FUZZY SOLUTIONS FOR BOUNDARY VALUE PROBLEMS WITH INTEGRAL BOUNDARY CONDITIONS

A. ARARA AND M. BENCHOHRA

ABSTRACT. The Banach fixed point theorem is used to investigate the existence and uniquenness of fuzzy solutions for a class of second order nonlinear boundary value problems with integral boundary conditions.

1. Introduction

In modelling real systems one can be frequently confronted with a differential equation

$$\begin{aligned} y'(t) &= f(t, y(t)), \qquad \text{for all } t \in [0, 1], \\ y(0) &= y_0 \end{aligned}$$

where the structure of the equation is known (represented by the vector field f) but the model parameters and the initial values y_0 are not known exactly. One method of treating this incertainty is to use a fuzzy set theory formulation of the problem. This note is concerned with the existence and uniqueness of fuzzy solutions for more general boundary value problems for second order differential equations with integral boundary conditions of the

Received January 28, 2005.
2000 *Mathematics Subject Classification.* Primary 34B15.
Key words and phrases. Fuzzy solution, integral boundary conditions, contraction, fixed point.

form

$$y''(t) = f(t, y(t)), \qquad \text{for all } t \in [0,1], \tag{1}$$

$$y(0) - k_1 y'(0) = \int_0^1 h_1(y(s))ds, \tag{2}$$

$$y(1) + k_2 y'(1) = \int_0^1 h_2(y(s))ds, \tag{3}$$

with $f : [0,1] \times E^n \to E^n$ a continuous function, where we let E^n be the set of all upper semi-continuous, convex, normal fuzzy numbers with α-level, $h_i : E^n \to E^n (i = 1,2)$ are continuous functions and k_i $(i = 1,2)$ are nonnegative constants. Fuzzy boundary value problems with integral boundary conditions constitute a very interesting and important class of problems. They include two, three, multipoint and nonlocal boundary value problems as special cases. The theory of fuzzy sets, fuzzy valued functions and necessary calculus of fuzzy functions have been investigated in the recent monograph by Lakshmikantham and Mohapatra [13] and the references cited therein, and in the papers [7, 8, 11, 15]. Recently, fuzzy differential equations have also been developed and the basic properties of solutions of fuzzy differential equations are available, see for instance, [6, 9, 10, 12, 16, 18, 21]. Balasubramaniam and Muralisankar [5] gave existence and uniqueness results for fuzzy integrodifferential equations of Volterra type. Seikkala [20] defined the concept of fuzzy derivative which is generalization of Hukuhara derivative [19]. Nieto [14] studied the Cauchy problem for first order fuzzy differential equations. In [1] Benchohra et al. studied the existence of fuzzy solutions for multipoint boundary value problems. Arara and Benchohra [2] established the existence of solutions for fuzzy neutral functional differential equations with nonlocal conditions. Park et al. [17] considered fuzzy differential equation with nonlocal condition. Balachandran and Prakash [4] proved the existence of global solutions for fuzzy integrodifferential equations. For recent results on fuzzy differential equations and inclusions, we refer to the monograph of Lakshmikantham and Mohapatra [13].

In this note we study the uniqueness of fuzzy solutions for boundary value problems with integral boundary conditions. Since the boundary conditions (2)–(3) are more general than those considered in the previous literature, the result of the present note can be considered as a contribution to the subject.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this note.

Definition 2.1. (fuzzy set). Let X be a nonempty set. A fuzzy set A in X is characterized by its membership function $A : X \to [0, 1]$ and $A(x)$ is interpreted as the degree of membership of element x in fuzzy set A for each $x \in X$.

The value zero is used to represent complete non-membership, the value one is used to represent complete membership, and values in between are used to represent intermediate degrees of membership. The mapping A is also called the membership function of fuzzy set A.

Example 2.2. The membership function of the fuzzy set of real numbers "close to one" can be defined as

$$A(t) = \exp(-\beta(t-1)^2),$$

where β is a positive real number.

Example 2.3. Let the membership function for the fuzzy set of real numbers "close to zero" defined as follows

$$B(t) = \frac{1}{1+x^3}.$$

Using this function, we can determine the membership grade of each real number in this fuzzy set, which signifies the degree to which that number is close to zero. For instance, the number 3 is assigned a grade of 0.035, the number 1 a grade of 0.5 and the number 0 a grade of 1.

$CC(\mathbb{R}^n)$ denotes the set of all nonempty compact, convex subsets of $\mathbb{R}^n$. Denote by

$$E^n = \{y : \mathbb{R}^n \to [0,1] \text{ such that they satisfy (i) to (iv) mentioned below}\},$$

(i) y is normal, that is there exists an $x_0 \in \mathbb{R}^n$ such that $y(x_0) = 1$;
(ii) y is fuzzy convex, that is for $x, z \in \mathbb{R}^n$ and $0 < \lambda \leq 1$,

$$y(\lambda x + (1-\lambda)z) \geq \min[y(x), y(z)];$$

(iii) y is upper semi-continuous;
(iv) $[y]^0 = \overline{\{x \in \mathbb{R}^n : y(x) > 0\}}$ is compact.

For $0 < \alpha \leq 1$, we denote $[y]^\alpha = \{x \in \mathbb{R}^n : y(x) \geq \alpha\}$. Then from (i) to (iv), it follows that the α-level sets $[y]^\alpha \in CC(\mathbb{R}^n)$. If $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a function, then, according to Zadeh's extension principle we can extend g to $E^n \times E^n \to E^n$ by the function defined by

$$g(y, \overline{y})(z) = \sup_{z = g(x, \bar{z})} \min\{y(x), \overline{y}(\bar{z})\}.$$

It is well known that

$$[g(y, \overline{y})]^\alpha = g([y]^\alpha, [\overline{y}]^\alpha)$$

for all $y,\ \overline{y} \in E^n$, $0 \leq \alpha \leq 1$ and continuous function g. Especially for addition and scalar multiplication, we have

$$[y + \overline{y}]^\alpha = [y]^\alpha + [\overline{y}]^\alpha, \qquad [ky]^\alpha = k[y]^\alpha,$$

where $y, \overline{y} \in E^n,\ k \in \mathbb{R},\ \ 0 \leq \alpha \leq 1$.

Let A and B be two nonempty bounded subsets of $\mathbb{R}^n$. The distance between A and B is defined by the Hausdorff metric

$$H_d(A, B) = \max\left\{\sup_{a \in A} \inf_{b \in B} \|a - b\|, \sup_{b \in B} \inf_{a \in A} \|a - b\|\right\}$$

where $\|\cdot\|$ denotes the usual Euclidean norm in $\mathbb{R}^n$. Then $(CC(\mathbb{R}^n), H_d)$ is a complete and separable metric space [19]. We define the supremum metric d_∞ on E^n by

$$d_\infty(u, \overline{u}) = \sup_{0<\alpha\leq 1} H_d([u]^\alpha, [\overline{u}]^\alpha)$$

for all $u, \overline{u} \in E^n$. (E^n, d_∞) is a complete metric space. The supremum metric H_1 on $C([0,1], E^n)$ is defined by

$$H_1(w, \overline{w}) = \sup_{t\in J} d_\infty(w(t), \overline{w}(t)).$$

$(C([0,1], E^n), H_1)$ is a complete metric space.

Definition 2.4. [19] A map $f : [0,1] \to E^n$ is strongly measurable if, for all $\alpha \in [0,1]$, the multi-valued map $f_\alpha : [0,1] \to CC(\mathbb{R}^n)$ defined by

$$f_\alpha(t) = [f(t)]^\alpha$$

is Lebesgue measurable, when $CC(\mathbb{R}^n)$ is endowed with the topology generated by the Hausdorff metric d.

Definition 2.5. [19] A map $f : [0,1] \to E^n$ is called levelwise continuous at $t_0 \in [0,1]$ if the multi-valued map $f_\alpha(t) = [f(t)]^\alpha$ is continuous at $t = t_0$ with respect to the Hausdorff metric d for all $\alpha \in [0,1]$.

A map $f : [0,1] \to E^n$ is called integrably bounded if there exists an integrable function h such that $\|y\| \leq h(t)$ for all $y \in f_0(t)$.

Definition 2.6. [3] Let $f : [0,1] \to E^n$. The integral of f over $[0,1]$, denoted $\int_0^1 f(t)dt$ is defined by the equation

$$\begin{aligned}\left(\int_0^1 f(t)dt\right)^\alpha &= \int_0^1 f_\alpha(t)dt \\ &= \left\{\int_0^1 v(t)dt \mid v : [0,1] \to R^n \text{ is a measurable selection for } f_\alpha\right\}\end{aligned}$$

for all $\alpha \in (0, 1]$.

A strongly measurable and integrably bounded map $f : [0,1] \to E^n$ is said to be *integrable* over $[0,1]$, if $\int_0^1 f(t)dt \in E^n$.

If $f : [0,1] \to E^n$ is strongly measurable and integrably bounded, then f is integrable.

Definition 2.7. A map $f : [0,1] \to E^n$ is called differentiable at $t_0 \in [0,1]$ if there exists a $f'(t_0) \in E^n$ such that the limits

$$\lim_{h \to 0+} \frac{f(t_0 + h) - f(t_0)}{h} \quad \text{and} \quad \lim_{h \to 0+} \frac{f(t_0) - f(t_0 - h)}{h}$$

exist and are equal to $f'(t_0)$. Here the limit is taken in the metric space (E^n, H_d). At the end points of $[0,1]$, we consider only the one-side derivatives.

If $f : [0,1] \to E^n$ is differentiable at $t_0 \in [0,1]$, then we say that $f'(t_0)$ is the *fuzzy derivative* of $f(t)$ at the point t_0 or the Hukuhara derivative of $f(t)$ at t_0, usually noted by $D_H f(t_0)$. For the concepts of fuzzy measurability and fuzzy continuity we refer to [11].

Definition 2.8. A map $f : [0,1] \times E^n \to E^n$ is called levelwise continuous at point $(t_0, x_0) \in [0,1] \times E^n$ provided, for any fixed $\alpha \in [0,1]$ and arbitrary $\epsilon > 0$, there exists a $\delta(\epsilon, \alpha) > 0$ such that

$$H_d\left([f(t,x)]^\alpha, [f(t,x_0)]^\alpha\right) < \epsilon$$

whenever $|t - t_0| < \delta(\epsilon, \alpha)$ and $H_d\left([x]^\alpha, [x_0]^\alpha\right) < \delta(\epsilon, \alpha)$ for all $t \in [0,1], x \in E^n$.

3. The Main Result

In this section, we are concerned with the existence and uniqueness of solutions for the problem (1)–(3).

Definition 3.1. A function $y \in C^2([0,1], E^n)$ is said to be a solution of (1)–(3) if y satisfies the equation $y''(t) = f(t, y(t))$ on $[0,1]$ and the conditions (2)–(3).

We need the following auxiliary result. Its proof uses a standard argument.

Lemma 3.2. *For any $\sigma(t)$, $\rho_1(t)$, $\rho_2(t) \in C([0,1], E^n)$, the nonhomogeneous linear problem*

$$x''(t) = \sigma(t), \qquad \text{for all } t \in [0,1],$$
$$x(0) - k_1 x'(0) = \int_0^1 \rho_1(s)ds, \qquad x(1) + k_2 x'(1) = \int_0^1 \rho_2(s)ds,$$

has a unique solution $x \in C^2((0,1), E^n)$ given by

$$x(t) = P(t) + \int_0^1 G(t,s)\sigma(s)ds,$$

where

$$P(t) = \frac{1}{1+k_1+k_2}\{(1-t+k_2)\int_0^1 \rho_1(s)ds + (k_1+t)\int_0^1 \rho_2(s)ds\}$$

is the unique solution of the problem

$$x''(t) = 0, \qquad \text{for all } t \in [0,1],$$
$$x(0) - k_1 x'(0) = \int_0^1 \rho_1(s)ds, \qquad x(1) + k_2 x'(1) = \int_0^1 \rho_2(s)ds,$$

and

$$G(t,s) = \frac{-1}{k_1+k_2+1}\begin{cases} (k_1+t)(1-s+k_2), & 0 \le t < s \le 1, \\ (k_1+s)(1-t+k_2), & 0 \le s < t \le 1 \end{cases}$$

is the Green's function of the homogeneous problem.

Let us introduce the following hypotheses which are assumed hereafter:

Theorem 3.3. *Assume that*

(H1) *There exists a constant* d *such that*

$$H_d([f(t,u)]^\alpha,[f(t,\overline{u})]^\alpha)\leq dH_d([u(t)]^\alpha,[\overline{u}(t)]^\alpha),$$

for all $t\in[0,1]$ *and all* $u,\ \overline{u}\in E^n$.

(H2) *There exist constants* $d_i, i=1,2$ *such that*

$$H_d([h_i(y(t))]^\alpha,[h_i(\overline{y}(t))]^\alpha)\leq d_iH_d([y(t)]^\alpha,[\overline{y}(t)]^\alpha).$$

If

$$\frac{1+k_2}{1+k_1+k_2}d_1+d_2(1+k_1)+d\sup_{(t,s)\in[0,1]\times[0,1]}|G(t,s)|<1,$$

then the BVP (1)–(3) *has a unique fuzzy solution on* $[0,1]$.

Proof. Transform the problem into a fixed point problem. It is clear that the solutions of the problem (1)–(3) are fixed points of the operator $N:C([0,1],E^n)\to$
$C([0,1],E^n)$ defined by:

$$N(y)(t)=P(y)(t)+\int_0^1 G(t,s)f(s,y(s))ds$$

with

$$P(y)(t)=\frac{1}{1+k_1+k_2}(1-t+k_2)\int_0^1 h_1(y(s))ds+(k_1+t)\int_0^1 h_2(y(s))ds.$$

We shall show that N is a contraction operator. Indeed, consider $y,\overline{y}\in C([0,1],E^n)$ and $\alpha\in(0,1]$, then

$$
\begin{aligned}
&H_d([N(y)(t)]^\alpha, [N(\overline{y})(t)]^\alpha) \\
&\quad = H_d\left(\left[\frac{1-t+k_2}{1+k_1+k_2}\int_0^1 h_1(y(s))ds + \ (k_1+t)\int_0^1 h_2(y(s))ds + \int_0^1 G(t,s)f(s,y(s))ds\right]^\alpha,\right. \\
&\qquad \left.\left[\frac{1-t+k_2}{1+k_1+k_2}\int_0^1 h_1(\overline{y}(s))ds + (k_1+t)\int_0^1 h_2(\overline{y}(s))ds + \int_0^1 G(t,s)f(s,\overline{y}(s))ds\right]^\alpha\right) \\
&\quad \le H_d\left(\left[\frac{1-t+k_2}{1+k_1+k_2}\int_0^1 h_1(y(s))ds\right]^\alpha, \left[\frac{1-t+k_2}{1+k_1+k_2}\int_0^1 h_1(\overline{y}(s))ds\right]^\alpha\right) \\
&\qquad + H_d\left(\left[(k_1+t)\int_0^1 h_2(y(s))ds\right]^\alpha, \left[(k_1+t)\int_0^1 h_2(\overline{y}(s))ds\right]^\alpha\right) \\
&\qquad + H_d\left(\left[\int_0^1 G(t,s)f(s,y(s))ds\right]^\alpha, \left[\int_0^1 G(t,s)f(s,\overline{y}(s))ds\right]^\alpha\right) \\
&\quad \le \ \frac{1-t+k_2}{1+k_1+k_2}\ H_d\left(\left[\int_0^1 h_1(y(s))ds\right]^\alpha, \left[\int_0^1 h_1(\overline{y}(s))ds\right]^\alpha\right) \\
&\qquad + (k_1+t)H_d\left(\left[\int_0^1 h_2(y(s))ds\right]^\alpha, \left[\int_0^1 h_2(\overline{y}(s))ds\right]^\alpha\right) \\
&\qquad + \sup_{(t,s)\in[0,1]\times[0,1]} |G(t,s)| H_d\left(\left[\int_0^1 f(s,y(s))ds\right]^\alpha, \left[\int_0^1 f(s,\overline{y}(s))ds\right]^\alpha\right)
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{1-t+k_2}{1+k_1+k_2}\int_0^1 H_d\left([h_1(y(s))]^\alpha, [h_1(\overline{y}(s))]^\alpha\right) ds \\
&\quad + (k_1+t)\int_0^1 H_d\left([h_2(y(s))]^\alpha, [h_2(\overline{y}(s))]^\alpha\right) ds \\
&\quad + \sup_{(t,s)\in[0,1]\times[0,1]} |G(t,s)| \int_0^1 H_d\left([f(s,y(s))]^\alpha, [f(s,\overline{y}(s))]^\alpha\right) ds \\
&\leq \frac{1+k_2}{1+k_1+k_2} d_1 \sup_{\alpha\in[0,1]} d_\infty\left(y(t),\overline{y}(t)\right) + (k_1+1)d_2 d_\infty\left(y(t),\overline{y}(t)\right) \\
&\quad + d \sup_{(t,s)\in[0,1]\times[0,1]} |G(t,s)| d_\infty\left(y(t),\overline{y}(t)\right) \\
&\leq \left(\frac{1+k_2}{1+k_1+k_2} d_1 + d_2(k_1+1) + d \sup_{(t,s)\in[0,1]\times[0,1]} |G(t,s)|\right) d_\infty(y(t),\overline{y}(t)) \\
&\leq \left(\frac{1+k_2}{1+k_1+k_2} d_1 + d_2(k_1+1) + d \sup_{(t,s)\in[0,1]\times[0,1]} |G(t,s)|\right) H_1(y,\overline{y}).
\end{aligned}
$$

Hence

$$
\begin{aligned}
&H_1(N(y),N(\overline{y})) \\
&\qquad \leq \left(\frac{1+k_2}{1+k_1+k_2} d_1 + d_2(k_1+1) + d \sup_{(t,s)\in[0,1]\times[0,1]} |G(t,s)|\right) H_1(y,\overline{y}).
\end{aligned}
$$

So, N is a contraction and thus, by Banach fixed point theorem, N has a unique fixed point which is solution to (1)–(3). □

4. An Example

In this section we present a simple example to show the advantage gained by the fuzzification of the differential operator in the differential equation.

Consider the crisp initial value problem with unknown initial value y_0, that is,

$$y' = -y, \qquad y(0) = y_0 \in [-1, 1]. \tag{4}$$

The solution of problem (4) when restricted to the interval $[-1, 1]$ is

$$y(t) = [-e^{-t}, e^{-t}], \quad t \geq 0.$$

The fuzzy differential equation corresponding to (4) in E^1 is

$$D_H y = -y \qquad y(0) = y_0 = [-1, 1],\ y_0 \in E^1. \tag{5}$$

Suppose that $[y]^\alpha = [y_1^\alpha, y_2^\alpha]$, $[D_H y]^\alpha = \left[\frac{dy_1^\alpha}{dt}, \frac{dy_2^\alpha}{dt}\right]$ are α-level sets for $0 \leq \alpha \leq 1$. By extension principle, (5) becomes

$$\frac{dy_1^\alpha}{dt} = -y_2^\alpha,\ \frac{dy_2^\alpha}{dt} = -y_1^\alpha, \qquad 0 \leq \alpha \leq 1. \tag{6}$$

The solution of (6) is given by

$$y_1^\alpha(t) = -e^t,\ y_2^\alpha(t) = e^t, \qquad t \geq 0$$

and therefore the fuzzy function $y(t)$ solving (5) is

$$y(t) = [-e^t, e^t], \quad t \geq 0,$$

which shows that the diameter $\operatorname{diam}(y(t)) \to \infty$ as $t \to \infty$. This may be interpreted as the increasing of incertainty to go by the time, which is, in fact, reasonable.

1. Agarwal R. P., Benchohra M., O'Regan D. and Ouahab A., *Fuzzy solutions for multipoint boundary value problems*, Mem. Differential Equations Math. Phys. **35** (2005), 1–14.

2. Arara A. and Benchohra M., *Fuzzy solutions for neutral functional differential equations with nonlocal conditions*, Georgian Math. J. **11** (2004), 35–42.

3. Aumann R. J., *Integrals of set valued functions*, J. Math. Anal. Appl. **12** (1965), 1–12.

4. Balachandran K. and Prakash P., *Existence of solutions of nonlinear fuzzy integral equations in Banach spaces*, Libertas Mathematica, **XXI** (2001), 91–97.

5. Balasubramaniam P. and Muralisankar S., *Existence and uniqueness of fuzzy solution for the nonlinear fuzzy Volterra integrodifferential equations*, Differential Equations Dynam. Systems **11** (2003), 369–383.

6. Buckley J. J. and Feuring Th., *Fuzzy differential equations*, Fuzzy Sets and Systems **110**(1) (2000), 43–54.

7. Dubois D. and Prade H., *Towards fuzzy differential calculus. I. Integration of fuzzy mappings*, Fuzzy Sets and Systems **8**(1) (1982) 1–17.

8. Joo S. Y. and Kim Y. K., *Topological properties on space of fuzzy sets*, J. Math. Anal. Appl. **246**2 (2000) 576–590.

9. Kaleva O., *The Peano theorem for fuzzy differential equations revisited*, Fuzzy Sets and Systems **98**(1) (1998), 147–148.

10. Kandel A. and Byartt W. J., *Fuzzy differential equations*, Proceeding of the International Conference on Cybernetics and Society, Vol. I, II (Tokio/Kyoto 1978), 1213–1216, *IEEE, New York,* 1978.

11. Kim Y. K., *Measurability for fuzzy valued functions*, Fuzzy Sets and Systems **129**(1) (2002), 105–109.

12. Kloeden P. E., Remarks on Peano-like theorems for fuzzy differential equations. *Fuzzy Sets and Systems* **44**(1) (1991), No. 1, 161-163.

13. Lakshmikantham V. and Mohapatra R., *Theory of Fuzzy Differential Equations and Inclusions*, Taylor and Francis Publishers, London, 2003.

14. Nieto J., *The Cauchy problem for continuous fuzzy differential equations*, Fuzzy Sets and Systems **102**(2) (1999), 259–262.

15. Novak V., *Fuzzy sets and their applications*. Translated from the Czech. Adam Hilger, Ltd, Bristol, 1989.

16. Park J. Y. and Han H. K., *Existence and uniqueness theorem for a solution of fuzzy differential equations*. Int. J. Math. Math. Sci. **22**(2) (1999), 271–279.

17. Park J. Y. and Han H. K. and Son K. H., *Fuzzy differential equation with nonlocal condition*, Fuzzy Sets and Systems **115** (2000), 365–369.

18. Pearson D. W., *A property of linear fuzzy differential equations*. Appl. Math. Lett. **10**(3) (1997), 99–103.
19. Puri M. and Ralescu D., *Fuzzy random variables*, J. Math. Anal. Appl. **114**(2) (1986), 409–22.
20. Seikkala S., *On the fuzzy initial value problem*, Fuzzy Sets and Systems **24**(3) (1987), 319–330.
21. Yan H. and Xu J., *A class of convex fuzzy mapping*, Fuzzy Sets and Systems **129**(1) (2002), 47–56.

A. Arara, Laboratoire de Mathématiques, Université de Sidi Bel Abbès, BP 89, 22000 Sidi Bel Abbès, Algérie,
e-mail: amaria_ar@yahoo.fr

M. Benchohra, Laboratoire de Mathématiques, Université de Sidi Bel Abbès, BP 89, 22000 Sidi Bel Abbès, Algérie,
e-mail: benchohra@univ-sba.dz