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NON-SINGULAR COCYCLES AND PIECEWISE LINEAR TIME CHANGES

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Abstract. Cocycles of Z^m-actions on compact metric spaces provide a means for constructing R^m-actions or flows, called suspension flows. It is known that all R^m flows with a free dense orbit have an almost one-to-one extension which is a suspension flow. In this paper we investigate when the space for a suspension flow depends only on the given Z^m-action and not on the actual cocycle. The identity map I of R^m determines perhaps the simplest cocycle for any Z^m-action. We introduce invertible cocycles, and show that they produce the same space as the cocycle determined by the identity map I. The main result, Theorem 5.2, establishes an integration test for invertibility using piecewise linear maps and related topological ideas. Finally, it applies to the known methods for modeling R^m flows as suspensions and leads to refinements of these results.

1. Introduction

Let X be a compact metric space and let Z^m act as a group of commuting homeomorphisms on X. That is, we have a Z^m-action on X giving us the discrete dynamical system (X, Z^m). For $a \in Z^m$, we denote the action of a on $x \in X$ by ax. A **cocycle** for (X, Z^m) is a continuous map $h\colon X \times Z^m \to R^m$ satisfying the cocycle equation:

$$h(x, a + b) = h(x, a) + h(ax, b)$$

for all $a, b \in Z^m$ and $x \in X$.

A cocycle $h\colon X \times Z^m \to R^m$ can be used to construct the **suspension** (X_h, R^m) of (X, Z^m). This is done as follows: for each $a \in Z^m$ we obtain a homeomorphism on $X \times R^m$ given by

$$T_a(x, v) = (ax, v - h(x, a))$$

for $a \in Z^m$, $x \in X$ and $v \in R^m$. Because h is a cocycle, it is easily checked that $T_{a+b} = T_a \circ T_b$. Hence, Z^m acts as a group of commuting homeomorphisms on $X \times R^m$, and we have a Z^m-action on $X \times R^m$. We also have a natural R^m-action

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on $X \times R^m$ given as follows: for each $w \in R^m$ we obtain a homeomorphism on $X \times R^m$ via

$$((x, v), w) \to (x, v + w)$$

which is continuous in all three variables and clearly defines an R^m-action on $X \times R^m$.

We form

$$X_h = X \times R^m / \{T_a : a \in Z^m\},$$

the quotient space of $X \times R^m$ modulo the Z^m-action on $X \times R^m$. Let

$$\pi \colon X \times R^m \to X_h$$

be the usual projection. It is easily verified that the Z^m- and R^m-actions on $X \times R^m$ commute, and thus, R^m acts as a group of commuting homeomorphisms on X_h via

$$\big(\pi(x, v), w\big) \to \pi(x, v + w).$$

So we have the R^m flow (X_h, R^m). This flow is referred to as the R^m suspension of (X, Z^m) built using the cocycle h and $O(y) = \{yv : v \in R^m\}$ denotes the orbit of $y \in X_h$. When $m = 1$, the suspension flow is the familiar flow under a function.

In this context it is natural to write our Z^m-actions on the left and R^m-actions on the right. A Z^m-action {an R^m-action } has a free orbit provided that for some x in the space $ax = x$ $\{xv = x\}$ only if $a = 0$ $\{v = 0\}$. The action itself is said to be free if every orbit is free. If for some $a \neq 0$ in Z^m we have $ax = x$ for all $x \in X$, then we do not really have a faithful Z^m-action on X. Consequently it is not surprising that most of our results require a free dense orbit.

Suspensions are of interest in the study of R^m-actions. We look to R^m suspensions as a way of generating examples of R^m flows with interesting dynamical behavior. Conversely, as the following theorem shows, suspensions play an important role in modeling general R^m flows [**6**].

Theorem 1.1. *An R^m flow (Y, R^m) on a compact metric space Y with a free dense orbit has an almost one-to-one extension which is a suspension.*

If $M \colon R^m \to R^m$ is linear then clearly we obtain a cocycle by $h(x, a) = M(a)$. Perhaps the simplest suspension of a discrete dynamical system (X, Z^m) is the one constructed from $I \colon R^m \to R^m$ the identity linear transformation. This suspension is denoted (X_I, R^m) and is referred to as the **constant one suspension**. Each element in X_I has a unique representative in the set

$$\{(x, u) : x \in X, u \in [0, 1)^m\},$$

and the R^m-action on X_I is given by

$$((x, u), v) \to ([u + v]x, \{u + v\})$$

where $x \in X$, $v \in R^m$, $[u+v] \in Z^m$, $u, \{u+v\} \in [0,1)^m$ and $u+v = [u+v] + \{u+v\}$.

The goal of this paper is to establish a necessary condition for the suspension flow constructed from a cocycle to be isomorphic to a time change of the constant one suspension. In other words, we want a criteria for the orbit structure of a suspension to be the same as the orbit structure of (X_I, R^m) which is completely determined by the Z^m-action on X. The necessary condition obtained in the main theorem is an analytic condition, but the proof depends primarily on piecewise linear topology.

Our main result is a partial answer to a deeper question, the unfolding problem, posed by H. Furstenberg. In one form the question is whether or not every suspension flow on a compact metric space is a time change of the constant one flow. In another form it asks when, after bounded modification, can an injective map from Z^m to R^m be extended to a homeomorphism of R^m to R^m. Extending the map $a \to h(x,a)$ to a homeomorphism will play a critical role in this paper.

After some preliminaries in Section 2, we will introduce invertible cocycles in Section 3. Roughly speaking, an invertible cocycle h is one that can be extended to a map $H\colon X \times R^m \to R^m$ satisfying a cocycle-like equation and yielding a homeomorphism $H(x,\cdot)\colon R^m \to R^m$ for each $x \in X$. A tractable subset of the invertible cocycles is the set of piecewise linear or PL invertible cocycles. In this case, the map $H(x,\cdot)\colon R^m \to R^m$ described above is obtained by triangulating R^m and extending $h(x,\cdot)\colon R^m \to R^m$ linearly over the simplices.

Our main result, Theorem 5.2, gives an integration test for PL invertibility. To this end, in Section 4 we review the concept of a cocycle integral first developed in [**4**]. When these cocycle integrals are nonsingular the cocycle h is said to be nonsingular. In Section 5 we will prove Theorem 5.2 which states that nonsingular cocycles are, up to a possible coboundary change, PL invertible.

Among the important examples of nonsingular cocycles are the cocycles used to construct the suspensions of Theorem 1.1. The nonsingularity of these cocycles is verified in Section 6. Then, as a consequence of Theorems 1.1 and 5.2, we conclude that any space that supports an R^m flow with a free dense orbit does not differ significantly from a constant one suspension of a Z^m discrete dynamical system with a free dense orbit. Thus from the point of view of Rudolph's theorem enough cocycles can be unfolded.

2. Preliminaries

Let $\mathcal{C}$ be the real vector space of R^m-valued cocycles on (X, Z^m) with the norm

$$\begin{aligned} ||h|| &= \sup\left\{\frac{|h(x,a)|}{|a|} : x \in X, a \in Z^m, a \neq 0\right\} \\ &= \sup\{|h(x,e_i)| : x \in X, 1 \leq i \leq m\} \end{aligned}$$

where $e_1, e_2, \dots, e_m$ is the standard basis for R^m. With this norm, $\mathcal{C}$ is a separable Banach space. There is also a natural Z^m-action on $\mathcal{C}$ given by $ah(x,b) = h(ax,b)$.

One of the goals of [**1**] was to study the relationship between the properties of h and the topological properties of X_h. This section contains a summary of some of the results we need from [**1**]. *A standing assumption for these results and for the remainder of this work is that (X, Z^m) has a free, dense orbit.*

Definition 2.1. A cocycle $h\colon X \times Z^m \to R^m$ is **embedding** if

(i) X_h is a Hausdorff space,
(ii) The projection $\pi\colon X \times R^m \to X_h$ is one-to-one on

$$X \times \{v \in R^m : |v| < \epsilon\}$$

for some $\epsilon > 0$ where $|v| = \sum |v_i|$ (i.e. X can be embedded in X_h as a global section of the R^m flow).

Thus, by definition, an embedding cocycle h is one which when used to construct a suspension (X_h, R^m) yields an appropriate analog to a flow under a function. In this setting X_h will have the minimum topological requirement of being Hausdorff and the original space X can be identified with a global section of X_h. We note that (ii) holds if and only if π is one-to-one on $X \times \{0\}$ and is a local homeomorphism on $X \times R^m$. The second of these properties suggests a fundamental covering-space like behavior and motivates the following additional definition.

Definition 2.2. A cocycle $h\colon X \times Z^m \to R^m$ is **covering** if

(i) X_h is a Hausdorff space,
(ii) The projection $\pi\colon X \times R^m \to X_h$ is a local homeomorphism on $X \times R^m$.

The salient feature of the covering and embedding cocycles is that they have "sufficient" growth. This is made precise in the following theorem from [**1**, Corollary 2.1]. The norm we will use in R^m is $|v| = \sum |v_i|$.

Theorem 2.1. *Let $h\colon X \times Z^m \to R^m$ be a cocycle. Then*

a) *h is covering if and only if there exists constants $A, B, B' > 0$ such that*

$$B|a| \le |h(x,a)| \le B'|a|$$

whenever $|a| \ge A$,

b) *h is embedding if and only if there exists constants $B, B' > 0$ such that*

$$B|a| \le |h(x,a)| \le B'|a|.$$

We have noted that embedding cocycles are the ones which give us the appropriate analog to a flow under a function when used to build a suspension. The larger class of covering cocycles does not yield as good an analogy; a suspension (X_h, R^m) built using a covering cocycle may lack the property of having X identified with a global section of X_h. However, the suspensions built with covering cocycles are still of interest because when h is covering, we are assured that X_h has "nice" topological properties ([**1**, Theorem 2.10]):

Theorem 2.2. *Let $h\colon X \times Z^m \to R^m$ be covering. Then X_h is compact.*

The following proposition follows easily from Theorem 2.1 and the definition of the norm on $\mathcal{C}$.

Proposition 2.1. *The embedding cocycles and the covering cocycles form open subsets of $\mathcal{C}$.*

There are two other natural subsets of $\mathcal{C}$ which are useful. First, for each $f \in C(X, R^m)$, the continuous R^m valued functions on X, we obtain an element of $\mathcal{C}$ by setting

$$h(x, a) = f(ax) - f(x).$$

Such a cocycle is called a **coboundary**. If two cocycles h and h' in $\mathcal{C}$ differ by a coboundary, they are said to be **cohomologous**, denoted $h \sim h'$. This notion is important because cohomologous cocycles result in topologically conjugate suspensions.

Another collection of maps occurring naturally in $\mathcal{C}$ as a closed subset is $\mathcal{L}$, the linear maps from R^m to R^m. For each $M \in \mathcal{L}$ we obtain a cocycle in $\mathcal{C}$ by $h(x, a) = M(a)$. An arbitrary element h in $\mathcal{C}$ is given by a linear transformation M as described above if and only if $h(x, a) = h(y, a)$ for any $x, y \in X$ and $a \in Z^m$. Such a cocycle is called a **constant cocycle**. The embedding constant cocycles are those for which M is nonsingular linear map.

3. Invertible Cocycles

When h is an embedding cocycle, X_h contains a global section $\pi_h(X \times \{0\})$ which is a homeomorphic copy of X. The return locations to this global section are given by the Z^m-action on X and the return times by the values of h. If we regard the latter as the least important part of this structure, we are led to consider homeomorphisms $\tilde{\Psi}\colon X \times R^m \to X \times R^m$ of the form $\tilde{\Psi}(x, v) = (x, \psi(x, v))$ satisfying for $a \in Z^m$

$$\tilde{\Psi} \circ S_a = T_a \circ \tilde{\Psi}$$

where $S_a(x, v) = (ax, v - g(x, a))$, $T_a(x, v) = (ax, v - h(x, a))$, $g, h \in \mathcal{C}$. Such a $\tilde{\Psi}$ induces a homeomorphism $\Psi\colon X_g \to X_h$ such that

$$\Psi(\mathcal{O}(\pi_g(x, 0))) = \mathcal{O}(\pi_h(x, 0)).$$

Observation 3.1. When the Z^m-action is free, the converse is true. That is, a homeomorphism $\Psi\colon X_g \to X_h$ with $\Psi(\mathcal{O}(\pi_g(x,0))) = \mathcal{O}(\pi_h(x,0))$ induces a homeomorphism $\tilde{\Psi}\colon X \times R^m \to X \times R^m$ as described above. The converse does not hold more generally.

Proof. For each x the map $v \to \pi_h(x,v)$ is one-to-one because the Z^m-action is free. One is forced to define $\tilde{\Psi}\colon X \times R^m \to X \times R^m$ by setting

$$\tilde{\Psi}(x,v) = \pi_h^{-1}\left(\Psi(\pi_g(x,v))\right) \cap (\{x\} \times R^m).$$

It follows that $\tilde{\Psi}$ is one-to-one, onto and, since both π_g and π_h are open and local homeomorphisms, $\tilde{\Psi}$ and $\tilde{\Psi}^{-1}$ are both continuous. That $\tilde{\Psi} \circ S_a = T_a \circ \tilde{\Psi}$ follows from the hypothesis that the Z^m-action is free.

When there are periodic points in X, the map $\tilde{\Psi}$ constructed as above for free orbits need not have a continuous extension to $X \times R^m$. For example, let X be the unit circle, let $m = 1$, and let $\phi(e^{2\pi\theta i}) = e^{2\pi\theta^2 i}$ define a Z-action on X. The map $\Psi\colon X_I \to X_I$ defined by

$$\Psi((e^{2\pi\theta i}, t)) = \left(\phi^{[\theta + t(1-\theta+\theta^2)]}(e^{2\pi\theta i}), \{\theta + t(1-\theta+\theta^2)\}\right)$$

for $0 \le \theta \le 1$, $0 \le t < 1$ is an orbit preserving homeomorphism of X_I onto itself. The map $\tilde{\Psi}$ as constructed above is continuous on $(X - \{1\}) \times R$ but does not extend continuously to $X \times R$. (Although X does not have a dense orbit, the same conclusion holds for ϕ restricted to the orbit closure of $e^{\pi i}$.) $\square$

Any homeomorphism Ψ of X_g onto X_h mapping orbits to orbits induces a map of the orbit space of (X, Z^m) onto itself via the global sections $\pi_g(X \times \{0\})$ and $\pi_h(X \times \{0\})$. This induced map is the identity if and only if $\Psi(\mathcal{O}(\pi_g(x,0))) = \mathcal{O}(\pi_h(x,0))$, and in this case the role of (X, Z^m) as a global section is preserved as completely as possible.

Definition 3.1. A covering cocycle $h \in \mathcal{C}$ is said to be **invertible** if there exists a continuous map $H\colon X \times R^m \to R^m$ such that for all $x \in X$, $a \in Z^m$ and $v \in R^m$, the following hold:

1. $H(x,a) = h(x,a)$,
2. $H(x, a+v) = h(x,a) + H(ax, v)$,
3. $H(x,\cdot)\colon R^m \to R^m$ is an onto homeomorphism.

Note that invertible cocycles will always be embedding cocycles because conditions 1 and 3 imply that $h(x,a) = 0$ only when $a = 0$.

Remark 3.1. If h is a covering cocycle and $H\colon X \times R^m \to R^m$ is a continuous map satisfying conditions 1 and 2 in Definition 3.1, then $|H(x,v)| \to \infty$ uniformly in x as $|v| \to \infty$.

Proof. For $v \in R^m$, let $v = [v] + \{v\}$ where $[v] \in Z^m$ and $\{v\} \in [0,1)^m$. We first note that $H(X \times [0,1]^m)$ is a compact subset of R^m by the continuity of H. Thus, if $|H(x_k, v_k)| = |h(x, [v_k]) + H([v_k]x, \{v_k\})| \to w$ with $|v_k| \to \infty$, then we would have $|h(x_k, [v_k])|$ bounded and $|[v_k]| \to \infty$, contradicting Theorem 2.1. $\square$

Theorem 3.1. *A cocycle $h \in \mathcal{C}$ is cohomologous to an invertible cocycle if and only if there exists a homeomorphism $\tilde{\Psi}\colon X \times R^m \to X \times R^m$ of the form $\tilde{\Psi}(x,v) = (x, \psi(x,v))$ satisfying for $a \in Z^m$*

$$\tilde{\Psi} \circ S_a = T_a \circ \tilde{\Psi}$$

where $S_a(x,v) = (ax, v - a)$ and $T_a(x,v) = (ax, v - h(x,a))$.

Proof. Suppose $h \sim h'$ with h' invertible. Then h must be a covering cocycle and $h'(x,a) = h(x,a) + f(ax) - f(x)$ for some $f \in C(X, R^m)$. Let H' be the extension of h' given by Definition 3.1 and define $\tilde{\Psi}\colon X \times R^m \to X \times R^m$ by $\tilde{\Psi}(x,v) = (x, H'(x,v) + f(x))$. Then

$$\begin{aligned}
\tilde{\Psi} \circ S_a(x,v) &= (ax, H'(ax, v-a) + f(ax)) \\
&= (ax, H'(x,v) - h'(x,a) + f(ax)) \\
&= (ax, H'(x,v) + f(x) - h(x,a)) = T_a \circ \tilde{\Psi}(x,v)
\end{aligned}$$

Clearly $\tilde{\Psi}$ is continuous, one-to-one, and onto. If $\tilde{\Psi}^{-1}$ were not continuous, there would exist a sequence (x_k, v_k) with $|H'(x_k, v_k)|$ bounded and $|v_k| \to \infty$ which is impossible by Remark 3.1.

For the converse suppose $\tilde{\Psi}$ exists as required and set $f(x) = \psi(x,0)$. Let $h'(x,a) = h(x,a) + f(ax) - f(x)$ and define $H'\colon X \times R^m \to R^m$ by

$$H'(x,v) = \psi(x,v) - f(x).$$

Since I is an embedding cocycle it follows from the earlier discussion that h is also embedding and h' is at least a covering cocycle. It is also obvious that $H'(x,\cdot)\colon R^m \to R^m$ is an onto homeomorphism. Since $H'(x,0) = 0$, condition 1 will follow from condition 2 in the definition of invertible. Finally applying $T_a \circ \tilde{\Psi} = \tilde{\Psi} \circ S_a$ to $v + a$ yields $\psi(x, v+a) - h(x,a) = \psi(ax, v)$ or $H'(x, v+a) = h'(x,a) + H'(ax, v)$ to complete the proof. $\square$

Using a piecewise linear extension of $h(x,\cdot)$ to R^m, it is easy to obtain the first two conditions in the definition of invertible. To take advantage of this property we must work with simplicial complexes. Let K be an abstract simplicial complex with vertex scheme $\{K_v, K_S\}$ where K_v is the set of vertices of K and K_S is the collection of subsets of K_v that span simplices in K. Any vertex map $\theta\colon K_v \to R^m$ extends by linearity to a continuous map $|\theta|\colon |K| \to R^m$. All our simplicial complexes are geometric in the sense that $K_v \subset R^m$ and the linear extension of the inclusion map is a homeomorphism. In this context

$$|K| = \left\{ \sum_{i=1}^{p} \lambda_i v_i : \lambda_i \geq 0,\ \sum_{i=1}^{p} \lambda_i = 1,\ \{v_1, \ldots, v_p\} \in K_S \right\}$$

and

$$|\theta| \left(\sum_{i=1}^{p} \lambda_i v_i \right) = \sum_{i=1}^{p} \lambda_i \theta(v_i)$$

when $\lambda_i \geq 0$, $\sum_{i=1}^{p} \lambda_i = 1$, and $\{v_1, \ldots, v_p\} \in K_S$. Such geometric complexes must be countable, locally finite, and of dimension at most m [**7**, Chapter 3 Sections 1 and 2]. Furthermore, $|K|$ is compact if and only if K_v is finite.

For example, let $K_v = \{a \in Z^m : a_i = 0 \text{ or } 1\}$ and let K_S consisting of all simplices of the form $(0, e_{\pi(1)}, e_{\pi(1)} + e_{\pi(2)}, \cdots, e_{\pi(1)} + \cdots + e_{\pi(m)})$ where π is any permutation of $\{1, 2, \ldots, m\}$ be the vertex scheme of a simplicial complex K. Clearly $|K| = I^m = [0,1]^m$. Let Π_m denote this set of simplices given by permutations. Now define the simplicial complex $K(Z^m)$ for which $|K(Z^m)| = R^m$ by the vertex scheme

$$\{Z^m, \{a + s : a \in Z^m, s \in \Pi_m\}\}.$$

We will be primarily interested in subcomplexes of $K(Z^m)$.

For fixed $x \in X$ we can view $h(x, \cdot)$ as a vertex map for $K(Z^m)$ and extend it piecewise linearly to $|h(x, \cdot)| \colon R^m \to R^m$. The resulting map is continuous on $X \times R^m$ and will be denoted by $H(x, v)$. It is easy to verify that when $H \colon X \times R^m \to R^m$ is constructed in this way, it is continuous and satisfies properties (1) and (2) in the definition of invertible. This suggests the following definition:

Definition 3.2. A covering cocycle h is a **piecewise linear invertible cocycle** (or simply a PL invertible cocycle) provided that when H is the PL extension of h constructed above, the map $H(x, \cdot) \colon R^m \to R^m$ is an onto homeomorphism for all x.

It follows that PL invertible cocycles are invertible and we have a stronger result.

Theorem 3.2. *If h is a covering cocycle, then for all x the PL extension $H(x, \cdot)$ maps R^m onto R^m. In particular, a covering cocycle h is PL invertible if and only if the PL extension $H(x, \cdot)$ is one-to-one on R^m.*

Proof. Suppose h is a covering cocycle and its PL extension $H(x, \cdot)$ is one-to-one on R^m. Clearly H is continuous and satisfies (1) and (2) in Definition 3.1. It remains to verify (3), that is, that $H(x, \cdot) \colon R^m \to R^m$ is an onto homeomorphism.

We will need to show that for all x, $H(x,\cdot)\colon R^m \to R^m$ is onto and has a continuous inverse.

The fact that $H(x,\cdot)\colon R^m \to R^m$ is onto for all $x \in X$ is a consequence of the Borsuk-Ulam Theorem. The idea is that if this map is not onto, its natural extension to the sphere will have a pair of antipodal points mapped to the same point violating the linear growth of covering cocycles. For more detail see the proof of Proposition 1 in [**3**] or the proof of Theorem 2.10 in [**1**].

To see that $H^{-1}(x,\cdot)\colon R^m \to R^m$ is continuous for all $x \in X$ suppose $w_n \to w$ for $w_n, w \in R^m$. Then, taking a subsequence if necessary, there are only two possibilities for $H^{-1}(x,w_n)$: either $H^{-1}(x,w_n) \to v$ where $|v| < \infty$ or $H^{-1}(x,w_n) \to \infty$.

If $H^{-1}(x,w_n) \to v$ then, by continuity of H,

$$H\left(x, H^{-1}(x,w_n)\right) = w_n \to H(x,v).$$

Thus $H(x,v) = w$ and $H^{-1}(x,w) = v$ as desired.

Otherwise, $H^{-1}(x,w_n) \to \infty$. Because $w_n \to w$, we may assume $|w_n| < B$ for all n. But then we have $|H^{-1}(x,w_n)| \to \infty$ and

$$\left|H\left(x, H^{-1}(x,w_n)\right)\right| = |w_n| < B$$

which contradicts the fact that h is covering. □

Clearly a constant cocycle is PL invertible if and only if it is an invertible linear map of R^m onto R^m. Furthermore, we have the following theorem from [**3**, Theorem 6].

Theorem 3.3. *The PL invertible cocycles are open in $\mathcal{C}$.*

It follows that the invertible cocycles are not insignificant in $\mathcal{C}$.

Corollary 3.1. *The nonsingular linear cocycles are in the interior of the invertible cocycles.*

Throughout the rest of the paper H will denote the piecewise linear extension of $h \in \mathcal{C}$ constructed in this section.

4. Nonsingular Cocycles

In this section we will define nonsingular cocycles and discuss some of their properties. In Section 5, our main result will establish the fact that nonsingular cocycles are cohomologous to PL invertible cocycles. The definition of nonsingularity relies on the concept of cocycle integration discussed in [**4**] and [**5**]. We begin with some of the terminology and notation found there.

It is easily checked that $\mathcal{C}$ is naturally an $\mathcal{L}$ module under composition which we will denote simply by juxtaposition. (So Tg will denote the cocycle $(x,a) \to T(g(x,a))$.) Since $\mathcal{L}$ is also an $\mathcal{L}$ module under composition, we can consider

$$\mathcal{H} = \{\rho \in \operatorname{Hom}_{\mathcal{L}}(\mathcal{C},\mathcal{L}) \,:\, \rho \text{ is continuous}\}\,.$$

Although $\mathcal{H}$ is not an $\mathcal{L}$ module because $\mathcal{L}$ is not commutative, it is a vector space over R and each $\rho \in \mathcal{H}$ is a bounded linear map.

In addition to being a vector space, $\mathcal{H}$ is a Banach space with norm

$$||\rho|| = \sup_{||h|| \le 1} |\rho(h)| = \sup_{||h|| \le 1} \; \sup_{|w| \le 1} |\rho(h)w|$$

for $h \in \mathcal{C}$ and $w \in R^m$. We also can put a weak-$*$ topology on $\mathcal{H}$ by

$$\rho_n \to \rho \quad \text{if and only if} \quad \rho_n(h) \to \rho(h)$$

for all $h \in \mathcal{C}$. Now a slight modification of the usual proof of Alaoglu's Theorem shows that $\{\rho \in \mathcal{H} \,:\, ||\rho|| \le 1\}$ is compact in the weak-$*$ topology.

An element $\rho \in \mathcal{H}$ is called an **invariant cocycle integral** provided

1. $||\rho|| = 1$,
2. $\rho(ah) = \rho(h)$ for all $h \in \mathcal{C}$ and $a \in Z^m$,
3. $\rho(I) = I$ where $I \in \mathcal{L}$ is the identity.

The weak-$*$ compact subset of $\mathcal{H}$ consisting of invariant cocycle integrals will be denoted $\mathcal{I}$. For a Z- action $\mathcal{I}$ is just the set of invariant measures, and not surprisingly $\mathcal{I}$ can in general be described in terms of invariant measures. For our purposes it will suffice to work with the matrices of the linear transformations $\rho(h)$. The next theorem provides the necessary matrix description of $\mathcal{I}$, and a proof of it can be found in [**5**] which builds on the work found in [**4**] for real valued cocycles. However, the reader can simply use Theorem 4.1 as the definition of $\mathcal{I}$.

Theorem 4.1. *Let $\rho \in \mathcal{H}$. Then $\rho \in \mathcal{I}$ if and only if there exist Z^m invariant Borel probability measures $\mu_1, \dots, \mu_m$ such that the matrix of $\rho(h)$ with respect to the standard basis is given by*

$$\begin{pmatrix} \int_X h_1(x,e_1)\,d\mu_1 & \cdots & \int_X h_1(x,e_m)\,d\mu_m \\ \vdots & & \vdots \\ \int_X h_m(x,e_1)\,d\mu_1 & \cdots & \int_X h_m(x,e_m)\,d\mu_m \end{pmatrix}$$

for all $h = (h_1, \dots, h_m) \in \mathcal{C}$.

Definition 4.1. A cocycle $h \in \mathcal{C}$ is said to be **nonsingular** if and only if $\rho(h)$ is nonsingular for all $\rho \in \mathcal{I}$. In other words, h is nonsingular if the matrix

$$\begin{pmatrix} \int_X h_1(x, e_1)\, d\mu_1 & \cdots & \int_X h_1(x, e_m)\, d\mu_m \\ \vdots & & \vdots \\ \int_X h_m(x, e_1)\, d\mu_1 & \cdots & \int_X h_m(x, e_m)\, d\mu_m \end{pmatrix}$$

is nonsingular whenever $\mu_1, \dots, \mu_m$ are Z^m-invariant Borel probability measures.

The next theorem tells us that nonsingularity is a sufficient condition for covering. However, it is not a necessary condition.

Theorem 4.2. *Let $h \in \mathcal{C}$. If h is nonsingular, then h is covering.*

The proof of this theorem is a straight forward application of Theorems 4.2 and 5.3 in [**4**] and is omitted. When $m = 1$ the converse is also true (see Theorem 1.12 in [**1**]), but the converse is not true in general. In [**5**] the first author constructed an example of a PL invertible cocycle with $m = 2$ for which $\rho(h)$ is not be invertible for all choices of μ_1 and μ_2 even when μ_1 and μ_2 are ergodic or when $\mu_1 = \mu_2$.

We conclude this section with three propositions describing some of the properties of nonsingular cocycles. These properties will be used in establishing the main theorem in Section 5.

For a cocycle h we define C_j for $1 \le j \le m$ by

$$\begin{aligned} C_j &= \{\rho(h)(e_j) \,:\, \rho \in \mathcal{I}\} \\ &= \left\{ \sum_{i=1}^m \left(\int_X h_i(x, e_j)\, d\mu \right) e_i \,:\, \mu \in \mathcal{M} \right\} \\ &= \left\{ \int_X h(x, e_j)\, d\mu \,:\, \mu \in \mathcal{M} \right\}. \end{aligned}$$

Obviously, the sets C_j, $1 \le j \le m$, are compact and convex in R^m. For each C_j, form the cone

$$\hat{C}_j = \{tv \,:\, v \in C_j \text{ and } t > 0\}.$$

Because C_j is convex, $\hat{C}_j$ is closed under vector addition and is convex.

Proposition 4.1. *If h is nonsingular and $b_j \in \left(\hat{C}_j \cup -\hat{C}_j\right)$ for $j = 1, \dots, m$, then $\{b_1, \dots, b_m\}$ are linearly independent.*

Proof. Suppose $\beta_1 b_1 + \cdots + \beta_m b_m = 0$ with the β_j real and not all zero. Then there exists $\mu_j \in \mathcal{M}$ and α_j not all zero such that

$$\sum_{j=1}^m \alpha_j \int_X h(x, e_j)\, d\mu_j = 0$$

and the matrix whose columns are given by $\int_X h(x, e_j)\, d\mu_j$ is singular. However, this matrix is nonsingular because h is nonsingular and $\mu_1, \dots, \mu_m$ determine $\rho \in \mathcal{I}$. $\square$

Notice that the above proof depends crucially on having different measures determine the columns of the matrix ρ.

The sets

$$B(C_j, \epsilon) = \{v : |v - w| < \epsilon \;\text{ for some } \quad w \in C_j\,\}$$

are convex, and we can choose ϵ small enough to ensure that the convex cones

$$\hat{B}_j = \{tv : v \in B(C_j, \epsilon) \;\text{ and }\; t > 0\,\}$$

also satisfy

$$\left(\hat{B}_i \cup -\hat{B}_i\right) \cap \left(\hat{B}_j \cup -\hat{B}_j\right) = \emptyset \qquad \text{for } i \neq j$$

and $\det(v_1, \dots, v_m) \neq 0$ for $v_j \in (\hat{B}_j \cup -\hat{B}_j)$.

Proposition 4.2. *If h is nonsingular, then there exists h' cohomologous to h such that for all $x \in X$,*

$$h'(x, ke_j) \in \hat{B}_j \quad \textit{for } k > 0$$

and

$$h'(x, ke_j) \in -\hat{B}_j \quad \textit{for } k < 0.$$

Proof. Since

$$h'(x, ke_j) = \sum_{p=0}^{k-1} h'\left((pe_j)x, e_j\right) \qquad \text{for } k > 0$$

and

$$h'(x, -e_j) = -h'\left((-e_j)x, e_j\right),$$

and since $\hat{B}_j$ is closed under vector addition, it will suffice to find h' cohomologous to h for which $h'(x, e_j) \in \hat{B}_j$ for all $x \in X$ and $1 \leq j \leq m$.

For each $M \in N$, define the continuous function $f_M \colon X \to R^m$ via

$$f_M(x) = -\frac{1}{M^m} \sum_{0 \leq a_i < M} h(x, a).$$

Set $h_M(x, a) = h(x, a) - (f_M(ax) - f_M(x))$. Then h_M is cohomologous to h and it is easy to check (see [**4**]) that

$$h_M(x, e_j) = \frac{1}{M^m} \sum_{0 \leq a_i < M} h(ax, e_j).$$

It follows that if $h_M(x, e_j) \notin \hat{B}_j$ for all $x \in X$ when M is large, then there exists $\mu \in \mathcal{M}$ such that $\int_X h(x, e_j)\, d\mu \notin \hat{B}_j$ and hence $\int_X h(x, e_j)\, d\mu \notin \hat{C}_j$, a contradiction. Therefore, $h_M \sim h$ and for M sufficiently large $h_M(x, e_j) \in \hat{B}_j$ for all $x \in X$ and $1 \leq j \leq m$. □

The cocycle h' obtained in the previous proposition has an additional property which will play an important role in the next section.

Proposition 4.3. *Let h be nonsingular and let $h' \sim h$ be as described in the previous proposition. Let $H' \colon X \times R^m \to R^m$ be the piecewise linear extension of h'. Then for all $x \in X$, $H'(x, \cdot) \colon R^m \to R^m$ is one-to-one on simplices.*

Proof. Because $H'(x, \cdot)$ is defined linearly over simplices, it will suffice to show that if π is any permutation of m symbols, then

$$\left\{ h'(x, e_{\pi(1)}), h'(x, e_{\pi(1)} + e_{\pi(2)}), \dots, h'\left(x, \sum_{i=1}^{m} e_{\pi(i)}\right) \right\}$$

is linearly independent for all $x \in X$.

Set

$$w_j = h'\left(x, \sum_{i=1}^{j} e_{\pi(i)}\right) \qquad \text{and} \qquad v_j = h'\left(\left(\sum_{i=1}^{j-1} e_{\pi(i)}\right) x, e_{\pi(j)}\right).$$

We observe that by the cocycle equation $w_j = w_{j-1} + v_j$. Also, $v_j \in \hat{B}_j$ for $1 \leq j \leq m$. Thus

$$\begin{aligned} &\det\left(h'(x, e_{\pi(1)}), h'(x, e_{\pi(1)} + e_{\pi(2)}), \dots, h'(x, \sum_{i=1}^{m} e_{\pi(i)}) \right) \\ &= \det(w_1, w_2, \dots, w_m) \\ &= \det(w_1, w_2, \dots, w_{m-1}, w_{m-1} + v_m) \\ &= \det(w_1, w_2, \dots, w_{m-1}, v_m) \\ &\quad \vdots \\ &= \det(v_1, v_2, \dots, v_m). \end{aligned}$$

It follows via Proposition 4.1 that $\det(v_1, v_2, \dots, v_m) \neq 0$ for $v_j \in \hat{B}_j$, $1 \leq j \leq m$. Thus, $H'(x, \cdot) \colon R^m \to R^m$ is one-to-one on simplices as desired. □

5. Nonsingular Implies PL-invertible

Let K be a geometric simplicial complex as discussed in Section 2. The **carrier** of a point $w \in |K|$ is the unique smallest simplex $s \in K_S$ such that $w \in |s|$. Let

$\xi(w)$ denote the carrier of w, and for $s, s' \in K_S$ let $s' \leq s$ denote that s' is a face of s. Define the **star** of a vertex v by

$$st(v) = \{w \in |K| : \{v\} \leq \xi(w)\}$$

which is an open subset of $|K|$ (see [**7**,p. 114]).

Next for $w \in |K|$ set

$$E(w) = \{s \in K_S : \exists\ s' \in K_S, s \leq s', \xi(w) \leq s'\}.$$

Clearly $E(w)$ defines a subcomplex of K and w is in the interior of $|E(w)|$ in $|K|$ because

$$w \in \bigcap_{i=1}^{p} st(v_i) \subset |E(w)|$$

where $\xi(w) = \{v_1, v_2, \ldots, v_p\}$.

Let $\varphi\colon K_v \to R^m$ be a vertex map. We say φ is **locally one-to-one** at $w \in |K|$ if $|\varphi|$ is one-to-one in an open neighborhood of $w \in |K|$. If for some $\epsilon > 0$ we have $B_\epsilon(w) \subset |K|$ and φ is locally one-to-one at w, then by invariance of domain $B_\delta(|\varphi|(w)) \subset |\varphi|(|K|)$ for some $\delta > 0$ (where $B_\epsilon(w) = \{z \in R^m : |z - w| < \epsilon\}$). In particular, if φ is locally one-to-one on an open subset U of R^m contained in $|K|$, then $|\varphi|(U)$ is an open subset of R^m. The following lemma is crucial for the main result:

Lemma 5.1. *The vertex map $\varphi\colon K_v \to R^m$ is locally one-to-one at a point $w \in R^m$ if and only if $|\varphi|$ is one-to-one on $|E(w)|$.*

Proof. Suppose $|\varphi|(w_1) = |\varphi|(w_2) = w'$ for some $w_1, w_2 \in |E(w)|$; then there exists simplices s_1, s_2 of $E(w)$ such that $w_i \in |s_i|$ and $w \in |s_i|$. Hence the line segments from w to w_1 and from w to w_2 are in $|E(w)|$ and map to the line segment from $|\varphi|(w)$ to w'. Unless w, w_1, and w_2 are collinear, $|\varphi|$ can not be locally one-to-one at w because the images of the line segments from w to w_i are identical. If they are collinear and all lie in $|s|$ for some $s \in K_S$, then the line segment containing them maps to a point. Otherwise w lies between w_1 and w_2 and $|\varphi|$ is not locally one-to-one as in the first case. The other direction is trivial.□

Theorem 5.1. *Let $\varphi\colon Z^m \to R^m$ be such that $|\varphi|$ is one-to-one on every simplex of $K(Z^m)$. If φ is not locally one-to-one, there exists a finite subset F of Z^m and $\delta > 0$ such that for any $\psi\colon Z^m \to R^m$ with $|\psi(a) - \varphi(a)| < \delta$ for all $a \in F$, ψ is not locally one-to-one.*

Proof. Choose $w \in R^m$ so that $|\varphi|$ is not locally one-to-one at w and the dimension, $\kappa(w)$, of $\xi(w)$ is maximal. By hypothesis $\kappa(w) < m$ and $|\varphi|$ is one-to-one on $|\xi(w)| \subset |E(w)|$. There exist $w_1, w_2 \in |E(w)|$ such that $|\varphi|(w_1) = |\varphi|(w_2)$. Again by the hypothesis we have $w_i \in |s_i|$, $s_i \in E(w)$ with $s_1 \neq s_2$. Without loss of generality $\xi(w) \le s_i$. In fact it must be a proper face or else $|\varphi|$ will not be one-to-one on either $|s_1|$ or $|s_2|$. Moreover, because the images of the line segments from w to w_i are identical, by moving w_i closer to w we can assume $\xi(w_i) = s_i$.

From the above it follows that $\kappa(w_i) > \kappa(w)$ and by our choice of w, $|\varphi|$ is one-to-one on $|E(w_i)|$. We also know that w_i is in the interior of $|E(w_i)|$ and hence $B_\epsilon(w_i) \subset |E(w_i)|$ for some $\epsilon > 0$ because $|K| = R^m$. It follows that there exists $\epsilon' > 0$ such that

$$B_{\epsilon'}(w') = B_{\epsilon'}(|\varphi|(w_i)) \subset |\varphi|(|E(w_1)|) \cap |\varphi|(|E(w_2)|).$$

Notice that $E(w_i) \subset E(w)$ which is a finite simplicial complex. Set F equal to the set of vertices of $E(w)$. There exists $\delta' > 0$ such that $|\psi(v) - \varphi(v)| \le \delta'$ for all $v \in F$ implies $|\psi|$ is one-to-one on $|E(w_i)|$. (A detailed proof of this intuitively clear idea appears as Lemma 3 in [**3**].)

Let $\mathbf{u} = (u_1, u_2, \ldots, u_p) \in (R^m)^p$ satisfy $|u_i - \varphi(v_i)| \le \delta'$ where $v_1, v_2, \ldots, v_p$ are the vertices in $E(w_1)$. Define $\psi_{\mathbf{u}}$ by $\psi_{\mathbf{u}}(v_i) = u_i$. Then by the above $|\psi_{\mathbf{u}}|$ is one-to-one on $|E(w_1)|$. Set $n = m + mp$ and define

$$\Psi\colon |E(w_1)| \times \{\mathbf{u} : |u_i - \varphi(v_i)| \le \delta'\} \to R^m$$

by $\Psi(x, \mathbf{u}) = (|\psi_{\mathbf{u}}|(x), \mathbf{u})$ which is one-to-one. By invariance of domain there exists an open neighborhood of

$$(|\varphi|(w_1), \varphi(v_1), \varphi(v_2), \ldots, \varphi(v_p))$$

in the image of Ψ. Using the product topology on $R^m \times (R^m)^p$ we can find δ such that given $y \in B_\delta(|\varphi|(w_1))$ and $|u_j - \varphi(v_j)| < \delta$ for $j = 1, \ldots, v_p$ there exists $x \in |E(w_1)|$ satisfying $|\psi_{\mathbf{u}}|(x) = y$. Finally do the same thing for w_2 and use the smaller δ. Then for any $\mathbf{u} \in (R^m)^p$ with $|u_j - \phi(v_j)| \le \delta$, $\Psi_{\mathbf{u}}\colon |E(w_i)| \to B_\delta(w)$ is onto for both $i = 1$ and $i = 2$. Therefore, given ψ as in the statement of the Theorem, $|\psi|$ is not one-to-one on $|E(w)|$ and not locally one-to-one by Lemma 5.1. □

Theorem 5.2. *If h is a nonsingular cocycle, then h is cohomologous to a PL-invertible cocycle.*

Proof. If h is nonsingular, we know that h is cohomologous to a cocycle h' as described in Proposition 4.2. So, we will assume that $h(x, ke_j) \in \hat{B}_j$ for $k > 0$, $h(x, ke_j) \in -\hat{B}_j$ for $k < 0$ and for $H\colon X \times R^m \to R^m$ the piecewise linear extension of h, $H(x, \cdot)$ is one-to-one on simplices for all $x \in X$ by Proposition 4.3.

Let $b_j \in \hat{B}_j$ and let $L \in \mathcal{L}$ be defined by $L(e_j) = b_j$ for $1 \le j \le m$. Consider the line segment $th + (1-t)L = h_t$, $0 \le t \le 1$ in $\mathcal{C}$. Clearly the cocycle h_t is nonsingular and therefore covering. Let $H_t(x, w)$ be the piecewise linear extension of h_t and note that

$$H_t(x, w) = tH(x, w) + (1-t)L(w)$$

where $H(x,w)$ is the piecewise linear extension of h. The set $\{t : h_t$ is PL-invertible$\}$ is open in $[0,1]$ by Theorem 6 in [**3**] and nonempty because L is PL-invertible. The following two lemmas will be used to complete the proof of the theorem:

Lemma 5.2. *If h is a covering cocycle and $H(x,\cdot)$ is locally one-to-one for all $x \in X$, then h is PL-invertible.*

Proof. Letting $H_x(w) = H(x,w)$, it suffices to show that $H_x\colon R^m \to R^m$ is a covering space for all x. By Theorem 3.2 it is onto, and by the remarks preceding Lemma 5.1 it is an open mapping. Thus it is a local homeomorphism of R^m onto itself.

Next we show that $H_x^{-1}(B_1(w))$ is a bounded set for all w. If it was unbounded there would exist a sequence w_i such that $|w_i| \to \infty$ and $H_x(w_i) \in B_1(w)$. Setting $a_i = [w_i]$ and $u_i = w_i - a_i$, we have

$$H_x(w_i) = h(x,a_i) + H(a_i x, u_i).$$

Since $|a_i| \to \infty$ and $u_i \in [0,1]^m$, $|h(x,a_i)| \to \infty$ because h is covering and $H(a_i x, u_i)$ is bounded. Hence we have $|H(x,w_i)| \to \infty$, contradicting the fact that $H(x,w_i) \in B_1(w)$.

It now follows that $H_x^{-1}(w) = \{w_1, w_2, \ldots, w_q\}$ is finite and there exists $\epsilon > 0$ such that $B_\epsilon(w_i) \cap B_\epsilon(w_j) = \emptyset$ when $i \neq j$. Finally setting

$$V = \bigcap_{i=1}^{q} H_x(B_\epsilon(w_i))$$

produces the required open neighborhood of w evenly covered by H_x and completes the proof of the lemma. □

Lemma 5.3. *If h is a nonsingular cocycle, then the set*

$$\{t : H_t(x,\cdot) \textit{ is not locally one-to-one for some } x\}$$

is an open subset of $[0,1]$.

Proof. Suppose $H_\tau(x_0,\cdot)$ is not locally one-to-one. Let

$$\varphi(a) = \tau h(x_0,a) + (1-\tau)L(a)$$

so that $H_\tau(x_0,w) = |\varphi|(w)$. Because h_τ is nonsingular H_τ is one-to-one on $|s|$ for all s and by Theorem 5.1 there exists a finite set F and $\delta > 0$ such that $|\psi(a) - \varphi(a)| \leq \delta$ for $a \in F$ implies ψ is not locally one-to-one. For $a \in F$ and t near τ we have

$$|th(x_0,a)+(1-t)L(a)-(\tau h(x_0,a)+(1-\tau)L(a))|$$
$$\leq |t-\tau|(||h||+|L|)(\sup\{|a|:a\in F\})<\delta$$

to complete the proof of Lemma 5.3. □

Returning to the proof of Theorem 5.2, because h_t is covering for all t, it follows from Lemma 5.2 that h_t is PL-invertible if and only if $H_t(x,\cdot)$ is locally one-to-one for all $x\in X$. Thus,

$$\{t\,:\,H_t(x,\cdot) \text{ is not locally one-to-one for some } x\in X\,\}$$
$$=\{t: h_t \text{ is not PL-invertible}\}$$

is also an open subset of $[0,1]$ by Lemma 5.3. Therefore, h_t is PL-invertible for $0\leq t\leq 1$ and $h_1=h$ is PL-invertible.

6. Modeling General R^m Flows

The goal of this section is to show how Theorem 5.2 extends the known results about modeling arbitrary R^m flows with suspensions. The seminal result relating R^m flows and suspensions is Rudolph's Theorem [**6**]:

Theorem 6.1. *Let (Y,R^m) be a flow with a free dense orbit. Then there is an almost one-to-one extension $(\hat{Y},R^m)$ of (Y,R^m) which has a suspension as a homomorphic image. That is, there exists a homomorphism $\phi\colon(\hat{Y},R^m)\to(\Gamma'_g,R^m)$ where the suspension (Γ'_g,R^m) is constructed from the space of Markov tilings Γ' and the tiling cocycle $g\colon\Gamma'\times Z^m\to R^m$.*

It is a small step to see that any flow $(\hat{Y},R^m)$ which has a suspension as a homomorphic image is a suspension itself [**7**, Lemma 3].

Lemma 6.1. *Let $\phi\colon(\hat{Y},R^m)\to(W_g,R^m)$ be a homomorphism from the flow $(\hat{Y},R^m)$ onto a suspension flow with base (W,Z^m) and given by embedding cocycle g. Then $(\hat{Y},R^m)$ is isomorphic to a suspension (U_h,R^m) where h is an embedding cocycle taking on the same values as g.*

To extend these results we will verify that the tiling cocycle is nonsingular. In order to do so, we need to understand the nature of the suspension (Γ'_g,R^m). For this purpose we now review some of the notation and terminology concerning Markov tilings of R^m. Readers unfamiliar with the details should consult [**6**] for a more thorough explanation.

Let $\alpha_1,\alpha_2,\ldots,\alpha_m$ be a sequence of irrationals and consider the collection of m dimensional rectangles given by

$$\{B_j\}_{j=1}^{2^m}=\left\{\prod_{i=1}^{m}[0,d_i)\right\}$$

where $d_i = 1$ or α_i. These are called basic tiles. A tiling of R^m is a partition

$$\mathcal{T} = \{T_1, T_2, \dots\}$$

of R^m such that each $T_i = v_i + B_{j(i)}$ is a translation of one of the basic tiles. The point $v_i \in R^m$ is called the vertex of the tile T_i. For our purposes, we will assume that the vertex of one of the tiles in a tiling $\mathcal{T}$ is zero.

A tiling $\mathcal{T}$ is called arithmetic if, for any vertex v,

$$v \in \prod_{i=1}^{m} (Z + \alpha_i Z)\,.$$

Arithmetic tilings have the property that there is a one-to-one correspondence ω between the vertices of $\mathcal{T}$ and Z^m. This is given via

$$\omega(v) = (a_1 + b_1, \dots , a_m + b_m)$$

where $v = (a_1 + \alpha_1 b_1, \dots , a_m + \alpha_m b_m)$ is a vertex. We will let $v_p(\mathcal{T})$ be the vertex of $\mathcal{T}$ for which $\omega(v_p(\mathcal{T})) = p$. This puts a Z^m-action on the vertices of $\mathcal{T}$ by

$$qv_p(\mathcal{T}) = v_{q+p}(\mathcal{T})$$

for $p, q \in Z^m$.

Notice that for $\mathcal{T}$ an arithmetic tiling, we have

$$\left(e_j v_p(\mathcal{T}) - v_p(\mathcal{T})\right)_i = a_i^{p,j} + b_i^{p,j}\alpha_i$$

with $a_i^{p,j} + b_i^{p,j} = \delta_{ij}$. An arithmetic tiling is said to Markov if for all $p \in Z^m$ and $1 \le i, j \le m$, we have

$$|a_i^{p,j}| \le 1 \quad \text{and} \quad |b_i^{p,j}| \le 1.$$

We will denote the collection of all Markov tilings with the usual tiling metric by Γ'. (Two tilings are within ϵ of each other if they agree, up to translation by $v \in R^m$ with $|v| \le \epsilon$, on a ball of radius $\frac{1}{\epsilon}$ about the origin.) It can be shown that Γ' is a compact metric space.

We have a Z^m-action on Γ' given by

$$p\mathcal{T} = \mathcal{T} - v_p(\mathcal{T})$$

and an embedding cocycle $g\colon \Gamma' \times Z^m \to R^m$ given by

$$g(\mathcal{T}, p) = v_p(\mathcal{T}).$$

We will refer to this cocycle as the tiling cocycle.

Proposition 6.1. *For $\alpha_1, \alpha_2, \ldots, \alpha_m$ irrationals with $1 \leq \alpha_i < 1 + \frac{1}{m}$, the tiling cocycle g as described is nonsingular.*

Proof. We observe that if $\mathcal{T}$ is a Markov tiling then

$$g(\mathcal{T}, e_i) \in \{v \in R^m \ : \ v_j \in \{0, \pm(1-\alpha_j)\} \text{ for } j \neq i, \ v_i \in \{1, \alpha_i\}\}\,;$$

that is, $g(\mathcal{T}, e_i)$ takes on one of $2(3^{m-1})$ values. For μ, a Z^m invariant Borel probability measure on Γ', when $j \neq i$,

$$1 - \alpha_j \leq \int_{\Gamma'} g_j(\mathcal{T}, e_i)\, d\mu \leq -1 + \alpha_j$$

and

$$1 \leq \int_{\Gamma'} g_i(\mathcal{T}, e_i)\, d\mu \leq \alpha_i.$$

Thus

$$\int_{\Gamma'} g(\mathcal{T}, e_i)\, d\mu \in \prod_{j=1}^{i-1} [1-\alpha_j, -1+\alpha_j] \times [1, \alpha_i] \times \prod_{j=i+1}^{m} [1-\alpha_j, -1+\alpha_j].$$

Let

$$C_i = \prod_{j=1}^{i-1} [1-\alpha_j, -1+\alpha_j] \times [1, \alpha_i] \times \prod_{j=i+1}^{m} [1-\alpha_j, -1+\alpha_j].$$

Since the i^{th} column in an invariant cocycle integral of g is of the form $\int_{\Gamma'} g(\mathcal{T}, e_i)\, d\mu$, the proposition follows from the following lemma.

Lemma 6.3. *If $v_i \in C_i$ for $1 \leq i \leq m$ then $\{v_i\}_{i=1}^m$ is a linearly independent set.*

Suppose that $a_1 v_1 + \cdots + a_m v_m = 0$ with the a_i's not all zero. Choose $a_i \neq 0$ so that $|a_i| \geq |a_j|$ for all $j \neq i$. Then

$$v_i = -\sum_{j \neq i} \frac{a_j}{a_i} v_j = \sum_{j \neq i} b_j v_j$$

where $0 \leq |b_j| \leq 1$.

We know that since the $v_i \in C_i$, the i^{th} coordinate of v_i is greater than or equal to 1. On the other hand, since $v_j \in C_j$ with $j \neq i$, the i^{th} coordinate of v_j is between $1 - \alpha_j$ and $-1 + \alpha_j$. But we chose our irrationals so that $1 \leq \alpha_j < 1 + \frac{1}{m}$. Thus we have a contradiction since

$$\left| \left(\sum_{j \neq i} b_j v_j \right)_i \right| \leq \frac{m-1}{m} < 1 \leq (v_i)_i.$$

This conclude the proof of the lemma and hence of Proposition 6.1. □

By combining our main result, Theorem 5.2, with Rudolph's tiling theorem stated above, we obtain the following theorem.

Theorem 6.2. *Let (Y, R^m) be a flow with a free dense orbit on a compact metric space. Then there exists a discrete dynamical system (X, Z^m) and an invertible cocycle $h\colon X \times Z^m \to R^m$ such that the suspension flow (X_h, R^m) is an almost one-to-one extension of (Y, R^m).*

Because suspensions of invertible cocycles are time changes of the unit one suspension, it follows that a space which supports an R^m flow with a free dense orbit does not differ significantly from a space supporting a unit one suspension of a Z^m discrete dynamical system with a free dense orbit. Specifically,

Corollary 6.1. *Given an R^m flow (Y, R^m) with a free dense orbit on a compact metric space Y, there exists a discrete dynamical system (X, Z^m) on a compact metric space X such that a time change of (X_I, R^m) is an almost one-to-one extension of (Y, R^m). In particular, there exists an orbit preserving map of (X_I, R^m) onto (Y, R^m).*

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