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GENERALIZED CENTERS OF FINITE SETS IN BANACH SPACES

L. VESELÝ

Abstract. We study mainly the class (GC) of all real Banach spaces X such that the set $E_f(a)$ of the minimizers of the function

$$X \ni x \mapsto f(\|x-a_1\|, \dots, \|x-a_N\|)$$

is nonempty whenever N is a positive integer, $a \in X^N$, and f is a continuous monotone coercive function on $[0,+\infty[^N$. For particular choices of f, the set $E_f(a)$ coincides with the set of Chebyshev centers of the set $\{a_i : i = 1, \dots, N\}$ or with the set of its medians. The class (GC) is stable under making c_0-, ℓ^p- and similar sums. Under some geometric conditions on X, the function spaces $C_b(T,X)$ or $L^p(\mu, X)$ belong to (GC). One of the main tools is a theorem which asserts that, in the definition of the class (GC), one can restrict himself to the functions f of the type $f(\xi_1, \dots, \xi_N) = \max \varrho_i \xi_i$ ($\varrho_i > 0$).

Introduction

Let X be a real Banach space, f a real-valued function of N variables defined at least on $\mathbf{R}_+^N = [0,+\infty[^N$. Instead of finite sets $A = \{a_1, \dots, a_N\} \subset X$ we consider ordered N-tuples $a = (a_1, \dots, a_N) \subset X^N$. (In this way we fix the order of the a_i's and allow repeatings.)

The minimizers of the function $\varphi: X \to \mathbf{R}$,

$$\varphi(x) = f(\|x-a_1\|, \dots, \|x-a_N\|), \tag{1}$$

are called **f-centers of** a and the value

$$r_f(a) := \inf \varphi(X)$$

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is the ***f*-radius of** a. The set of f-centers of a will be denoted by $E_f(a)$. Thus

$$E_f(a) = \{x \in X : \varphi(x) = r_f(a)\}.$$

The problem of finding minimizers of functions as in (1) occurs sometimes in economical questions as "optimal location problem". The most common particular cases are **Chebyshev centers** (solutions of the "mini-max problem", i.e. f-centers for $f(\xi) = \max\{\xi_1 \dots, \xi_N\}$) and **medians** (solutions of the "mini-sum problem", i.e. f-centers for $f(\xi) = \xi_1 + \dots + \xi_N$). Some results connected with various types of f-centers (characterizations, properties, …) appeared, for example, in [**Du1**], [**Du2**], [**B-C-P1**], [**B-C-P2**], [**Ko**], [**Ve1**], [**Du3**], [**Ve2**]. Of course, it has sense to define Chebyshev centers also for bounded sets which are not finite. This subject was widely studied by various authors (see e.g. [**Am1**], [**Am2**], [**A-M-S**] and references therein).

The aim of the present paper is to study generalized centers (f-centers) of finite sets for a wide class of functions f (namely, monotone convex and coercive), and to state general results, especially for the existence of generalized centers in vector-valued sequence and function spaces. We obtain some results which are new even for Chebyshev centers of finite sets or for medians (e.g., Theorems 3.7, 4.8, 5.10).

We prove that the f-radius of $a \in X^N$ always coincides with the f-radius of a calculated in X^{**}. (This is not true in general for the Chebyshev radius of infinite sets.) The generalized centers of the finite sets exist in X if and only if the "weighted Chebyshev centers" of the finite sets exist in X. This, together with a result on upper semicontinuity of the multivalued mapping $a \mapsto E_f(a)$, yields sufficient geometric conditions on X for the existence of the generalized centers of finite sets in the space $C_b(T, X)$ of all bounded continuous X-valued functions on a topological space T. In particular, in $C_b(T) = C_b(T, \mathbf{R})$ such centers always exist. We present an example (due to J. Kolář) of a three-dimensional X such that a three-point set in $C_b([0,1], X)$ has no Chebyshev center. The class (GC) of the spaces, in which f-centers exist for every monotone convex coercive function f on $\mathbf{R}^N_+$, is stable under making arbitrary c_0 and ℓ^p sums ($1 \le p \le \infty$), and more general types of sums. We discuss also the Lebesgue-Bochner spaces $L^p(\mu, X)$.

1. Definitions and Auxiliary Results

Let X be a real Banach space. We shall always consider X canonically embedded in its second dual X^{**}. For a positive integer N, the space X^N will be endowed with the 2-norm $\|a\|^2 = \sum \|a_i\|^2$.

The unit ball and sphere of X will be denoted by B_X and S_X respectively. By $B(c, r)$ and $B^0(c, r)$ we denote respectively the closed and open ball in X centered in $c \in X$ with radius r. The same balls in X^{**} will be denoted by $\hat{B}(c, r)$ and $\hat{B}^0(c, r)$. (We put $B(c, 0) = \{c\}$, $B^0(c, 0) = \emptyset$.)

1.1. Definition.

(a) For $a \in X^N$ and $f\colon \mathbf{R}_+^N \to \mathbf{R}$ we define the **f-radius** and the **set of f-centers** of a by

$$r_f(a) = \inf \varphi(X), \quad E_f(a) = \{x \in X :\ \varphi(x) = r_f(a)\},$$

where φ is as in (1).

Considering a as an element of $(X^{**})^N$, we put

$$\hat{r}_f(a) = \inf \varphi(X^{**}), \quad \hat{E}_f(a) = \{x^{**} \in X^{**} :\ \varphi(x^{**}) = \hat{r}_f(a)\}.$$

(b) If f is of the form

$$f(\xi) = \max_{1 \le i \le N} \varrho_i \xi_i \ ,$$

where $\varrho := (\varrho_1, \dots, \varrho_N) \in]0, +\infty[^N$, we shall use the notation $r_\varrho(a)$, $\hat{r}_\varrho(a)$, $E_\varrho(a)$, $\hat{E}_\varrho(a)$ for the f-radius of a in X and in X^{**} and for the set of f-centers of a in X and in X^{**} respectively. In this case, the f-centers are called **weighted Chebyshev centers**. (Classical **Chebyshev centers** are weighted Chebyshev centers for the case of a constant weight $\varrho = (1, \dots, 1)$.)

(c) If f is of the form

$$f(\xi) = \sum_{i=1}^N \varrho_i \xi_i^p \ ,$$

where $\varrho := (\varrho_1, \dots, \varrho_N) \in]0, +\infty[^N$, the corresponding f-centers are called **weighted p-medians**. (**Medians** are weighted 1-medians for the case of a constant weight $\varrho = (1, \dots, 1)$.)

1.2. Example. Let $X = \mathbf{R}$, $a \in \mathbf{R}^N$. It is easy to see that there exists a unique Chebyshev center of a, namely the point $x_0 = (1/2)\min a_i + (1/2)\max a_i$. Moreover, if $a_1 \le \dots \le a_N$, the medians of a are given as follows. If $N = 2k+1$ then $E_f(a) = \{a_{k+1}\}$. If $N = 2k$ then $E_f(a) = [a_k, a_{k+1}]$ (this interval can be degenerated to one point if $a_k = a_{k+1}$).

The rest of this section is devoted to definitions and properties of some auxiliary notions.

We shall consider the coordinate-wise ordering on $\mathbf{R}^N$:

$$\xi \le \eta \quad \overset{\text{def}}{\iff} \quad \xi_i \le \eta_i \ \forall i \in \{1, \dots, N\}.$$

Thus, e.g., $\xi \vee 0 = (\max\{\xi_1, 0\}, \dots, \max\{\xi_N, 0\})$ and $|\xi| = (|\xi_1|, \dots, |\xi_N|)$.

1.3. Definition. A function $f\colon \mathbf{R}^N \supset S \to \mathbf{R}$ is said to be

(a) **monotonic** if $f(\xi) \le f(\eta)$ whenever $\xi, \eta \in S$, $|\xi| \le |\eta|$;
(b) **monotone** if $f(\xi) \le f(\eta)$ whenever $\xi, \eta \in S$, $\xi \le \eta$;
(c) **strictly monotone** if $f(\xi) < f(\eta)$ whenever $\xi, \eta \in S$, $\xi \le \eta$ and $\xi \ne \eta$;
(d) **weakly strictly monotone** if f is monotone, and $f(\xi) < f(\eta)$ whenever $\xi, \eta \in S$, $\xi_i < \eta_i$ for all $i \in \{1, \dots, N\}$;
(e) **coercive** if $f(\xi)$ tends to $+\infty$ as $\|\xi\| \to +\infty$, $\xi \in S$.

We collect some useful properties of convex monotone functions on $\mathbf{R}^N_+$ in the following Proposition.

1.4. Proposition. *Each convex monotone $f\colon \mathbf{R}^N_+ \to \mathbf{R}$ satisfies the following properties.*

(i) *f is Lipschitz on bounded subsets of $\mathbf{R}^N_+$ and attains its minimum over $\mathbf{R}^N_+$ at the origin.*
(ii) *If the set $f^{-1}(f(0))$ of the points of minimum is bounded, then f is coercive (and hence every minimizing sequence has a convergent subsequence).*
(iii) *If 0 is the unique minimum point for f, then f is weakly strictly monotone.*
(iv) *Let T be a topological space and $g_i\colon T \to [0, +\infty[$ lower semicontinuous functions ($i = 1, \dots, N$). Then the function $\varphi\colon T \to \mathbf{R}$, $\varphi(t) = f(g_1(t), \dots, g_N(t))$, is lower semicontinuous.*

Proof. **(i)** The function $F\colon \mathbf{R}^N \to \mathbf{R}$, given by $F(\xi) = f(\xi \vee 0)$ is a convex extension of f; hence it is locally Lipschitz (see e.g. [**Ph**]).

(ii) Fix $r > 0$ such that $f(\xi) > f(0)$ whenever $\|\xi\| = r$. Then $m := \inf_{\|\xi\|=r} f(\xi) > f(0)$ by (i) and the compactness of $\{\|\xi\| = r\}$.

For every $\xi \in \mathbf{R}^N_+$ with $\|\xi\| > r$ we have by convexity that

$$m \le f\left(\frac{r\xi}{\|\xi\|}\right) = f\left(\frac{r}{\|\xi\|}\xi + \left(1 - \frac{r}{\|\xi\|}\right)0\right) \le \frac{r}{\|\xi\|} f(\xi) + \left(1 - \frac{r}{\|\xi\|}\right) f(0).$$

An elementary calculation gives $f(\xi) \ge f(0) + \frac{m - f(0)}{r}\|\xi\|$, hence f is coercive. This also implies that any sequence that minimizes f is bounded, and hence relatively compact.

(iii) Suppose, on the contrary, that there exist $\xi, \eta \in \mathbf{R}^N_+$ such that $\xi_i < \eta_i$ ($i = 1, \dots, N$) and $f(\xi) = f(\eta) =: p$. Since the value $f(0)$ is attained only at 0, we must have $p > f(0)$. The order interval $I = \{x \in \mathbf{R}^N_+ : \xi < x < \eta\}$ is a nonempty open (convex) set in $\mathbf{R}^N_+$ and, by monotonicity, $f(x) = p$ for every $x \in I$. Choose any $x_0 \in I$. Then $f_1(t) := f(tx_0)$ is a convex nondecreasing function on $\mathbf{R}_+$ which is constant on a nontrivial interval containing the point $t = 1$. Thus f_1 must be constant on $[0, 1]$. But this implies $f(0) = f(x_0) = p$, a contradiction.

(iv) Let $(t_\gamma) \subset T$ be a net converging to $t_0 \in T$. We have to prove that $\varphi(t_0) \le \liminf \varphi(t_\gamma)$.

Take an arbitrary $\varepsilon > 0$. There exists an index γ_0 such that $g_i(t_\gamma) \geq g_i(t_0) - \varepsilon$ whenever $\gamma \geq \gamma_0$ and $1 \leq i \leq N$. Put $\delta_i(\varepsilon) := \max\{g_i(t_0) - \varepsilon, 0\}$ and observe that $g_i(t_\gamma) \geq \delta_i(\varepsilon)$ whenever $\gamma \geq \gamma_0$ and $1 \leq i \leq N$. Thus, $f(\delta_1(\varepsilon), \ldots, \delta_N(\varepsilon)) \leq \varphi(t_\gamma)$ for $\gamma \geq \gamma_0$. This implies that

$$f(\delta_1(\varepsilon), \ldots, \delta_N(\varepsilon)) \leq \liminf \varphi(t_\gamma), \text{ for every } \varepsilon > 0.$$

The left-hand side tends to $\varphi(t_0)$ as $\varepsilon \to 0$ since f is continuous by (i). □

1.5. Definition. Let $x \in S_X$, $x^* \in S_{X^*}$. We shall say that

(a) x^* **strongly exposes** B_X if every sequence $(x_n) \subset B_X$ such that $x^*(x_n) \to 1$ is norm-convergent;

(b) x^* **compactly strongly exposes** B_X if every sequence $(x_n) \subset B_X$ such that $x^*(x_n) \to 1$ has a norm-convergent subsequence;

(c) x is **(compactly) strongly exposed by** x^* if x^* (compactly) strongly exposes B_X and $x^*(x) = 1$;

(d) X satisfies (CSE) if every norm-attaining element of S_{X^*} compactly strongly exposes B_X.

(e) X satisfies (w^*K) if X is isometric to a dual of a normed space Z and the corresponding **weak*** topology $\sigma(X, Z)$ coincides with the **norm** topology on the unit sphere S_X.

We state the following Proposition 1.6 to illustrate the relation of the properties (CSE), (w^*K) with some more common geometric properties of Banach spaces. We omit the standard proofs.

1.6. Proposition.

(a) *If X satisfies* (CSE) *or (w^*K) then it has the Kadets property (i.e., the weak and the norm topologies coincide on S_X).*

(b) *Every locally uniformly convex space satisfies* (CSE).

(c) *Every dual locally uniformly convex space satisfies (w^*K).*

(d) *Every reflexive space with the Kadets-Klee property (i.e. the convergence of sequences in (S_X, weak) and in (S_X, norm) coincide) satisfies* (CSE) *and (w^*K).*

(e) *Every finite-dimensional space satisfies* (CSE) *and (w^*K).*

1.7. Remark. The property (CSE) is not equivalent to the following (weaker) property (CSE$_0$): *each $x \in S_X$ is compactly strongly exposed by some element of S_{X^*}.*

This was shown by J. Saint Raymond [**SR**] who constructed a counterexample. However, in the class of reflexive Banach spaces the properties (CSE) and (CSE$_0$) coincide. It follows from Proposition 1.6(d) and from the fact that (CSE$_0$) implies the Kadets-Klee property.

The last part of this section is dedicated to some basic facts about subdifferentials of convex functions. For our purposes, it will be sufficient to consider continuous functions only. Basic information can be found in [**Ph**], an intensive study of subdifferentials is contained in [**I-L**].

Let $U \subset X$ be an open convex set and $f\colon U \to \mathbf{R}$ be continuous and convex. For $x \in U$, the **subdifferential** of f at x is the set

$$\partial f(x) = \{x^* \in X^* \,:\, f(y) \geq f(x) + x^*(y-x) \;\forall y \in U\}.$$

It is well known that $\partial f(x)$ is a nonempty **weak***-compact set and it is a local notion in the sense that if $f = g$ on some neighborhood of x then $\partial f(x) = \partial g(x)$.

It follows directly from the definition that $0 \in \partial f(x)$ if and only if f attains its minimum at x.

It is a well known and not difficult consequence of the Hahn-Banach theorem that the subdifferential of the norm can be written as $\partial\|\cdot\|(x) = D(x)$ where

$$D(x) = \{x^* \in X \,:\, \|x^*\| \leq 1 \text{ and } x^*(x) = \|x\|\}.$$

The following "chain rule formula" for subdifferentials is a particular case of [**I-L**, Theorem 2 of §8, p. 44]; cf. also [**Pa**, Theorem 4.3].

1.8. Theorem. *Let X be a Banach space, $N \in \mathbf{N}$. Let $f\colon \mathbf{R}^N \to \mathbf{R}$ be convex and monotone. For $i = 1, \dots, N$, let $g_i\colon X \to \mathbf{R}$ be convex and continuous. Let $\varphi\colon X \to \mathbf{R}$ be the composed function $\varphi(x) = f(g_1(x), \dots, g_N(x))$. Then for every $x_0 \in X$*

$$\partial\varphi(x_0) = \Big\{x^* \in X^* :\, x^* = \sum_{i=1}^{N} \lambda_i u_i^*,\ \lambda \in \partial f(g_1(x_0), \dots, g_N(x_0)),\\ u_i^* \in \partial g_i(x_0)\ \forall i\Big\}.$$

2. Properties of Generalized Centers, the Class (GC)

We shall consider f-centers for convex monotone functions f on $\mathbf{R}_+^N$. It is easy to prove that, in this case, the corresponding function $\varphi(x) = f(\|x-a_1\|, \dots, \|x-a_N\|)$ is continuous (by Proposition 1.4(i)) and convex. Hence the set $E_f(a)$ of the f-centers is closed and convex.

The following two propositions collect some easy facts which are basically well known for the cases of Chebyshev centers and medians.

2.1. Proposition. *For $a \in X^N$, let us denote $\Delta(a) = \min_k \max_i \|a_k - a_i\|$, $d(a) = \max_{k,i} \|a_k - a_i\|$, $\|a\|_\infty = \max_i \|a_i\|$. Let f be a monotone function on $\mathbf{R}_+^N$, and let φ be as in* (1). *Then*

(a) $\Delta(a) \leq d(a)$;

(b) $r_f(a) = \inf\{\varphi(x) : \ x \in \bigcup_1^N B(a_i, \Delta(a))\}$
$= \inf\{\varphi(x) : \ \|x\| \le \|a\|_\infty + \Delta(a)\}$;

(c) $E_f(a) \subset \displaystyle\bigcup_{i=1}^N B(a_i, \Delta(a))$ *whenever f is weakly strictly monotone;*

(d) $E_f(a) \subset \displaystyle\bigcap_{i=1}^N B(a_i, \Delta(a) + d(a))$ *whenever f is weakly strictly monotone.*

Proof. The inequality **(a)** is obvious.

Let $x \notin \bigcup_1^N B(a_i, \Delta(a))$. Choose $k_0 \in \{1, \dots, N\}$ such that $\max\limits_i \|a_{k_0} - a_i\| = \Delta(a)$. Then we have

$$\|a_{k_0} - a_i\| \le \Delta(a) < \|x - a_i\| \quad \text{for all } i.$$

Consequently, $\varphi(a_{k_0}) \le \varphi(x)$ if f is only monotone, or $\varphi(a_{k_0}) < \varphi(x)$ if f is weakly strictly monotone. This easily implies the first equality in **(b)**, and **(c)**.

If $x \in \bigcup_1^N B(a_i, \Delta(a))$, choose an index j such that $\|x - a_j\| \le \Delta(a)$. Then

$$\|x\| \le \|a_j\| + \|x - a_j\| \le \|a\|_\infty + \Delta(a).$$

This proves the second equality in **(b)**.

Let us prove **(d)**. Let $x \in E_f(a)$. By **(c)**, there exists an index j such that $\|x - a_j\| \le \Delta(a)$. Then, for each $i \in \{1, \dots, N\}$, we have

$$\|x - a_i\| \le \|x - a_j\| + \|a_j - a_i\| \le \Delta(a) + d(a). \qquad \square$$

2.2. Proposition (Existence). *Let X be a Banach space that is norm-one complemented in its bidual. Let $f: \mathbf{R}_+^N \to \mathbf{R}$ be continuous, monotone and coercive. Then $E_f(a)$ is nonempty for each $a \in X^N$.*

Proof. Let $a \in X^N$. The function φ from (1) is coercive and **weak*** lower semicontinuous on X^{**} (cf. Proposition 1.4(iv)). By a standard **weak***-compactness argument, φ attains its minimum over X^{**}. In other words, there exists $c^{**} \in \hat{E}_f(a)$.

Let $P: X^{**} \to X$ be a projection of norm one. Put $c = P(c^{**})$. For all i we have $\|c - a_i\| = \|P(c^{**} - a_i)\| \le \|c^{**} - a_i\|$. Consequently,

$$\begin{aligned} r_f(a) &\le \varphi(c) = f(\|c - a_1\|, \dots, \|c - a_N\|) \le f(\|c^{**} - a_1\|, \dots, \|c^{**} - a_N\|) \\ &= \hat{\varphi}(c^{**}) = \hat{r}_f(a) \le r_f(a) \end{aligned}$$

since f is monotone. Hence $c \in E_f(a)$. $\square$

2.3. Remark. Besides reflexive spaces, the class of spaces which are norm-one complemented in their biduals contains also:

— all dual spaces (because of the-well known decomposition $X^{***} = X^* \oplus X^\perp$),

— the spaces $L^1(\mu)$ (cf. [**H-W-W**, p. 158]),
— the subspaces X of $L^1(\mu)$ such that the unit ball B_X of X is closed in $L^1(\mu)$ w.r.t. convergence in measure μ ([**H-W-W**, p. 183]).

For other examples see Theorem 5.1.

Various examples of finite sets that lack Chebyshev centers are known. The first such example was found by A. L. Garkavi [**Ga**] who defined a closed hyperplane X in $C[0,1]$ and a three-points set in X without Chebyshev centers in X (see also [**A-M-S**, p. 514 and §3]).

S. V. Konjagin [**Ko**] proved that such an example can be found in any nonreflexive Banach space X after a suitable renorming. The author of the present paper generalized Konyagin's construction to the following result. A norm f on $\mathbf{R}^3$ is **symmetric** if $f = f \circ \pi$ for every permutation π of the three variables.

2.4. Theorem [**Ve1**]. *Let X be a nonreflexive Banach space. There exists $a \in X^3$ with the following property: if f is a norm on $\mathbf{R}^3$ which is symmetric and monotonic, then X admits an equivalent norm $\|\!|\cdot|\!\|$ such that $E_f(a) = \emptyset$ in $(X, \|\!|\cdot|\!\|)$.*

Obviously, $\hat{r}_f(a) \le r_f(a)$ holds always true. We are going to prove that equality holds in fact (Theorem 2.6). We shall need the following lemma. Recall that $\hat{B}(c,r)$ denotes the closed r-ball in X^{**}, centered in c.

2.5. Lemma [**Li**, Lemma 5.8, p. 59]. *Let $a \in X^N$, $r \in]0,\infty[^N$. Suppose*

$$\bigcap_{i=1}^{N} \hat{B}(a_i, r_i) \neq \emptyset.$$

Then for every $\varepsilon > 0$ one has $\displaystyle\bigcap_{i=1}^{N} B(a_i, r_i + \varepsilon) \neq \emptyset$.

2.6. Theorem. *Let f be a continuous monotone function on $\mathbf{R}_+^N$, X be an arbitrary Banach space. Then for every $a \in X^N$*

$$r_f(a) = \hat{r}_f(a).$$

Proof. For $\varepsilon \in (0,1)$ choose $x_\varepsilon^{**} \in X^{**}$ such that $f(\|x_\varepsilon^{**} - a_1\|, \ldots, \|x_\varepsilon^{**} - a_n\|) < \hat{r}_f(a) + \varepsilon$ and $r_i(\varepsilon) := \|x_\varepsilon^{**} - a_i\| \le \|a\|_\infty + \Delta(a)$ (cf. Proposition 2.1(b)). Since $x_\varepsilon^{**} \in \bigcap_i \hat{B}(a_i, r_i(\varepsilon))$, by Lemma 2.5 there exists $x_\varepsilon \in \bigcap_i B(a_i, r_i(\varepsilon) + \varepsilon)$. Then the assumptions allow us to write

$$\begin{aligned}
0 &\le r_f(a) - \hat{r}_f(a) \\
&\le f(\|x_\varepsilon - a_1\|, \ldots, \|x_\varepsilon - a_n\|) - f(\|x_\varepsilon^{**} - a_1\|, \ldots, \|x_\varepsilon^{**} - a_n\|) + \varepsilon \\
&\le f(r_1(\varepsilon) + \varepsilon, \ldots, r_N(\varepsilon) + \varepsilon) - f(r_1(\varepsilon), \ldots, r_N(\varepsilon)) + \varepsilon.
\end{aligned}$$

But the right-hand side of this chain of inequalities tends to zero as $\varepsilon \to 0^+$, since $r_i(\varepsilon) \leq 2\|a\|_\infty + \Delta(a)$ and f is uniformly continuous on compact subsets of $\mathbf{R}_+^N$.$\square$

Let us remark that the equality of the Chebyshev radii in X and in X^{**} does not hold for **infinite** bounded sets in general. To see this, consider $X = c_0$ and $A = \{e_n\}_1^\infty$ the canonical basis of c_0. It is easy to see that the Chebyshev radii of A satisfy

$$r(A) := \inf_{x \in c_0} \sup_n \|x - e_n\| = 1, \quad \hat{r}(A) := \inf_{x^{**} \in \ell^\infty} \sup_n \|x^{**} - e_n\| = 1/2.$$

In a similar way as in Theorem 2.6, we shall prove that for the existence of a large class of generalized centers, it is sufficient the existence of all weighted Chebyshev centers.

2.7. Theorem. *For a Banach space X and $a \in X^N$, the following assertions are equivalent.*

(i) *If $r \in]0, +\infty[^N$ and $\bigcap_{i=1}^N \hat{B}(a_i, r_i) \neq \emptyset$, then also $\bigcap_{i=1}^N B(a_i, r_i) \neq \emptyset$.*

(ii) *a admits weighted Chebyshev centers for all weights $\varrho \in]0, +\infty[^N$.*

(iii) *a admits f-centers for every continuous monotone coercive function f on $\mathbf{R}_+^N$.*

Proof. (i) $\Rightarrow$ (iii). Suppose (i) holds. Let f be as in (iii). By Proposition 2.2 and Remark 2.3, there exists $x^{**} \in \hat{E}_f(a)$. Put $r_i = \|x^{**} - a_i\|$ and observe that $x^{**} \in \bigcap_i \hat{B}(a_i, r_i)$. By (i), there exists $x \in \bigcap_i B(a_i, r_i)$. In the same way as in the proof of Theorem 2.6, now with $\varepsilon = 0$, it is possible to see that $\varphi(x) = \varphi(x^{**})$ where φ is as in (1). Then clearly $x \in E_f(a)$, since $\varphi(x) = \varphi(x^{**}) = \hat{r}_f(a) \leq r_f(a)$.

(iii) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (i). Suppose a has all weighted Chebyshev centers. Let x^{**} belong to $\bigcap_i \hat{B}(a_i, r_i)$. Define $\varrho \in]0, +\infty[^N$ by $\varrho_i = 1/r_i$. By Theorem 2.6,

$$r_\varrho(a) = \hat{r}_\varrho(a) \leq \max_i (1/r_i)\|x^{**} - a_i\| \leq 1.$$

By (ii) there exists $x \in E_\varrho(a)$. This point satisfies $\max_i(1/r_i)\|x - a_i\| = r_\varrho(a) \leq 1$, and hence it belongs to $\bigcap_i B(a_i, r_i)$. $\square$

2.8. Definition (The class (GC)). We shall denote by (GC) the class of all Banach spaces X such that for every positive integer N and every $a \in X^N$, one of the equivalent conditions (i), (ii), (iii) of Theorem 2.7 is satisfied. ("GC" stands for "generalized centers".)

By Proposition 2.2, the class (GC) contains all the spaces which are norm-one complemented in their biduals (in particular, all dual spaces). As we shall see later, $c_0 \in$ (GC), though c_0 is known to be uncomplemented in $(c_0)^{**} = \ell^\infty$ [**So**].

2.9. Problem. *Does there exist a Banach space X which does not belong to* (GC) *but any finite set in X admits a (classical) Chebyshev center?*

3. Uniqueness and Stability of Generalized Centers

3.1. Remark. It is natural to ask that the set of f-centers (f monotone on $\mathbf{R}_+^N$) of any singleton coincide with the singleton, i.e., $E_f(a) = \{x_0\}$ whenever $a_1 = \ldots = a_N = x_0$. It is easy to ee that this condition is equivalent to the fact that 0 is the only minimum point for f. (This happens, for instance, if f is strictly monotone.)

Moreover, if 0 is the only minimum point for f, then $A := \{a_i : i = 1, \ldots, N\}$ is a singleton if and only if $r_f(a) = f(0)$. (Indeed, denote by e_i the i-th vector of the canonical basis of $\mathbf{R}^n$ and suppose that $r_f(a) = f(0)$. The function $f_i(t) := f(te_i)$ is nondecreasing on $\mathbf{R}_+$ with 0 as the unique minimum point. Take a sequence (x_n) in X such that $\varphi(x_n) \equiv f(\|x_n - a_1\|, \ldots, \|x_n - a_N\|) < f(0) + (1/n)$. Then $f_i(0) \le f_i(\|x_n - a_i\|) \le \varphi(x_n) < f_i(0) + (1/n)$; consequently $\|x_n - a_i\| \to 0$ as $n \to \infty$. Since this holds for all i, all a_i's must be equal.)

For a set M we denote by $\operatorname{aff} M$ and $\operatorname{ri} M$ the affine hull of M and the relative interior of M (i.e., the interior of M with respect to $\operatorname{aff} M$).

3.2. Theorem (Uniqueness). *Let X be a strictly convex Banach space, $a \in X^N$, $A = \{a_i : i = 1, \ldots, N\}$.*

(a) *For every $\varrho \in]0, +\infty[^N$, the set $E_\varrho(a)$ contains at most one point.*

(b) *Let f be a convex monotone function on $\mathbf{R}_+^N$ with 0 as a unique minimum point. If $E_f(a)$ contains more than one point, then $E_f(a)$ is a closed segment which is contained in the segment $[a_j, a_k]$ for some $j, k \in \{1, \ldots, N\}$.*

(c) *Let f be a convex strictly monotone function on $\mathbf{R}_+^N$. If $E_f(a)$ contains more than one point, then $E_f(a)$ is a closed segment such that $A \subset (\operatorname{aff} E_f(a) \setminus \operatorname{ri} E_f(a))$. (In particular, A is contained in a line.)*

Proof.

Case (a). Let $x, y \in E_\varrho(a)$ be two distinct points. Clearly, $r := r_\varrho(a) > 0$ (Remark 3.1). Put $\varphi(u) = \max_i \varrho_i \|u - a_i\|$. Since $\varphi(x) = \varphi(y) = r$, the points x, y belong to $B(a_i, r/\varrho_i)$ $(i = 1, \ldots, N)$. By rotundity, the point $z = (x+y)/2$ belongs to the interior of $B(a_i, r/\varrho_i)$; in other words, $\varrho_i \|z - a_i\| < r$ for every i. This implies $\varphi(z) < r = \inf \varphi(X)$, a contradiction.

Case (b). Suppose that $E_f(a)$ is not contained in a line. Since it is convex (see the beginning of Section 2), the set $E_f(a) \setminus A$ must contain three affinely independent points x_0, x_1, y (the vertices of a nondegenerate triangle). For $t \in]0, 1[$ put $L_t = \operatorname{aff}\{(1-t)x_0 + tx_1, y\}$. Since $y \notin A$ and any pair of lines L_t,$L_{t'}$ intersects only in y, there exists $t \in]0, 1[$ such that L_t contains no point of A (A is finite!). Put $x = (1-t)x_0 + tx_1$, $z = (x+y)/2$. For each $i = 1, \ldots, N$ the points x, y, a_i are affinely independent, hence by rotundity

$$\|z - a_i\| < \frac{1}{2}\|x - a_i\| + \frac{1}{2}\|y - a_i\|. \tag{2}$$

But Proposition 1.4(iii) gives, for φ as in (1),

$$\begin{aligned}\varphi(z) &< f\left(\frac{\|x-a_1\|+\|y-a_1\|}{2},\dots,\frac{\|x-a_N\|+\|y-a_N\|}{2}\right)\\ &\le \frac{1}{2}\varphi(x)+\frac{1}{2}\varphi(y)=r_f(a),\end{aligned}$$

a contradiction.

Thus $E_f(a)$ is a line segment since it is bounded (cf. Proposition 1.4(ii)). Denote by x, y its endpoints and put $L := \operatorname{aff} E_f(a)$, $z = (x+y)/2$. If $(L\backslash]x,y[) \cap A = \emptyset$ then, either the points x, y, a_i are affinely independent $(i = 1\dots,N)$ or $a_i \in]x,y[$ $(i = 1,\dots,N)$. Thus (2) holds for each i and, as above, we arrive to a contradiction.

It remains to show that $A_0 := (L\backslash]x,y[) \cap A$ cannot be contained in one of the two components of $L\backslash]x,y[$. Suppose, on the contrary, that A_0 is contained in the component of $L\backslash]x,y[$ which contains x. As we observed above, (2) holds whenever $a_i \in A\setminus A_0$. Thus there exists $\varepsilon \in]0,1[$ so small that

$$\|z-a_i\| < \frac{1-\varepsilon}{2}\|x-a_i\| + \frac{1+\varepsilon}{2}\|y-a_i\| \tag{2'}$$

holds whenever $a_i \in A\setminus A_0$. Moreover, for $a_i \in A_0$, we have $\|z-a_i\| < \|y-a_i\|$; consequently

$$\begin{aligned}\|z-a_i\| &= (1-\varepsilon)\|z-a_i\| + \varepsilon\|z-a_i\|\\ &< (1-\varepsilon)\|z-a_i\| + \varepsilon\|y-a_i\|\\ &\le \frac{1-\varepsilon}{2}\|x-a_i\| + \frac{1-\varepsilon}{2}\|y-a_i\| + \varepsilon\|y-a_i\|\\ &= \frac{1-\varepsilon}{2}\|x-a_i\| + \frac{1+\varepsilon}{2}\|y-a_i\|.\end{aligned}$$

We conclude that (2') holds for every $i = 1,\dots,N$. As above, since f is weakly strictly monotone and convex, we obtain

$$\begin{aligned}\varphi(z) < f\Big(&\frac{(1-\varepsilon)\|x-a_1\|+(1+\varepsilon)\|y-a_1\|}{2},\dots,\\ &\frac{(1-\varepsilon)\|x-a_N\|+(1+\varepsilon)\|y-a_N\|}{2}\Big)\\ \le\ &\frac{1-\varepsilon}{2}\varphi(x)+\frac{1+\varepsilon}{2}\varphi(y)=r_f(a),\end{aligned}$$

a contradiction.

Case (c). Let x, y be two distinct points of $E_f(a)$ and let φ be as in (1). Remark 3.1 implies that A is not a singleton. If there exists $i_0 \in \{1,\dots,N\}$ such that either $\{x, y, a_{i_0}\}$ is not contained in a line or $a_{i_0} \in]x, y[$, then (2) holds for $i = i_0$. As above, using strict monotonicity instead of weak strict monotonicity, we get a contradiction.

Hence $L := \operatorname{aff} A = \operatorname{aff} E_f(a)$ is a line and no point of A can lie strictly between two distinct points of $E_f(a)$. This completes the proof of (c). □

3.3. Example. Theorem 3.2(c) covers the case of medians. For medians in rotund Banach spaces, we can say much more combining Theorem 3.2 with Example 1.2. In particular, if there are more than one medians then the endpoints of the interval $E_f(a)$ belong to $\{a_i : i = 1, \dots, N\}$. But this is not the case in general, as the following simple example shows:

Let $N = 2$, $f(\xi_1, \xi_2) = \xi_1 + \xi_2 + \max\{\xi_1, \xi_2, (3/4)(\xi_1 + \xi_2)\}$, $X = \mathbf{R}$, $a = (-2, 2) \in \mathbf{R}^2$. Clearly, f is a convex symmetric strictly monotone function on $\mathbf{R}^2_+$. It is easy to verify that the function $\varphi(x) = f(|x+2|, |x-2|)$ attains its minimum exactly at the points $-1 \leq x \leq 1$. In other words, $E_f(a) = [-1, 1]$.

We are going to prove two stability results that will be used in the next section for weighted Chebyshev centers.

3.4. Theorem. *Let X be a Banach space with the property (w^*K) (i.e., $X = Z^*$ for some Banach space Z and the corresponding* **weak*** *topology coincides with the* **norm** *topology on S_X), $a \in X^N$. Let f be a convex monotone function on $\mathbf{R}^N_+$ having 0 as a unique point of minimum. Then every minimizing sequence for the function*

$$\varphi(x) = f(\|x - a_1\|, \dots, \|x - a_N\|)$$

*(i.e., a sequence (x_n) in X with $\varphi(x_n) \to r_f(a)$) has a (***norm***) convergent subsequence. In particular, $E_f(a)$ is a nonempty compact set.*

Proof. It is sufficient to prove that every minimizing sequence (x_n) has a **norm**-cluster point. By Proposition 1.4(ii), f is coercive; hence (x_n) is bounded. Passing to a subsequence, if necessary, we can suppose that for each i there exists $\mu_i := \lim_{n\to\infty} \|x_n - a_i\|$. Moreover, (x_n) admits a subnet (z_γ) that **weak*** converges to a point $x_0 \in X$. By **weak*** lower semicontinuity of the norm (X is dual!) and of φ (Proposition 1.4(iv)), we have $\|x_0 - a_i\| \leq \mu_i$ for all i, and

$$r_f(a) \leq \varphi(x_0) \leq \liminf_{\gamma} \varphi(z_\gamma) = r_f(a).$$

Consequently, $\varphi(x_0) = r_f(a)$ and $x_0 \in E_f(a)$.

Claim. There exists an index $k \in \{1, \dots, N\}$ such that $\mu_k = \|x_0 - a_k\|$.
If this is not the case, we have $\mu_i > \|x_0 - a_i\|$ for all i. By Proposition 1.4(iii), f is weakly strictly monotone, hence we have

$$\varphi(x_0) < f(\mu_1, \dots, \mu_N) = \lim_{\gamma} \varphi(z_\gamma) = r_f(a),$$

a contradiction that proves our Claim.

We have $z_\gamma - a_k \overset{w^*}{\to} x_0 - a_k$ and $\|z_\gamma - a_k\| \to \|x_0 - a_k\|$. The property ($w^*$K) implies that the **weak*** convergence is in fact **norm** convergence. Consequently x_0 is a cluster point of (z_γ), and hence also of (x_n). □

The following result concerns spaces in which **weak** or **weak*** compactness arguments cannot be used. Thus we have to **assume** the existence of the corresponding generalized centers.

3.5. Theorem. *Let X have the property* (CSE) *(i.e. every norm-attaining element of S_{X^*} compactly strongly exposes B_X). Let f be a convex monotone function on $\mathbf{R}_+^N$ having* 0 *as a unique point of minimum. Let $a \in X^N$ be such that $E_f(a)$ is not empty. Then every minimizing sequence for the function*

$$\varphi(x) = f(\|x - a_1\|, \dots, \|x - a_N\|)$$

*has a (***norm***) convergent subsequence. In particular, $E_f(a)$ is compact.*

Proof. If $a_1 = \ldots = a_N$, the result easily follows from Remark 3.1 and Proposition 1.4(ii).

Suppose that the set $A = \{a_i : \ i = 1, \dots, N\}$ is not a singleton. We want to prove that every minimizing sequence (x_n) has a cluster point in the **norm** topology. Proposition 1.4(ii) implies that (x_n) is bounded. Consider the convex monotone extension of f to the whole $\mathbf{R}^N$, given by $F(\xi) = f(\xi \vee 0)$. Then obviously $\varphi(x) = F(\|x - a_1\|, \dots, \|x - a_N\|)$ for each $x \in X$.

Choose an arbitrary point $x_0 \in E_f(a)$. Since φ attains its minimum at x_0, we must have $0 \in \partial\varphi(x_0)$. The formula for the subdifferential of φ (Theorem 1.8) implies that there exist

$$\lambda \in \partial F(\|x_0 - a_1\|, \dots, \|x_0 - a_N\|) \text{ and } u_i^* \in D(x_0 - a_i) \ (i = 1, \dots, N) \tag{3}$$

such that

$$\sum_{i=1}^{N} \lambda_i u_i^* = 0 \ . \tag{4}$$

(Recall that $D = \partial\|\cdot\|$.) Since F is nondecreasing in each coordinate, we must have $\lambda_i \geq 0$ for all i.

Claim. There exists an index $i_0 \in \{1, \dots, N\}$ such that $\lambda_{i_0} > 0$ and $\|x_0 - a_{i_0}\| > 0$.

Since A contains more than one element, there is at least one index i such that $a_i \neq x_0$. Without any loss of generality we can suppose that, for some $K \in \{1, \dots, N\}$,

$$\begin{cases} a_i \neq x_0 & \text{if } 1 \leq i \leq K; \\ a_i = x_0 & \text{if } K < i \leq N. \end{cases}$$

For any $\xi \in \mathbf{R}^K$ denote by $\overline{\xi}$ the element of $\mathbf{R}^N$ whose first K coordinates are those of ξ and the remaining ones, if any, are zeros. Put $\xi_0 = (\|x_0 - a_1\|, \dots, \|x_0 - a_K\|)$ and define a function $g\colon \mathbf{R}^K \to \mathbf{R}$ by $g(\xi) = F(\overline{\xi})$. Then g is convex and monotone, and the only point of minimum of g over $\mathbf{R}^K_+$ is 0. For each $\xi \in \mathbf{R}^K$ we have $g(\xi) - g(\xi_0) = F(\overline{\xi}) - F(\overline{\xi}_0) \geq \langle \lambda, \overline{\xi} - \overline{\xi}_0 \rangle = \sum_{i=1}^K \lambda_i(\xi_i - \|x_0 - a_i\|)$. This means that $(\lambda_1, \dots, \lambda_K) \in \partial g(\xi_0)$. Since ξ_0 is not a point of minimum for g, there exists $i_0 \in \{1, \dots, K\}$ such that $\lambda_{i_0} \neq 0$. This proves our Claim.

Using (4) we can write

$$
\begin{aligned}
(5) \qquad \varepsilon_n &:= \varphi(x_n) - \varphi(x_0) \\
&= f(\|x_n - a_1\|, \dots, \|x_n - a_N\|) - f(\|x_0 - a_1\|, \dots, \|x_0 - a_N\|) \\
&= \Big[f(\|x_n - a_1\|, \dots, \|x_n - a_N\|) - f(\|x_0 - a_1\|, \dots, \|x_0 - a_N\|) \\
&\quad - \sum_{i=1}^N \lambda_i(\|x_n - a_i\| - \|x_0 - a_i\|) \Big] \\
&\quad + \sum_{i=1}^N \lambda_i \big[\|x_n - a_i\| - \|x_0 - a_i\| - u_i^*(x_n - x_0) \big].
\end{aligned}
$$

By the definition of subdifferential, all square brackets in (5) are nonnegative. Thus, using the fact that $\|x_0 - a_{i_0}\| = u_{i_0}^*(x_0 - a_{i_0})$, we obtain $\varepsilon_n \geq \lambda_{i_0}[\|x_n - a_{i_0}\| - u_{i_0}^*(x_n - a_{i_0})] \geq 0$. Consequently

$$\|x_n - a_{i_0}\| - u_{i_0}^*(x_n - a_{i_0}) \to 0.$$

If $\liminf \|x_n - a_{i_0}\| = 0$, then (x_n) has a subsequence converging to a_{i_0}, and we are done.

Suppose $\liminf \|x_n - a_{i_0}\| > 0$. This implies

$$0 \leq 1 - u_{i_0}^* \left(\frac{x_n - a_{i_0}}{\|x_n - a_{i_0}\|} \right) = \frac{1}{\|x_n - a_{i_0}\|} [\|x_n - a_{i_0}\| - u_{i_0}^*(x_n - a_{i_0})] \to 0.$$

Since $u_{i_0}^*$ compactly strongly exposes B_X and (x_n) is bounded, the sequence (x_n) admits a subsequence $(\tilde{x}_m)$ such that

$$\frac{\tilde{x}_m - a_{i_0}}{\|\tilde{x}_m - a_{i_0}\|} \to y \text{ and } \|\tilde{x}_m - a_{i_0}\| \to t$$

for some $y \in S_X$ and $t > 0$. But this implies that $(\tilde{x}_m)$ converges to $ty + a_{i_0}$. □

3.6. Lemma. *Let f be a convex weakly strictly monotone function on $\mathbf{R}^N_+$. Then the function $a \mapsto r_f(a)$ is Lipschitz on any bounded subset of X^N.*

Proof. Let $m > 0$. We shall prove that r_f is Lipschitz on the set

$$B = \{a \in X^N : \ \|a_i\| \leq m \text{ for all } i\}.$$

First observe that $d(a) \le 2m$ for every $a \in B$. For $a \in B$, Proposition 2.1(a,b) implies $\|a\|_\infty + \Delta(a) \le 3m$ and

$$r_f(a) = \inf\{f(\|x - a_1\|, \dots, \|x - a_N\|) : \|x\| \le 3m\}. \tag{6}$$

Moreover, we have $\|x - a_i\| \le 4m$ whenever $\|x\| \le 3m$ and $a \in B$. Let L be the Lipschitz constant of f on $[0, 4m]^N$ (Proposition 1.4(i)). For $a, \tilde{a} \in B$ and $\|x\| \le 3m$ we have

$$\begin{aligned} &|f(\|x - a_1\|, \dots, \|x - a_N\|) - f(\|x - \tilde{a}_1\|, \dots, \|x - \tilde{a}_N\|)| \\ &\le L\left(\sum_{i=1}^N (\|x - a_i\| - \|x - \tilde{a}_i\|)^2\right)^{1/2} \le L\left(\sum_{i=1}^N \|a_i - \tilde{a}_i\|^2\right)^{1/2}. \end{aligned}$$

Consequently, by (6), the function r_f on B is the pointwise infimum of the functions $a \mapsto f(\|x - a_1\|, \dots, \|x - a_N\|)$ (w.r.t. $x \in X$, $\|x\| \le 3m$) which are all Lipschitz on B with the same Lipschitz constant L. This implies that r_f is Lipschitz on B with constant L, too. □

Let us recall the definition of upper semicontinuity of multivalued mappings. Let F be a multivalued mapping from a topological space T into another topological space S, with $F(t)$ nonempty for all $t \in T$. The mapping F is said to be **upper semicontinuous at a point** $t_0 \in T$ if for every open set V containing $F(t_0)$ there exists an open neighborhood U of t_0 such that $F(t) \subset V$ whenever $t \in U$.

F is said to be **upper semicontinuous** on T if it is upper semicontinuous at each point of T.

Obviously, if F is singlevalued then F is upper semicontinuous if and only if F is continuous in the classical sense.

3.7. Theorem. *Let f be a convex monotone function on $\mathbf{R}_+^N$ with 0 as a unique point of minimum. Denote*

$$\mathcal{D}(E_f) = \{a \in X^N : E_f(a) \ne \emptyset\}.$$

Suppose that X has at least one of the properties (w^*K), (CSE). *Then the multivalued mapping $a \mapsto E_f(a)$ (from $\mathcal{D}(E_f)$ into X) is upper semicontinuous on $\mathcal{D}(E_f)$ with respect to the* **norm** *topologies. Moreover, $\mathcal{D}(E_f) = X$ whenever X is (w^*K).*

Proof. Suppose that, on the contrary, $E_f(\cdot)$ is not upper semicontinuous at some $a \in \mathcal{D}(E_f)$. Then there exist an open set V containing $E_f(a)$ and a sequence $(a^{(n)}) \subset \mathcal{D}(E_f)$ such that $a^{(n)} \to a$ and $E_f(a^{(n)}) \setminus V \ne \emptyset$ for all n. For each n choose $x_n \in E_f(a^{(n)}) \setminus V$.

For simplicity, put $r = r_f(a)$, $r_n = r_f(a^{(n)})$, $\varphi(x) = f(\|x-a_1\|, \dots, \|x-a_N\|)$, $\varphi_n(x) = f(\|x-a_1^{(n)}\|, \dots, \|x-a_N^{(n)}\|)$. Take $R > 0$ such that $\|a^{(n)}\|_\infty \le R$ for every n. By Proposition 1.4(iii) and Proposition 2.1 we have $\|x_n - a_i^{(n)}\| \le \Delta(a_i^{(n)}) + d(a_i^{(n)}) \le 2d(a_i^{(n)}) \le 4R$, and hence $\|x_n - a_i\| \le \|x_n - a_i^{(n)}\| + \|a_i^{(n)}\| + \|a_i\| \le 6R$. Let $L > 0$ be a Lipschitz constant for f on $[0, 6R]^N$ and for $r_f(\cdot)$ on $\{b \in X^N : \|b\|_\infty \le R\}$ (cf. Proposition 1.4(i) and Lemma 3.6). Then we have

$$r \le \varphi(x_n) = r + (r_n - r) + (\varphi(x_n) - \varphi_n(x_n)) \le r + 2L\|a^{(n)} - a\|.$$

This implies $\varphi(x_n) \to r$. By Theorems 3.4 and 3.5, there exists a subsequence (x_k) of (x_n) converging (in **norm**) to some $x \in X$. Obviously, $x \in E_f(a) \subset V$. But this is contradiction since V is open and $x_k \notin V$ for any k. $\square$

Let us remark that P. Smith proved the following related result ([**Ho**, p. 188]): *If X is a reflexive strictly convex Banach space with the Kadec-Klee property, then the Chebyshev-center map is a singlevalued continuous mapping from the space of compact subsets of X (equipped with the Hausdorff metric) into X.*

4. Products and Vector-Valued Sequence Spaces

The purpose of this section is to prove that the class (GC) is stable under making arbitrary c_0- and ℓ^p-sums ($1 \le p \le \infty$). Since practically the same proof works also for other types of sums of spaces, we state it in a general form (Theorem 4.7). This requires some definitions.

In this section Γ denotes a nonempty set, X and X_γ ($\gamma \in \Gamma$) are Banach spaces. By e_γ we denote the characteristic function of the singleton $\{\gamma\} \subset \Gamma$ (i.e., $e_\gamma(\gamma') = \delta_{\gamma\gamma'}$).

Let Y be a linear space, $\Gamma_0 \subset \Gamma$, $y \in Y^\Gamma$. We denote by $y_{|\Gamma_0}$ the element of Y^Γ defined by

$$y_{|\Gamma_0}(\gamma) = \begin{cases} y(\gamma) & \text{for } \gamma \in \Gamma_0; \\ 0 & \text{otherwise.} \end{cases}$$

Hence $y_{|\Gamma_0}$ is the canonical projection of y onto the subspace of functions whose support is contained in Γ_0.

When non specified explicitely, the norms are considered to be finite. However, in this section, from formal reasons, we use also norms on $\mathbf{R}^\Gamma$ that can attain also the value $+\infty$, i.e. functions $\pi: \mathbf{R}^\Gamma \to [0, +\infty]$ which are convex, even, positively homogeneous and attain the value 0 only at the zero element of $\mathbf{R}^\Gamma$.

By a **sequence space on** Γ we mean a normed linear space (V, ν) such that V is a linear subspace of $\mathbf{R}^\Gamma$.

4.1. Definition. Let (V, ν) be a sequence space on Γ such that ν is monotone on the nonnegative elements of V (i.e., $\nu(\xi) \le \nu(\eta)$ whenever $\xi, \eta \in V$ and $0 \le$

$\xi \leq \eta$). We denote by $(\bigoplus X_\gamma)_V$ the linear space

$$\left(\bigoplus X_\gamma\right)_V = \left\{x \in \left[\bigcup X_\gamma\right]^\Gamma : x(\gamma) \in X_\gamma \text{ for all } \gamma \in \Gamma, \text{ and } \|x(\cdot)\| \in V\right\}$$

equipped with the norm

$$\|x\|_V = \nu(\|x(\cdot)\|).$$

(By $\|x(\cdot)\|$ we mean the function $\gamma \mapsto \|x(\gamma)\|_{X_\gamma}$.) If $X_\gamma = X$ for all $\gamma \in \Gamma$, the space $(\bigoplus X_\gamma)_V$ will be denoted by $V(X)$.

4.2. Definition.

1. Let $\pi \colon \mathbf{R}^\Gamma \to [0, +\infty]$ be a norm on $\mathbf{R}^\Gamma$ which is finite on the elements with finite support. By $S_\pi(\Gamma)$ we denote the linear space

$$S_\pi(\Gamma) = \{\xi \in \mathbf{R}^\Gamma : \ \pi(\xi) < +\infty\}$$

   equipped with the norm π.
2. We shall say that V is an **ideal in** $S_\pi(\Gamma)$ if the following three conditions are satisfied.
   (a) V is a closed linear subspace of $S_\pi(\Gamma)$;
   (b) $\xi \in V$ whenever $|\xi| \leq |\eta|$ for some $\eta \in V$;
   (c) $\{e_\gamma : \ \gamma \in \Gamma\} \subset V$.
3. If $V = S_\pi(\Gamma)$ in Definition 4.1, then we shall write $(\bigoplus X_\gamma)_\pi$, $S_\pi(\Gamma, X)$ and $\| \cdot \|_\pi$ instead of $(\bigoplus X_\gamma)_V$, $V(X)$ and $\| \cdot \|_V$.

4.3. Remark. The condition (c) of our definition of an ideal in $S_\pi(\Gamma)$ is not standard. It means that V contains all elements of $S_\pi(\Gamma)$ having finite support.

4.4. Definition. A norm $\pi \colon \mathbf{R}^\Gamma \to [0, +\infty]$ will be called

(a) **proper** if it is finite on the elements with finite support;
(b) **finitely determined** if for every $\xi \in \mathbf{R}^\Gamma$ we have

$$\pi(\xi) = \sup\left\{\pi\left(\xi_{|\Gamma_0}\right)\right\} \Gamma_0 \text{ is a finite subset of } \Gamma.$$

(c) **monotonic** if $\pi(\xi) \leq \pi(\eta)$ whenever $|\xi| \leq |\eta|$, $\xi, \eta \in \mathbf{R}^\Gamma$.
(d) **dual norm of a sequence space on** Γ if there exists (V, ν) a sequence space on Γ, containing all sequences with finite support, such that its dual V^* is isometric with $S_\pi(\Gamma)$ and the isometric correspondence between $v^* \in V^*$ and $\omega \in S_\pi(\Gamma)$ is given by

$$v^*(\xi) = \sum_{\gamma \in \Gamma} \xi(\gamma)\omega(\gamma) \qquad (\xi \in V).$$

4.5. Example. Let $1 \le p \le \infty$. Let $\pi\colon \mathbf{R}^\Gamma \to [0,+\infty]$ be the classical ℓ^p-norm. Then π is monotonic, proper and finitely determined, and we have

$$S_\pi(\Gamma) = \ell^p(\Gamma), \quad \Big(\bigoplus X_\gamma\Big)_\pi = \Big(\bigoplus X_\gamma\Big)_{\ell^p}, \quad S_\pi(\Gamma, X) = \ell^p(\Gamma, X).$$

The space $V = c_0(\Gamma)$ is an ideal in $\ell^\infty(\Gamma)$ and we have

$$\Big(\bigoplus X_\gamma\Big)_V = \Big(\bigoplus X_\gamma\Big)_{c_0}, \quad V(X) = c_0(\Gamma, X).$$

Each classical ℓ^p norm is a dual norm of a sequence space on Γ, with the predual V given by

$$V = \begin{cases} c_0(\Gamma) & \text{if } p = 1; \\ \ell^q(\Gamma) & \text{if } 1 < p < \infty,\ p^{-1} + q^{-1} = 1; \\ \ell^1(\Gamma) & \text{if } p = \infty. \end{cases}$$

The following lemma states what everybody would expect. We postpone its proof to Appendix.

4.6. Lemma. *Let $\pi\colon \mathbf{R}^\Gamma \to [0,+\infty]$ be a norm which is monotonic, proper and finitely determined. For $\gamma \in \Gamma$, let X_γ be a Banach space.*

(a) *$(\bigoplus X_\gamma)_\pi$ and $S_\pi(\Gamma)$ are Banach spaces.*
(b) *If π is a dual norm of a sequence space on Γ, then the space $(\bigoplus X^*_\gamma)_\pi$ is isometric to a dual space.*

Proof. See Appendix. □

We are ready to prove the main result of this section.

4.7. Theorem. *Let $\pi\colon \mathbf{R}^\Gamma \to [0,+\infty]$ be a norm which is monotonic, proper and finitely determined. Let V be an ideal in $S_\pi(\Gamma)$ (in the sense of Definition 4.2). Then the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) hold between the three assertions below. Moreover, if in addition $V = S_\pi(\Gamma)$ and π is a dual norm of a sequence space on Γ, then (i), (ii), (iii) are equivalent.*

(i) *$X_\gamma \in$ (GC) for each $\gamma \in \Gamma$, and $(\bigoplus X^{**}_\gamma)_V \in$ (GC).*
(ii) *$(\bigoplus X_\gamma)_V \in$ (GC).*
(iii) *$X_\gamma \in$ (GC) for each $\gamma \in \Gamma$.*

Proof. For simplicity, let us denote $W = (\bigoplus_\gamma X_\gamma)_V$, $\tilde{W} = (\bigoplus_\gamma X^{**}_\gamma)_V$.

(i) $\Rightarrow$ (ii). Suppose (i). By Theorem 2.7, it suffices to show that finite subsets of W admit all weighted Chebyshev centers (in W). Fix $a \in W^N$ and $\varrho \in]0,+\infty[^N$. There exists $z \in \tilde{W}$ a ϱ-center of a in $\tilde{W}$. For $\gamma \in \Gamma$ and $1 \le i \le N$ put

$$\delta_i(\gamma) = \|z(\gamma) - a_i(\gamma)\| \quad (\text{norm in } X^{**}_\gamma).$$

For each $\gamma \in \Gamma$ we have $z(\gamma) \in \bigcap_{i=1}^N \hat{B}(a_i(\gamma), \delta_i(\gamma))$, hence by Theorem 2.7 there exists

$$x(\gamma) \in \bigcap_{i=1}^N B(a_i(\gamma), \delta_i(\gamma)).$$

Thus we have a function $x\colon \Gamma \to \bigcup_\gamma X_\gamma$. It belongs to W since $\|z(\cdot)\|+2\|a_1(\cdot)\| \in V$ and

$$\begin{aligned}\|x(\cdot)\| &\le \|x(\cdot) - a_1(\cdot)\| + \|a_1(\cdot)\| \le \|z(\cdot) - a_1(\cdot)\| + \|a_1(\cdot)\| \\ &\le \|z(\cdot)\| + 2\|a_1(\cdot)\|.\end{aligned}$$

Denote by r_ϱ and $\tilde{r}_\varrho$ the ϱ-radius of a in W and in $\tilde{W}$. Then we have

$$\begin{aligned}\tilde{r}_\varrho \le r_\varrho &\le \max_{1\le i\le N} \varrho_i \|x - a_i\|_\pi = \max_{1\le i\le N} \varrho_i \pi(\|x(\cdot) - a_i(\cdot)\|) \\ &\le \max_{1\le i\le N} \varrho_i \pi(\|z(\cdot) - a_i(\cdot)\|) = \max_{1\le i\le N} \varrho_i \|z - a_i\|_\pi = \tilde{r}_\varrho.\end{aligned}$$

Consequently all inequalities are in fact equalities and x is a ϱ-center of a in W.

(ii) $\Rightarrow$ (iii). Suppose $W \in$ (GC). Fix $\gamma_0 \in \Gamma$. Let $a \in (X_{\gamma_0})^N$ and $\varrho \in]0, +\infty[^N$. The function

$$\overline{a}_i(\gamma) = \begin{cases} a_i, & \text{if } \gamma = \gamma_0 \\ 0, & \text{otherwise} \end{cases}$$

belongs to W (by property (c) of the definition of ideal). There exists a ϱ-center $z \in W$ for $\overline{a} = (\overline{a}_1, \dots, \overline{a}_N)$. Define

$$x(\gamma) = \begin{cases} z(\gamma), & \text{if } \gamma = \gamma_0 \\ 0, & \text{otherwise.} \end{cases}$$

Since x also belongs to W and $\|x - \overline{a}_i\|_\pi \le \|z - \overline{a}_i\|_\pi$, necessarily x is also a ϱ-center for $\overline{a}$. Now it is easy to see that $x(\gamma_0)$ is a ϱ-center of a in X_{γ_0}. It follows from the fact that $\|ue_{\gamma_0}\|_\pi = \pi(e_{\gamma_0}) \cdot \|u\|$ for all $u \in X_{\gamma_0}$. Indeed, for $u \in X_{\gamma_0}$ we have $\max_i \varrho_i \|u - a_i\| = [\pi(e_{\gamma_0})]^{-1} \max_i \varrho_i \|ue_{\gamma_0} - \overline{a}_i\|_\pi \ge [\pi(e_{\gamma_0})]^{-1} \max_i \varrho_i \|x - \overline{a}_i\|_\pi = \max_i \varrho_i \|x(\gamma_0) - a_i\|$.

Finally, suppose that $V = S_\pi(\Gamma)$ and π is a dual norm of a sequence space on Γ. By Lemma 4.6 the space $(\bigoplus X_\gamma^{**})_\pi$ is dual, and hence it is of class (GC) by Proposition 2.2. $\square$

4.8. Corollary. *Let $1 \le p \le \infty$. Then the following assertions are equivalent.*

(i) $X_\gamma \in$ (GC) *for every* $\gamma \in \Gamma$*;*
(ii) $(\bigoplus X_\gamma)_{\ell^p} \in$ (GC)*;*
(iii) $(\bigoplus X_\gamma)_{c_0} \in$ (GC).

Proof. The equivalence (i) $\Leftrightarrow$ (ii) and the implication (iii) $\Rightarrow$ (i) follow directly from Example 4.5 and Theorem 4.7.

(i) $\Rightarrow$ (iii). Suppose (i) holds. By Theorem 2.7, it suffices to show that finite sets in $(\bigoplus X_\gamma)_{c_0}$ admit weighted Chebyshev centers. Fix $a = (a_1, \dots, a_N) \in [(\bigoplus X_\gamma)_{c_0}]^N$ and $\varrho \in]0, +\infty[^N$. For every γ, put $a(\gamma) = (a_1(\gamma), \dots, a_N(\gamma)) \in (X_\gamma)^N$. By the assumption, there exists $x(\gamma) \in X_\gamma$ that belongs to $E_\varrho(a(\gamma))$. Denote $r = \max \varrho_i$. For each γ we have

$$\begin{aligned}
\|x(\gamma)\| &\le \|a_1(\gamma)\| + (\varrho_1)^{-1}(\varrho_1\|x(\gamma) - a_1(\gamma)\|) \\
&\le \|a_1(\gamma)\| + (\varrho_1)^{-1}\big(\max_i \varrho_i\|x(\gamma) - a_i(\gamma)\|\big) \\
&\le \|a_1(\gamma)\| + (\varrho_1)^{-1}\big(\max_i \varrho_i\|a_1(\gamma) - a_i(\gamma)\|\big) \\
&\le \|a_1(\gamma)\| + (\varrho_1)^{-1}\big(r\|a_1(\gamma)\| + r\max_{i\ne 1}\|a_i(\gamma)\|\big) \\
&\le \big(1 + (r/\varrho_1)\big) \cdot \sum_{i=1}^N \|a_i(\gamma)\|.
\end{aligned}$$

This implies that the function $x\colon \gamma \mapsto x(\gamma)$ belongs to $(\bigoplus X_\gamma)_{c_0}$. It remains to show that $x \in E_\varrho(a)$. But this is easy since, for any $z \in (\bigoplus X_\gamma)_{c_0}$, we have

$$\begin{aligned}
\max_i \varrho_i\|z - a_i\|_\infty &= \sup_\gamma \max_i \varrho_i\|z(\gamma) - a_i(\gamma)\| \ge \sup_\gamma \max_i \varrho_i\|x(\gamma) - a_i(\gamma)\| \\
&= \max_i \varrho_i\|x - a_i\|_\infty. \qquad \square
\end{aligned}$$

4.9. Corollary. *Let $1 \le p \le \infty$. Then the following assertions are equivalent.*

(i) $X \in (\mathrm{GC})$;
(ii) $\ell^p(\Gamma, X) \in (\mathrm{GC})$;
(iii) $c_0(\Gamma, X) \in (\mathrm{GC})$.

5. Spaces of Vector-Valued Functions

Let X be a real Banach space. First, we shall state a theorem about Lebesgue-Bochner spaces $L^p(\mu, X)$ of X-valued functions defined on a complete positive finite measure space (Ω, Σ, μ). We refer the reader to the book [**D-U**] for definitions and basic properties.

5.1. Theorem. *Let $1 \le p < \infty$. If X has the Radon-Nikodým property and is norm-one complemented in its bidual, then $L^p(\mu, X)$ is also norm-one complemented in its bidual; in particular, $L^p(\mu, X)$ belongs to the class* (GC).

Proof. The case of $p = 1$ was proved in [**Rao1**] (see also [**Rao2**], [**Em**]).

The case of $1 < p < \infty$. If X is a dual space, then $L^p(\mu, X)$ is also dual by [**D-U**, Theorem IV.1.1]; hence $L^p(\mu, X)$ is norm-one complemented in its bidual.

The general case (X not necessarily dual) was communicated to the author by T. S. S. R. K. Rao [**Rao3**]. □

In what follows, T stands for an arbitrary topological space.

By $C_b(T,X)$ we denote the Banach space of all continuous bounded X-valued functions on T equipped with the supremum norm. The space $C_b(T,X)$ is a subspace of $\ell^\infty(T,X)$. For $a \in [C_b(T,X)]^N$, let us denote by $r_f^C(a)$ and $r_f^\infty(a)$ the f-radius of a in $C_b(T,X)$ and in $\ell^\infty(T,X)$, respectively. Clearly, $r_f^C(a) \geq r_f^\infty(a)$.

The main results of the present section are contained in Theorem 5.3, Theorem 5.4 and Theorem 5.10.

Let us start with a simple selection lemma whose proof uses a standard partition-of-unity technique. A variant of it can be found in [**A-C**, p. 81].

5.2. Lemma. *For every $i \in \{1,\dots,N\}$, let $a_i\colon T \to X$ be a continuous function, $s_i > 0$. Suppose that the set $\psi(t) := \bigcap_i B^0(a_i(t), s_i)$ is nonempty for each $t \in T$. Then the multivalued mapping ψ has a continuous selection.*

Proof. Let $D = \{b \in X^N : \bigcap_i B^0(b_i, s_i) \neq \emptyset\}$. Observe that D is open in X^N. Define a multivalued mapping Ψ from D into X by

$$\Psi(b) = \bigcap_{i=1}^{N} B^0(b_i, s_i).$$

If $b \in D$, there exists a point $x_b \in \Psi(b)$. It is easy to see that the constant function with value x_b is a (continuous) selection of Ψ on a certain open neighborhood U_b of b in D. The set D, being a metric space, is paracompact [**Eng**]. Hence there exists a locally finite open covering $\{V_\gamma : \gamma \in \Gamma\}$ of D and continuous functions $p_\gamma\colon D \to \mathbf{R}_+$ such that

— for each $\gamma \in \Gamma$ there exists $b_\gamma \in D$ such that $V_\gamma \subset U_{b_\gamma}$;
— for each $\gamma \in \Gamma$ the function p_γ is null outside of V_γ;
— $\sum_\gamma p_\gamma \equiv 1$.

Clearly, the function

$$g(b) = \sum_{\gamma \in \Gamma} p_\gamma(b) x_{b_\gamma}$$

is a continuous selection of Ψ on D. Then the function $g_0(t) = g(a_1(t),\dots,a_N(t))$ is a continuous selection of ψ. □

5.3. Theorem.

(a) *If $C_b(T,X) \in$ (GC), then $X \in$ (GC).*
(b) *Let X be the ℓ^∞-sum of a family of Banach spaces X_γ ($\gamma \in \Gamma$). Then $C_b(T,X) \in$ (GC) if and only if $C_b(T,X_\gamma) \in$ (GC) for each $\gamma \in \Gamma$.*
(c) *Let f be a continuous monotone function on $\mathbf{R}_+^N$. Then for every $a \in [C_b(T,X)]^N$ we have $r_f^C(a) = r_f^\infty(a)$.*

Proof. **(a)** Let $a \in X^N$, $\varrho \in]0, +\infty[^N$. Consider X canonically embedded (as constant functions) in $C_b(T, X)$. Let r and $\overline{r}$ denote the ϱ-radius of a in X and in $C_b(T, X)$ respectively. Clearly, $\overline{r} \le r$. Suppose $C_b(T, X) \in$ (GC) and take a function x which is a ϱ-center of a in $C_b(T, X)$. Then for every $t \in T$ we have $\max_i \varrho_i \|x(t) - a_i\| \le \overline{r} \le r$. Consequently, $x(t) \in E_\varrho(a)$ (in X) for each $t \in T$. Hence $X \in$ (GC) by Theorem 2.7.

(b) Observe that $C_b(T, X)$ is isometric with the ℓ^∞-sum of the spaces $C_b(T, X_\gamma)$ $(\gamma \in \Gamma)$, then use Corollary 4.8.

(c) For $\varepsilon > 0$ choose $x_\varepsilon \in \ell^\infty(T, X)$ such that $f(\|x_\varepsilon - a_1\|_\infty, \dots, \|x_\varepsilon - a_N\|_\infty) < r_f^\infty(a) + \varepsilon$. Put $r_i = \|x - a_i\|_\infty$ and observe that $x_\varepsilon(t) \in \bigcap_i B^0(a_i(t), r_i + \varepsilon)$ for every $t \in T$. By Lemma 5.2 there exists a continuous function $z_\varepsilon \colon T \to X$ such that $z_\varepsilon(t) \in \bigcap_i B^0(a_i(t), r_i + \varepsilon)$ for each $t \in T$. Then z_ε belongs to $C_b(T, X)$ since a_i's are bounded. Moreover, we have

$$\begin{aligned} f(r_1, \dots, r_N) &< r_f^\infty(a) + \varepsilon \le r_f^C(a) + \varepsilon \le f(\|z_\varepsilon - a_1\|_\infty, \dots, \|z_\varepsilon - a_N\|_\infty) + \varepsilon \\ &\le f(r_1 + \varepsilon, \dots, r_N + \varepsilon) + \varepsilon. \end{aligned}$$

The result follows from the fact that the last term tends to $f(r_1, \dots, r_N)$ as $\varepsilon \to 0^+$. □

5.4. Theorem. *The space $C_b(T, X)$ belongs to the class* (GC) *provided any of the following two conditions is satisfied.*

(a) *X is strictly convex and satisfies the property (w^*K).*
(b) *$X \in$ (GC) and every norm-attaining element of S_{X^*} strongly exposes B_X (i.e., X is strictly convex and satisfies the property* (CSE)*).*

Proof. By Theorem 2.7 it suffices to show that each $a \in [C_b(T, X)]^N$ admits weighted Chebyshev centers. Let $\varrho \in]0, +\infty[^N$. By Proposition 2.2 and Theorem 3.2(a), for each $t \in T$ the set $E_\varrho(a(t))$ is a singleton in X. By Theorem 3.7, $x(t) := E_\varrho(a(t))$ depends continuously on t. Moreover, x is bounded by Proposition 2.1 (a_i's are bounded!). Now, as in the very end of the proof of Corollary 4.8, it is elementary to see that x is a ϱ-center of a in $C_b(T, X)$. □

5.5. Remark. D. Amir [**Am1**] proved that if X is uniformly rotund and T compact then each bounded subset in $C_b(T, X)$ admits a Chebyshev center. Thus, for finite sets, Amir's result follows from our Theorem 5.4, since any uniformly convex space is reflexive and hence satisfies the assumptions (a), (b) from Theorem 5.4.

The first example of a finite-dimensional Banach space X such that a finite set in $C_b([0, 1], X)$ admits no Chebyshev center, is due to J. Kolář [**Kol**]. We present a simplified version of it, essentially due to P. Holický and J. Kolář.

5.6. Example. There exists a three-dimensional Banach space X and a three-point set in $C_b([0, 1], X)$ that has no Chebyshev center.

Put $D = \{(x,y,z) \in \mathbf{R}^3 : x \geq 0,\ x^2+y^2 \leq 1,\ z=1\}$ and $B = \mathrm{co}\big[D \cup (-D)\big]$. The set B is a closed bounded convex symmetric neighborhood of the origin, hence it is the unit ball of an equivalent norm $\|\cdot\|$ on $\mathbf{R}^3$. Let $X = (\mathbf{R}^3, \|\cdot\|)$ and denote $e_2 = (0,1,0) \in X$.

Observe that the intersection $(e_2+B) \cap (-e_2+B)$ is exactly the line segment $L := \{(0,0,s) : s \in [-1,1]\}$. For $t \in [0,1]$ define

$$\begin{aligned} a_1(t) &= (\cos \pi t, 1 + \sin \pi t, 0) = (\cos \pi t, \sin \pi t, 0) + e_2 \\ a_2(t) &= (\cos \pi t, -1 + \sin \pi t, 0) = (\cos \pi t, \sin \pi t, 0) - e_2 \\ a_3(t) &= (0,0,0). \end{aligned}$$

Then $a_i \in C_b([0,1], X)$ for $i = 1,2,3$. Moreover, for every $t \in [0,1]$ we have

$$\begin{aligned} F(t) := \bigcap_{i=1}^{3} [a_i(t) + B] &= \big[(\cos \pi t, \sin \pi t, 0) + L\big] \cap B \\ &= \begin{cases} \{(\cos \pi t, \sin \pi t, 1)\} & \text{for } 0 \leq t < 1/2; \\ (\cos \pi t, \sin \pi t, 0) + L & \text{for } t = 1/2; \\ \{(\cos \pi t, \sin \pi t, -1)\} & \text{for } 1/2 < t \leq 1. \end{cases} \end{aligned}$$

Obviously, F admits no continuous selection because of its “jump” at the point $t = 1/2$.

Since $F(t)$ belongs to $\bigcap [a_i(t) + S_X]$ for every t, it is easy to see that the Chebyshev radius of $a(t) = (a_1(t), a_2(t), a_3(t))$ in X is equal to 1 for every t. This easily implies that the Chebyshev radius of $a = (a_1, a_2, a_3)$ in $\ell^\infty([0,1], X)$, and hence also in $C_b([0,1], X)$ (see Theorem 5.3(c)), equals 1. Thus any Chebyshev center of a in $C_b([0,1], X)$ has to be a continuous selection of F. Consequently, a has no Chebyshev center in $C_b([0,1], X)$.

5.7. Proposition. *The following two assertions are equivalent.*

(i) $C_b(T, X) \in$ (GC) *for every topological space* T.

(ii) $X \in$ (GC) *and for every* $N \in \mathbf{N}$ *and every* $r \in]0, +\infty[^N$ *there exists a continuous selection of the multivalued mapping*

$$F(b) = \bigcap_{i=1}^{N} B(b_i, r_i) \ ,$$

defined on the set $D = \{b \in X^N : F(b) \neq \emptyset\}$.

Proof. (i) $\Rightarrow$ (ii). Suppose (i) holds. Then $X \in$ (GC) by Theorem 5.3(a). Let r, F, D be as in (ii). Denote $\rho = (1/r_1, \dots, 1/r_N)$. Let D_0 be an arbitrary bounded relatively open subset of D. Put $T = D_0$, $a_i(b) = b_i$ $(i = 1, \dots, N)$ and observe that the functions a_i belong to $C_b(T,X)$. By (i), there exists $x \in E_\rho(a)$. We are going to show that x is a continuous selection of F on D_0.

Let $y\colon D_0 \to X$ be an arbitrary selection of F on D_0. Then $y \in \ell^\infty(D_0,X)$, and hence we have

$$r_\rho^C(a) = r_\rho^\infty(a) \le \max_i r_i^{-1}\|y-a_i\|_\infty = \max_i \sup_{b\in D_0} r_i^{-1}\|y(b)-b_i\| \le 1$$

by Theorem 5.3(c) and the definition of D. Consequently, for every i and each $b \in D_0$ we have $r_i^{-1}\|x(b)-b_i\| \le r_\rho^C(a) \le 1$. But this implies that $x(b) \in F(b)$ for every $b \in D_0$.

What we have proved implies that each point of (the metric space) D has an open neighborhood on which F has a continuous selection. The same partition-of-unity argument as in the proof of Lemma 5.2 gives a continuous selection of F on the whole set D.

(ii) $\Rightarrow$ (i). Suppose that (ii) holds. By Theorem 2.7 it suffices to show that the finite subsets of $C_b(T,X)$ admit weighted Chebyshev centers. Fix $a \in [C_b(T,X)]^N$ and $\rho \in]0,+\infty[^N$. For every $t \in T$, let $z(t)$ be a ρ-center of $a(t) := (a_1(t),\dots, a_n(t))$. In this way we have defined an element $z \in \ell^\infty(T,X)$. For any $t \in T$ and any i we have

$$\rho_i\|z(t)-a_i(t)\| \le \rho_i\|z-a_i\|_\infty \le r_\rho^\infty(a) \le r_\rho^C(a) \equiv r.$$

Consequently, $z(t) \in F(a(t))$ for every $t \in T$, where $F(t)$ is as in (ii) with $r_i = r/\rho_i$. Hence $a(t) \in D$ for every $t \in T$. Let ψ be a continuous selection of F defined on D. Then $x(t) = \psi(a(t))$ defines a continuous X-valued function on T. Moreover, $\rho_i\|x-a_i\|_\infty \le r = r_\rho^C(a)$. Thus x belongs to $C_b(T,X)$ and is a ρ-center for a. $\square$

5.8. Remark. Repeating the proof of Proposition 5.7 with $r_1 = \dots = r_N$ and $\rho_1 = \dots = \rho_N$, we get the following criterion of the existence of (classical) Chebyshev centers in $C_b(T,X)$.

The following two assertions are equivalent:

- (i) for every topological space T each finite set in $C_b(T,X)$ admits a Chebyshev center;
- (ii) each finite set in X admits a Chebyshev center and for every $N \in \mathbf{N}$ the mapping $F(a) = \bigcap_{i=1}^N (a_i + B_X)$ has a continuous selection on $D = \{a \in X^N : F(a) \neq \emptyset\}$.

Let us recall the notion of a lower semicontinuous multivalued mapping. Let F be a multivalued mapping from T into another topological space T' such that $F(t) \neq \emptyset$ for every $t \in T$. We shall say that T is **lower semicontinuous** at a point t_0 if for every open set V with $F(t_0) \cap V \neq \emptyset$ there exists a neighborhood U of t_0 such that $F(t) \cap V \neq \emptyset$ whenever $t \in U$. If F is lower semicontinuous at each point of T, we shall say simply that F is lower semicontinuous (on T).

5.9. Remark. Let F be as above.

(a) It is well known (and easy to see) that F is lower semicontinuous if and only if the inverse image $F^{-1}(\Omega) := \{t \in T \,:\, F(t) \cap \Omega \neq \emptyset\}$ of any open set $\Omega \subset T'$ is open (in T).

(b) It is also easy to see that for lower semicontinuity of F at t_0 it is sufficient the following condition: *for every $x_0 \in F(t_0)$ there exists a selection of F such that it is continuous at t_0 and its value at t_0 is x_0.*

The rest of this section is dedicated to proving the following theorem. A finite-dimensional Banach space X is called **polyhedral** if its unit ball B_X is a polytope.

5.10. Theorem. *Let X be a finite-dimensional Banach space. Then each multivalued mapping of the form $F(a) = \bigcap_{i=1}^{N} B(a_i, r_i)$ (acting from X^N into X) is lower semicontinuous on the set $D = \{a \in X^N \,:\, F(a) \neq \emptyset\}$ provided any of the following three conditions is satisfied.*

(a) *X is strictly convex.*

(b) *X is polyhedral.*

(c) *X is two-dimensional.*

In particular, each of the conditions **(a), (b), (c)** *is sufficient for $C_b(T, X) \in$ (GC) for any topological space T.*

5.11. Remark.

(a) The final assertion of Theorem 5.10 follows easily from Proposition 5.7. Indeed, if the mapping F from Theorem 5.10 is lower semicontinuous on D then there exists a continuous selection of F on D by Michael's selection theorem [**Mi**].

(b) The mapping F from Theorem 5.10 is upper semicontinuous on D in any finite-dimensional space X. To prove this, suppose the contrary. This means that there exists an open set $V \subset X$, points $a^n \in D$ $(n \geq 0)$ and $x_n \in F(a^n)$ $(n \geq 1)$ such that $a^n \to a^0$, $F(a^0) \subset V$ and $x_n \notin V$ for any $n \geq 1$. The sequence (x_n) has a cluster point $x_0 \in X$ since F is locally bounded. Clearly, x_0 does not belong to V. But for every $1 \leq i \leq N$, $\|a_i^0 - x_0\|$ is a cluster point of the sequence $(\|a_i^n - x_n\|)$ which is contained in $[0, r_i]$. This implies $x_0 \in F(a^0) \subset V$, a contradiction.

Proof of Theorem 5.10(a). Let X be strictly convex and $a \in D$. If $F(a)$ is a singleton, then F is upper semicontinuous (and hence also lower semicontinuous) in a by Remark 5.11(b). Now, suppose that $F(a)$ contains two distinct points x, y. The strict convexity of B_X and the definition of F easily imply that $z := \frac{x+y}{2}$ is an interior point of $F(a)$. Then F is Lipschitz (with respect to the Hausdorff metric), and hence lower semicontinuous, on a neighborhood of a (in D). This follows easily from the following particular case of [**P-Y**, Theorem 4]: *Let Φ be a multivalued mapping from a subset S of X^N into X, which has nonempty closed* *convex values and is Lipschitz (w.r.t. the Hausdorff metric). If $a \in S$, $r > 0$, $i \in \{1, \dots, N\}$ and $\Phi(a) \cap B^0(a_i, r) \neq \emptyset$, then the mapping $b \mapsto \Phi(b) \cap B(b_i, r)$ is Lipschitz on a neighborhood of a (in S).* $\square$

Proof of Theorem 5.10(b). If B_X is a polytope, then it is the intersection of finitely many closed halfspaces $H_1, \dots, H_k$. Thus the mapping F from Theorem 5.10 can be written in the form

$$F(a) = \bigcap_{i=1}^{N} \bigcap_{j=1}^{k} (a_i + r_i H_j).$$

Consequently, it is sufficient to prove the following proposition.

5.12. Proposition. *Let $H_1, \dots, H_N$ be closed halfspaces in X, $\dim X = d$. Then the multivalued mapping F from X^N into X, given by*

$$F(a) = \bigcap_{i=1}^{N} (a_i + H_i) \quad \text{for any } a = (a_1, \dots, a_N) \in X^N ,$$

is lower semicontinuous on $\mathcal{D}(F) = \{a \in X^N \,:\, F(a) \neq \emptyset\}$.

Proof. We shall proceed by induction with respect to the dimension d. Without any loss of generality we can suppose that the boundary hyperplane of each H_i passes through the origin.

Case $d = 1$**.** If $X = \mathbf{R}$, denote by I_+, I_- the set of indices $i \in \{1 \dots, N\} \equiv I$ such that, respectively, $H_i = [0, +\infty[$, $H_i =]-\infty, 0]$. Then for every $a \in \mathbf{R}^N$ we have

$$F(a) = \begin{cases} [\max\limits_{i \in I_+} a_i, \min\limits_{i \in I_-} a_i] & \text{if } I_+ \neq \emptyset,\ I_- \neq \emptyset; \\ [\max\limits_{i \in I} a_i, +\infty[& \text{if } I_- = \emptyset; \\ ]-\infty, \min\limits_{i \in I} a_i] & \text{if } I_+ = \emptyset. \end{cases}$$

In all three cases F is lower semicontinuous on $\mathcal{D}(F)$.

Induction step. Let n be a positive integer such that the assertion of Proposition 12 holds for any dimension $d < n$. We shall prove that it holds also for $d = n$.

First, let us consider the case $N = n$, i.e. the number of the halfspaces is the same as the dimension of X. For $1 \leq i \leq n$, let $f_i \in X^*$ be such that $H_i = f_i^{-1}([0, +\infty[)$. Denote $K = \bigcap_{i=1}^{n} f_i^{-1}(0)$.

a) If $\dim K > 0$, consider the quotient map $q\colon X \to X/K$. Then

$$\begin{aligned} F(a) &= q^{-1}\left(\bigcap_{i=1}^{n} q(a_i + H_i) \right) = q^{-1}\left(\bigcap_{i=1}^{n} [q(a_i) + q(H_i)] \right) \\ &= \left[q^{-1} \circ G \circ Q \right](a), \end{aligned}$$

where $Q: X^n \to (X/K)^n$ and $G: (X/K)^n \to X/K$ are given by

$$Q(a) = (q(a_1), \dots, q(a_n)) ,$$
$$G(b) = \bigcap_{i=1}^{n} \big(b_i + q(H_i)\big) .$$

By the induction assumption, G is lower semicontinuous on $\mathcal{D}(G)$. Since q is open and Q is continuous, the mapping $F = q^{-1} \circ G \circ Q$ is lower semicontinuous on $\mathcal{D}(F)$ (Remark 5.9(a)).

b) If $\dim K = 0$ then the linear mapping $L\colon X \to \mathbf{R}^n$, given by $L(x) = (f_1(x), \dots, f_n(x))$, is one-to-one and hence surjective. Observe that

$$\begin{aligned} F(a) &= \{x \in X \,:\, x - a_i \in H_i \ \text{ for } i = 1, \dots, n\} \\ &= \{x \in X \,:\, f_i(x) \geq f_i(a_i) \ \text{ for } i = 1, \dots, n\} \\ &= L^{-1}\Big([f_1(a_1), +\infty[\times \cdots \times [f_n(a_n), +\infty[\Big) \\ &= \big[L^{-1} \circ S \circ \Lambda\big](a) , \end{aligned}$$

where $\Lambda: X^n \to \mathbf{R}^n$ is the linear mapping given by $\Lambda(a) = (f_1(a_1), \dots, f_n(a_n))$, and $S(b) = b + \mathbf{R}^n_+$ ($b \in \mathbf{R}^n$) is the "shifting" of the positive cone in $\mathbf{R}^n$. Observe that L is open, S is lower semicontinuous and Λ is continuous. Hence, by Remark 5.9(a), F is lower semicontinuous.

Now, let N be arbitrary. If $N < n$, define $H_i = H_N$ for $N < i \leq n$, and apply what was proved above to get easily that F is lower semicontinuous also in this case.

Let $N > n$. Fix $a \in \mathcal{D}(F)$, $x_0 \in F(a)$ and $\epsilon > 0$. By lower semicontinuity proved above for the number of halfspaces equal to the dimension n, there exists a neighborhood U of a in $\mathcal{D}(F)$ such that for each $E \subset \{1, \dots, N\}$ of cardinality n we have

$$\bigcap_{i \in E} (b_i + H_i) \cap B(x_0, \epsilon) \neq \emptyset \quad \text{whenever } b \in U.$$

Consequently, for $b \in U$, each $n+1$ of the closed convex sets

$$B(x_0, \epsilon), b_1 + H_1, \dots, b_N + H_N$$

have nonempty intersection. By Helly's theorem [**Va**], all these sets have nonempty intersection. In other words, $F(b) \cap B(x_0, \epsilon) \neq \emptyset$ whenever $b \in U$. This completes the proof. $\square$

Theorem 5.10(b) is proved.

To prove Theorem 5.10(c), it is possible to use Helly's theorem in a similar way as in the proof of Proposition 5.12. Thus it suffices to prove the lower semicontinuity of the intersection of two balls only (i.e., the case $N = 2$). And this is what says the following lemma.

5.13. Lemma. *Let* $\dim X = 2$, $r > 0$, $D = \{a \in X : B(0,1) \cap B(a,r) \neq \emptyset\} = B(0, 1+r)$. *Then the multivalued mapping* F *from* D *into* X, *given by*

$$F(a) = B(0,1) \cap B(a,r)\,,$$

is lower semicontinuous.

Proof. Fix $a_0 \in D$ and $x_0 \in F(a_0)$. We shall consider several cases that cover all possible situations. (Note that the cases are not disjoint.)

Case 1: $a_0 \in B^0(0, 1+r)$**, i.e.** $B^0(0,1) \cap B^0(a_0, r) \neq \emptyset$**.** Then by [**P-Y**, Theorem 4] F is Lipschitz with respect to the Haussdorff metrics on some neighborhood of a_0. This implies that F is lower semicontinuous at a_0.

Case 2: $\|a_0\| = 1 + r$ **and** $F(a_0) = \{x_0\}$**.** In this case, F is upper semicontinuous, and hence also lower semicontinuous, at a_0 (cf. Remark 5.11(b)).

Case 3: $\|a_0\| = 1+r$ **and** $x_0 = \frac{a_0}{1+r}$**.** Then $f(a) = \frac{a}{1+r}$ is a continuous selection of F with $f(a_0) = x_0$. Use Remark 5.9(b).

Case 4: $\|a_0\| = 1+r$ **and** $x_0 \neq \frac{a_0}{1+r}$**.** The set $F(a_0)$ is a nondegenerate closed line segment that contains $\frac{a_0}{1+r}$. Put

$$v = \frac{\frac{a_0}{1+r} - x_0}{\left\|\frac{a_0}{1+r} - x_0\right\|} = \frac{a_0 - (1+r)x_0}{\|a_0 - (1+r)x_0\|}.$$

Let L be the affine hull of $F(a_0)$. The line L separates $B(0,1)$ and $B(a_0, r)$. Since the vector x_0 is not parallel to L, every point $a \in X$ can be written (in a unique way) in the form

$$a = a_0 + tv + sx_0 \quad (t, s \in \mathbf{R}).$$

Observe that the line $L + (a_0 - x_0)$ supports $B(0, 1+r)$ at a_0. From this fact, it easily follows that $s \leq 0$ whenever $a = a_0 + tv + sx_0 \in D$. Fix $\delta \in (0,1)$ so small that

$$x_0 + tv \in F(a_0) \quad \text{for} \quad 0 \leq t < \delta, \tag{7}$$

and

$$x_0 + tv + sx_0 \in B(0,1) \quad \text{whenever} \quad -\delta < s \leq 0 \leq t < \delta \tag{8}$$

(this is possible since the triangle $\mathrm{co}\{0, x_0, \frac{a_0}{1+r}\}$ is contained in $B(0,1)$).

Put $U = \{a_0 + tv + sx_0 :\ -\delta < s \leq 0,\ \ |t| < \delta\}$. Then $U \cap D$ is a neighborhood of a_0 in D. Define a mapping $f\colon U \to X$ by

$$f(a_0 + tv + sx_0) = \begin{cases} (1+s)x_0 + tv & \text{if } -\delta < s \leq 0 \leq t < \delta, \\ (1+s)x_0 & \text{if } -\delta < s \leq 0,\ -\delta < t \leq 0. \end{cases}$$

Then f is continuous and $f(a_0) = x_0$. We shall show that $f(a) \in F(a)$ for every $a \in U$.

Let $a = a_0 + tv + sx_0 \in U$. If $t \geq 0$ then $f(a) \in B(0,1)$ by (8), and $\|f(a) - a\| = \|x_0 - a_0\| = r$. For $t < 0$ we have $\|f(a)\| = \|(1+s)x_0\| = 1 + s \leq 1$, and $\|f(a) - a\| = \|(x_0 + |t|v) - a_0\| \leq r$ by (7).

Also in this case, F is lower semicontinuous at a_0 by Remark 5.9(b). The proof is complete. □

Proof of Theorem 5.10(c). Since the assertion is obvious if $r_i = 0$ for some i, let us suppose that $r_i > 0$ for each i. The case $N = 2$ (two balls) is an easy consequence of Lemma 5.13.

Suppose $N > 2$. Fix $a_0 \in D$, $x_0 \in F(a_0)$, $\varepsilon > 0$. Since the assertion is valid for two balls, there exists a neighborhood U of a_0 in D such that for every $a \in U$ and every pair of distinct indices $j, k \in \{1, \dots, N\}$ one has

$$B(a_j, r_j) \cap B(a_k, r_k) \cap B(x_0, \varepsilon) \neq \emptyset.$$

Consequently, for every $a \in U$ each three of the balls

$$B(a_1, r_1), \dots, B(a_N, r_N), B(x_0, \varepsilon)$$

have nonempty intersection. By Helly's theorem [**Va**], the intersection of all these balls is nonempty; in other words, $F(a) \cap B(x_0, \varepsilon) \neq \emptyset$ for $a \in U$. □

5.14. Remark. Lemma 5.13 does not hold in general if $\dim X > 2$. For every $n > 2$, it is easy to construct an n-dimensional Banach space X such that the set of the extreme points of B_X is not closed. Put $D = B(0,2)$, $F(a) = B(0,1) \cap B(a,1)$ for $a \in D$. We shall show that F is not lower semicontinuous at some point of D.

Let (x_n) be a sequence of extreme points of B_X that converges to a point x which is not extreme, i.e. $x \pm u \in S_X$ for some nonzero vector $u \in X$. If we denote $a_n = 2x_n$ and $a_0 = 2x$, we have $a_n \to a_0$, $F(a_n) = \{x_n\}$, $x \in [x+u, x-u] \subset F(a_0)$. Such situation could not happen if F were lower semicontinuous at a_0.

5.15. Remark. Our proofs of the existence of weighted Chebyshev centers of finite sets in $C_b(T,X)$ are based on the fact that "max" (which defines the function we want to minimize) and "sup" (which defines the norm on $C_b(T,X)$) are interchangeable, since this implies that any continuous selection of the point-by-point ϱ-center map $E_\varrho(a(\cdot))$ is a ϱ-center of a in $C_b(T,X)$.

A similar idea can be used to prove some results on the existence of weighted p-medians in $L^p(\mu, X)$ for $1 \leq p < \infty$. It is easy to see that, since sum and integral are interchangeable, any Bochner-measurable selection of the point-by-point weighted p-median map (for a fixed weight) is a weighted p-median in $L^p(\mu, X)$ for the same weight. Using selection theorems for **weak** or **weak*** upper semicontinuous maps (due to V. V. Srivatsa and to J. E. Jayne and C. A. Rogers, cf.

[**J-O-P-V**, Theorem 19, Theorem 16]), and versions of Theorem 3.7 for **weak** or **weak*** topologies (instead of **norm** topology), it is possible to prove the folowing result:

Let $1 \le p < \infty$. Then weighted p-medians of the finite sets in $L^p(\mu, X)$ exist in any of the following three cases:

(a) *X is a dual space with the Radon-Nikodým property* (this follows from Theorem 5.1)*;*

(b) *$X = Z^*$ for some space Z and the corresponding* **weak** *and* **weak*** *topologies coincide on S_X;*

(c) *X admits the weighted p-medians of the finite sets and every norm-attaining element x^* of S_{X^*} weakly compactly strongly exposes B_X, in the sense that any sequence $(x_n) \subset B_X$ with $x^*(x_n) \to 1$ has a* **weak** *cluster point.*

6. Appendix (Proof of Lemma 4.6)

Proof of Lemma 4.6(a). Let (x_n) be a Cauchy sequence in $(\bigoplus X_\gamma)_\pi$. Let $C > 0$ be such that $\|x_n\|_\pi \le C$ for every n. Fix $\gamma \in \Gamma$. Since $\|x_n(\gamma) - x_m(\gamma)\|\pi(e_\gamma) = \pi(\|x_n(\gamma) - x_m(\gamma)\|e_\gamma) \le \|x_n - x_m\|_\pi$, the sequence $(x_n(\gamma))$ is Cauchy in X_γ, and hence convergent to some $x(\gamma)$.

In this way we have defined a function $x\colon \Gamma \to \bigcup X_\gamma$. For every finite subset Γ_0 of Γ we have

$$\left\|x_{|\Gamma_0}\right\|_\pi = \lim_n \left\|(x_n)_{|\Gamma_0}\right\|_\pi \le \limsup_n \|x_n\|_\pi \le C.$$

Taking supremum w.r.t. all finite subsets $\Gamma_0 \subset \Gamma$ we obtain $\|x\|_\pi \le C$, and hence $x \in (\bigoplus X_\gamma)_\pi$.

It remains to prove that (x_n) converges to x. Fix an arbitrary $\varepsilon > 0$. There exists n_0 such that $\|x_n - x_m\|_\pi < \varepsilon$ whenever $n, m > n_0$. Fix an arbitrary n greater that n_0. There exists a finite set $\Gamma_n \subset \Gamma$ such that $\|x_n - x\|_\pi < \left\|(x_n - x)_{|\Gamma_n}\right\|_\pi + \varepsilon$. Then, for every $m > n_0$, we have

$$\begin{aligned}\|x_n - x\|_\pi &< \left\|(x_m - x)_{|\Gamma_n}\right\|_\pi + \left\|(x_n - x_m)_{|\Gamma_n}\right\|_\pi + \varepsilon \\ &< \left\|(x_m - x)_{|\Gamma_n}\right\|_\pi + 2\varepsilon.\end{aligned}$$

Passing to the limit as $m \to \infty$ we obtain $\|x_n - x\|_\pi \le 2\varepsilon$ $(n > n_0)$. This completes the proof. (The completeness of $S_\pi(\Gamma)$ follows from the fact that $S_\pi(\Gamma) = (\bigoplus X_\gamma)_\pi$ with $X_\gamma = \mathbf{R}$ for each γ.) □

Proof of Lemma 4.6(b). Let (V, ν) be a sequence space on Γ, containing all e_γ's, such that $V^* = S_\pi(\Gamma)$ in the sense of Definition 4.4(d). We shall show that $(\bigoplus X_\gamma^*)_\pi$ is isometric with the dual of $(\bigoplus X_\gamma)_V$.

First, observe that $\nu(\xi)=\nu(|\xi|)$ for each $\xi\in V$, since $\pi(\omega)=\pi(|\omega|)$ ($\omega\in S_\pi(\Gamma)$):

$$\begin{aligned}\nu(\xi) &= \sup\left\{\sum_\gamma \xi(\gamma)\omega(\gamma) : \omega\in S_\pi(\Gamma),\ \pi(\omega)=1\right\}\\ &= \sup\left\{\sum_\gamma |\xi(\gamma)\omega(\gamma)| : \omega\in S_\pi(\Gamma),\ \pi(\omega)=1\right\}\\ &= \sup\left\{\sum_\gamma |\xi(\gamma)|\omega(\gamma) : \omega\in S_\pi(\Gamma),\ \pi(\omega)=1\right\}\\ &= \nu(|\xi|).\end{aligned}$$

Second, the linear space V_0 of all elements of V with finite support is dense in V. Indeed, if a functional $f\in V^*$ is null on V_0 then it is representable by some $\omega\in S_\pi(\Gamma)$, but then $\omega(\gamma)=f(e_\gamma)=0$ for every γ, hence $f=0$.

Take $\Phi\in[(\bigoplus X_\gamma)_V]^*$. For each $\gamma\in\Gamma$ define $u^*(\gamma)\in X_\gamma^*$ by $\langle u^*(\gamma),x_\gamma\rangle=\Phi(x_\gamma e_\gamma)$ ($x_\gamma\in X_\gamma$). This definition is correct since

$$|\Phi(x_\gamma e_\gamma)|\le\|\Phi\|_{V^*}\,\|x_\gamma e_\gamma\|_V=\|\Phi\|_{V^*}\nu(\|x_\gamma\|e_\gamma)=\|\Phi\|_{V^*}\nu(e_\gamma)\|x_\gamma\|.$$

We claim that $u^*\in(\bigoplus X_\gamma^*)_\pi$. Let $0<\varepsilon<1$ and $\xi\in V$ be arbitrary. For each γ find $x(\gamma)\in X_\gamma$ such that $\|x(\gamma)\|=|\xi(\gamma)|$ and $\langle u^*(\gamma),x(\gamma)\rangle\ge(1-\varepsilon)\|u^*(\gamma)\|\,\|x(\gamma)\|$. We have $x=x(\cdot)\in(\bigoplus X_\gamma)_V$ since $\nu(\|x(\cdot)\|)=\nu(|\xi|)=\nu(\xi)$. Moreover, for every finite set $\Gamma_0\subset\Gamma$,

$$\begin{aligned}\sum_{\Gamma_0}\|u^*(\gamma)\|\xi(\gamma) &\le (1-\varepsilon)^{-1}\sum_{\Gamma_0}\Phi(x(\gamma)e_\gamma)\\ &= (1-\varepsilon)^{-1}\Phi\left(\sum_{\Gamma_0}x(\gamma)e_\gamma\right)\\ &\le (1-\varepsilon)^{-1}\|\Phi\|_{V^*}\,\|x_{|\Gamma_0}\|_V\\ &\le (1-\varepsilon)^{-1}\|\Phi\|_{V^*}\,\|x\|_V=(1-\varepsilon)^{-1}\|\Phi\|_{V^*}\,\nu(\xi).\end{aligned}$$

Thus $\sum_\Gamma\|u^*(\gamma)\|\xi(\gamma)\le\|\Phi\|_{V^*}\nu(\xi)$ for every $\xi\in V$. Hence $\|u^*\|_\pi\le\|\Phi\|_{V^*}$.

If $x=\sum x(\gamma)e_\gamma\in(\bigoplus X_\gamma)_{V_0}$ then $\Phi(x)=\sum\Phi(x(\gamma)e_\gamma)=\sum\langle u^*(\gamma),x(\gamma)\rangle$ and $|\Phi(x)|\le\sum\|u^*(\gamma)\|\,\|x(\gamma)\|\le\|u^*\|_\pi\|x\|_V$. The density of V_0 in V implies the density of $(\bigoplus X_\gamma)_{V_0}$ in $(\bigoplus X_\gamma)_V$, hence $\Phi(x)=\sum\langle u^*(\gamma),x(\gamma)\rangle$ for all $x\in V$, and $\|\Phi\|_{V^*}\le\|u^*\|_\pi$. This completes the proof. □

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L. Veselý, Dipartimento di Matematica, Università degli Studi, Via C. Saldini 50, 20133 Milano, Italy, *e-mail:* libor@vmimat.mat.unimi.it