = x^{\alpha}\int_0^\infty\frac{\ln y-\ln x}{y-x}y^{-\alpha}f_i(y)\,\mathrm{d}y.$$



Substituting $g_i(x)$, $(i=1,2)$ in (2.2), we get

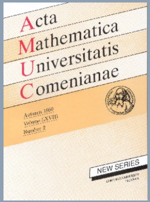
$$\left(\int_0^\infty u(x)\left[\Phi\left(\frac{\int_0^\infty \frac{\ln y-\ln x}{y-x} y^{-\alpha} f_1(y)\,\mathrm{d}y}{\int_0^\infty \frac{\ln y-\ln x}{y-x} y^{-\alpha} f_2(y)\,\mathrm{d}y}\right)\right]^{\frac{q}{p}} \mathrm{d}x\right)^{\frac{1}{q}} \leq \left(\int_0^\infty v(y)\Phi\left(\frac{f_1(y)}{f_2(y)}\right)\mathrm{d}y\right)^{\frac{1}{p}}. \tag{2.27}$$

Writing $f_i(y)$ instead of $y^{-\alpha} f_i(y)$ in (2.27), we obtain (2.26). □

Example 2.13. For $\alpha\in(0,1)$ and for the particular weight function $u(x)=x^{-\alpha}\left(\int_0^\infty \frac{\ln y-\ln x}{y-x} f_2(y)\,\mathrm{d}y\right)^{\frac{q}{p}}$, $x\in(0,\infty)$, in (2.25), we obtain $v(y)=y^{(1-\alpha)\frac{p}{q}-1} f_2(y)C$, where $C=\left(\int_0^\infty z^{-\alpha}\left(\frac{\ln z}{z-1}\right)^{\frac{q}{p}} dz\right)^{\frac{p}{q}}$. Let $p\geq 1$ and the function $\Phi\colon \mathbb{R}_+\to\mathbb{R}$ be defined by $\Phi(x)=x^p$. Then the inequality (2.26) becomes

$$\begin{aligned}&\left(\int_0^\infty x^{-\alpha}\left(\int_0^\infty \frac{\ln y-\ln x}{y-x} f_1(y)\,\mathrm{d}y\right)^q\left(\int_0^\infty \frac{\ln y-\ln x}{y-x} f_2(y)\,\mathrm{d}y\right)^{q\left(\frac{1}{p}-1\right)}\mathrm{d}x\right)^{\frac{1}{q}}\\ &\qquad\leq \left(C\int_0^\infty y^{(1-\alpha)\frac{p}{q}-1} f_1^p(y) f_2^{1-p}(y)\,\mathrm{d}y\right)^{\frac{1}{p}}.\end{aligned} \tag{2.28}$$

In the upcoming theorem we give the result for a multidimensional case.



Theorem 2.14. *Let $0 < p \le q < \infty$, l and u be non-negative measurable functions on $\mathbb{R}^n_+$. Suppose that the function*

$$\boldsymbol{x} \to u(\boldsymbol{x})\Big(f_2(\boldsymbol{y})l\Big(\frac{\boldsymbol{y}}{\boldsymbol{x}}\Big)\Big(\int\limits_{\mathbb{R}^n_+} l\Big(\frac{\boldsymbol{y}}{\boldsymbol{x}}\Big)f_2(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\Big)^{-1}\Big)^{\frac{q}{p}}$$

is integrable on $\mathbb{R}^n_+$ for each fixed $\boldsymbol{y} \in \mathbb{R}^n_+$. Let v be defined on $\mathbb{R}^n_+$ by

$$v(\boldsymbol{y}) = \Big(\int\limits_{\mathbb{R}^n_+} u(\boldsymbol{x})\Big(f_2(\boldsymbol{y})l\Big(\frac{\boldsymbol{y}}{\boldsymbol{x}}\Big)\Big(\int\limits_{\mathbb{R}^n_+} l\Big(\frac{\boldsymbol{y}}{\boldsymbol{x}}\Big)f_2(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\Big)^{-1}\Big)^{\frac{q}{p}}\mathrm{d}\boldsymbol{x}\Big)^{\frac{p}{q}}. \tag{2.29}$$

If Φ is a positive convex function on the interval $I \subseteq \mathbb{R}$, then the following inequality

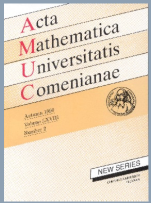
$$\left(\int\limits_{\mathbb{R}^n_+} u(\boldsymbol{x})\left[\Phi\left(\frac{\int_{\mathbb{R}^n_+} l\left(\frac{\boldsymbol{y}}{\boldsymbol{x}}\right) f_1(\boldsymbol{y})\mathrm{d}\boldsymbol{y}}{\int_{\mathbb{R}^n_+} l\left(\frac{\boldsymbol{y}}{\boldsymbol{x}}\right) f_2(\boldsymbol{y})\mathrm{d}\boldsymbol{y}}\right)\right]^{\frac{q}{p}}\mathrm{d}\boldsymbol{x}\right)^{\frac{1}{q}} \le \left(\int\limits_{\mathbb{R}^n_+} v(\boldsymbol{y})\Phi\left(\frac{f_1(\boldsymbol{y})}{f_2(\boldsymbol{y})}\right)\mathrm{d}\boldsymbol{y}\right)^{\frac{1}{p}} \tag{2.30}$$

holds for all measurable functions $f_i\colon \mathbb{R}^n_+ \to \mathbb{R}$, $(i = 1, 2)$, such that $\frac{f_1(\boldsymbol{y})}{f_2(\boldsymbol{y})} \in I$.

Proof. Apply Theorem 2.1 with $\Omega_1 = \Omega_2 = \mathbb{R}^n_+$, the Lebesgue measure $\mathrm{d}\mu_1(\boldsymbol{x}) = \mathrm{d}\boldsymbol{x}$, $\mathrm{d}\mu_2(\boldsymbol{y}) = \mathrm{d}\boldsymbol{y}$, and the kernel $k\colon \mathbb{R}^n_+ \times \mathbb{R}^n_+ \to \mathbb{R}$ of the form $k(\boldsymbol{x}, \boldsymbol{y}) = l\big(\frac{\boldsymbol{y}}{\boldsymbol{x}}\big)$, where $l\colon \mathbb{R}^n_+ \to \mathbb{R}$ is a non-negative measurable function. So $g_i(\boldsymbol{x})$ takes the form

$$g_i(\boldsymbol{x}) = \int\limits_{\mathbb{R}^n_+} l\Big(\frac{\boldsymbol{y}}{\boldsymbol{x}}\Big)f_i(\boldsymbol{y})\mathrm{d}\boldsymbol{y}, \qquad (i = 1, 2),$$

and inequality (2.30) follows. $\square$



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Applying Theorem 2.14 to the power function, we get the following corollary.

Corollary 2.15. *Let $p > 1$ and suppose that the assumptions in Theorem 2.14 are satisfied. Let v be defined by* (2.29)*. Then the following inequality*

$$\left(\int_{\mathbb{R}^n_+} u(\boldsymbol{x}) \left(\frac{\int_{\mathbb{R}^n_+} l\left(\frac{\boldsymbol{y}}{\boldsymbol{x}}\right) f_1(\boldsymbol{y}) \mathrm{d}\boldsymbol{y}}{\int_{\mathbb{R}^n_+} l\left(\frac{\boldsymbol{y}}{\boldsymbol{x}}\right) f_2(\boldsymbol{y}) \mathrm{d}\boldsymbol{y}} \right)^q \mathrm{d}\boldsymbol{x}\right)^{\frac{1}{q}} \leq \left(\int_{\mathbb{R}^n_+} v(\boldsymbol{y}) \left(\frac{f_1(\boldsymbol{y})}{f_2(\boldsymbol{y})}\right)^p \mathrm{d}\boldsymbol{y}\right)^{\frac{1}{p}} \tag{2.31}$$

holds for all measurable functions $f_i \colon \mathbb{R}^n_+ \to \mathbb{R}$, $(i = 1, 2)$.

Remark 2.16. If we take $f_2 = 1$ in Theorem 2.14, we obtain inequality (1.9) given in Corollary 1.5. So Theorem 2.14 is the quotient form of Corollary 1.5.

Remark 2.17. Particularly, if we take $p = q$ in Theorem 2.4, Corollary 2.6, Theorem 2.8, Theorem 2.10 and Theorem 2.12, we can obtain the corresponding results of Corollary 2.2 in quotients for Hardy's inequality, Pólya-Knopp's inequality, Hardy-Hilbert's inequality, Hardy-Littlewood-Pólya inequality and Hardy-Hilbert-type inequality, respectively, but here we omit the details.

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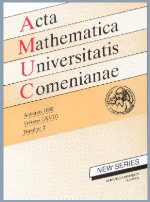

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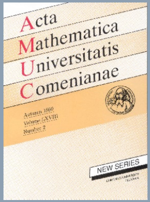

S. Iqbal, Department of Mathematics, University of Sargodha (Sub-Campus Bhakkar), Bhakkar, Pakistan, *e-mail*: `sajid_uos2000@yahoo.com`

K. Krulić Himmelreich, 2-Faculty of Textile Technology, University of Zagreb, Prilaz baruna Filipovića 28a, 10000 Zagreb, Croatia, *e-mail*: `kkrulic@ttf.hr`

J. Pečarić, 2-Faculty of Textile Technology, University of Zagreb, Prilaz baruna Filipovića 28a, 10000 Zagreb, Croatia, *e-mail*: `pecaric@element.hr`