Adv. Oper. Theory 1 (2016), no. 1, 8–14
http://doi.org/10.22034/aot.1610.1021
ISSN: 2538-225X (electronic)
http://aot-math.org

OPERATORS REVERSING ORTHOGONALITY IN NORMED SPACES

JACEK CHMIELIŃSKI

Communicated by P. Aiena

Abstract. We consider linear operators $T\colon X \to X$ on a normed space X which reverse orthogonality, i.e., satisfy the condition

$$x \perp y \quad \Longrightarrow \quad Ty \perp Tx, \qquad x, y \in X,$$

where $\perp$ stands for Birkhoff orthogonality.

1. Introduction

In the present paper we deal with linear operators, defined on a normed space, and their properties connected with the notion of orthogonality. The problem we propose to consider is related to the orthogonality preserving property.

1.1. **Birkhoff orthogonality.** Let $(X, \|\cdot\|)$ be a normed space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Unless the norm comes from an inner product there is no unique notion of orthogonality. However, one of the most significant is that of Birkhoff (or Birkhoff–James), introduced by Birkhoff [2] and developed by James [7, 8, 9]. (Actually, this relation was much earlier considered by Carathéodory — a good source for this topic is the survey [1].) The definition of the Birkhoff orthogonality reads as follows.

Date: Received: Oct. 2, 2016; Accepted: Oct. 24, 2016.

2010 *Mathematics Subject Classification.* Primary 15A86; Secondary 15A04, 46B04, 46B20, 46C15, 47A62, 52A21.

Key words and phrases. Birkhoff orthogonality, orthogonality preserving mappings, orthogonality reversing mappings, linear similarities, characterizations of inner product spaces.

Definition 1.1. Let $x, y \in X$. We say that x is orthogonal to y (we will use the symbol $x \perp y$) if and only if

$$\forall \lambda \in \mathbb{K}: \ \|x + \lambda y\| \geq \|x\|.$$

The geometrical interpretation is that the line passing through x in the direction of y supports (at the point x) the ball centred at 0 and with radius $\|x\|$. The above relation is generally not symmetric. If $\dim X \geq 3$, then the symmetry of $\perp$ yields that the norm comes from an inner product ([6, 9], cf. also [1]). Conversely, if X is an inner product space, $\perp$ coincides with the standard orthogonality relation ($x \perp y \ \Leftrightarrow \ \langle x|y \rangle = 0$) which obviously is symmetric. If $\dim X = 2$ the symmetry of $\perp$ is possible for norms which need not come from an inner product. These are the so-called Radon norms [12] (cf. also [1]) (we deal with this case in Section 3).

1.2. **Preserving orthogonality.** A natural linear preserver problem connected with the orthogonality relation is to characterize all linear mappings on X which preserve orthogonality.

Definition 1.2. We say that a given (nonzero) linear mapping $T\colon X \to X$ has the *orthogonality preserving property* if and only if

$$x \perp y \quad \Longrightarrow \quad Tx \perp Ty, \qquad x, y \in X. \tag{1.1}$$

It turns out that any nonzero linear mapping preserving orthogonality has to be a linear similarity, that is, with some positive constant γ

$$\|Tx\| = \gamma \|x\|, \qquad x \in X. \tag{1.2}$$

The implication (1.1) $\Rightarrow$ (1.2) is easy to verify in an inner product space (cf. e.g. [5]) but much more difficult for normed spaces in general. It was proved first by Koldobsky [10] only for real spaces and then by Blanco and Turnšek [3] for the general case and for mappings between two, possibly different, normed spaces. It is easy to see that (1.2) implies not only (1.1), but also the orthogonality preserving property *in both directions*:

$$x \perp y \quad \Longleftrightarrow \quad Tx \perp Ty, \qquad x, y \in X. \tag{1.3}$$

Thus we can conclude

Theorem 1.3. *For an arbitrary normed space X and a nonzero linear operator $T\colon X \to X$ the conditions* (1.1), (1.2) *(with some $\gamma > 0$) and* (1.3) *are equivalent. Moreover, an analogous result is true for mappings $T\colon X \to Y$ between two normed spaces X, Y.*

In particular, any nonzero linear operator preserving orthogonality is injective.

2. Reversing orthogonality

Since the orthogonality relation is generally not symmetric it makes sense to consider the problem of *reversing* orthogonality.

Definition 2.1. We say that a nonzero linear operator $T\colon X \to X$ *reverses orthogonality* if and only if

$$x \perp y \quad \Longrightarrow \quad Ty \perp Tx, \qquad x, y \in X. \tag{2.1}$$

The above property will be the main subject of our considerations. Let us notice that if T satisfies (2.1) then its iteration T^2 satisfies (1.1) (whence also (1.2) and (1.3)). In particular, T^2 is injective, hence T is injective as well. Besides, if $Ty \perp Tx$ then, by (2.1), $T^2x \perp T^2y$ which is equivalent to $x \perp y$. Thus (2.1) is, in fact, equivalent to:

$$x \perp y \quad \Longleftrightarrow \quad Ty \perp Tx, \qquad x, y \in X. \tag{2.2}$$

Remark 2.2. Notice that in the above considerations the linearity of mappings can by replaced by its conjugate linearity. Moreover, one can consider a much wider class of all operators which are phase equivalent to a linear or a conjugate-linear one. (By conjugate-linearity we mean $T(ax+by) = \overline{a}Tx + \overline{b}Ty$ ($x, y \in X$, $a, b \in \mathbb{K}$), and $T, S\colon X \to X$ are phase-equivalent if $Tx = \sigma(x)Sx$, $x \in X$ with some $\sigma\colon X \to \mathbb{K}$ such that $|\sigma(x)| = 1$, $x \in X$.)

Obviously, if X is an inner product space, then (2.1) is equivalent to (1.1) and (1.2). Thus, in this case, the class of solutions of (2.1) consists of all linear similarities which means that the reversing and preserving orthogonality properties are equivalent. However, some spaces admit operators which *essentially* reverse orthogonality, which means that they satisfy (2.1) but not (1.1).

Example 2.3. Let $X = \mathbb{R}^2$ with the *maximum* norm (i.e., $\|x\|_\infty = \|(x_1, x_2)\|_\infty := \max\{|x_1|, |x_2|\}$ for $x = (x_1, x_2) \in \mathbb{R}^2$). The mapping $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T(x) = T(x_1, x_2) := (x_1 - x_2, x_1 + x_2), \qquad x = (x_1, x_2) \in \mathbb{R}^2$$

is a nontrivial linear mapping essentially reversing orthogonality.

Indeed, for $x = (1, 1)$ and $y = (1, 0)$ we have $Tx = (0, 2)$, $Ty = (1, 1)$ whence

$$x \perp y, \quad Ty \perp Tx, \quad Tx \not\perp Ty.$$

Thus T does not satisfy (1.1). Now one could check directly the property (2.1), but we will wait until the next section where a short proof of it will be given (Remark 3.4).

On the other hand, as we will see in subsequent sections of the paper, there are spaces which do not admit nontrivial linear mappings reversing orthogonality.

Remark 2.4. One can consider a property which is stronger than (2.1). Namely, assume that a nonzero linear operator $T\colon X \to X$ satisfies

$$x \perp y \quad \Longrightarrow \quad Tx \perp Ty \text{ and } Ty \perp Tx, \qquad x, y \in X. \tag{2.3}$$

Notice that for bijective operators the condition (2.3) is equivalent to

$$(x \not\perp y \text{ or } y \not\perp x) \ \Rightarrow\ (Tx \not\perp Ty \text{ and } Ty \not\perp Tx).$$

Let $x, y \in X$ and assume $x \perp y$. Since (2.3) implies (1.1), T must be a similarity, and then $Ty \perp Tx$ implies $y \perp x$. This means that the orthogonality relation $\perp$ is symmetric. Therefore mappings satisfying property (2.3) may exists only on

Radon planes (two dimensional spaces with the Radon norm — see the next section) and, if the dimension is greater than 2, only in inner product spaces.

3. Minkowski plane

The case of two-dimensional normed spaces (Minkowski planes) is particular for our considerations. Only in this case there exist norms (namely, the so-called Radon norms [12]) which do not necessarily come from an inner product, but whose Birkhoff orthogonality relation is symmetric. A simple example is the l_1-l_∞-norm or the l_p-l_q-norm (with conjugated parameters $p, q > 1$) which is even strictly convex. For such spaces the condition (2.1) is equivalent to (1.1) and the class of nonzero linear operators reversing orthogonality coincides with that of all linear similarities. We will show that the reverse is also true (see Theorem 3.6).

3.1. **Smoothness and strict convexity.** It is known (cf. [8, 1]) that Birkhoff orthogonality is additive on the right (which means that if $x \perp y$ and $x \perp z$, then $x \perp y + z$) if and only if X is smooth. It is also known (cf. [9] or [1, Theorem 4.18 (i)]) that for two-dimensional normed spaces analogous additivity on the left is equivalent to the strict convexity of X.

Theorem 3.1. *Let X be a Minkowski plane which is either:* (i) *smooth and not strictly convex or:* (ii) *strictly convex but not smooth. Then there are no nontrivial linear operators reversing orthogonality.*

Proof. Assume (i). Let $x, y, z \in X$ be chosen such that $x \perp z$, $y \perp z$ and $x + y \not\perp z$ (this is possible since X is not strictly convex). If there existed a nonzero, linear mapping $T\colon X \to X$ satisfying (2.1), we would have $Tz \perp Tx$ and $Tz \perp Ty$. Since X is smooth, that would give $Tz \perp T(x + y)$, whence $T^2(x + y) \perp T^2 z$. But T^2 satisfies (1.3), so we would get $x + y \perp z$ — a contradiction. The case (ii) can be proved analogously (see also Remark 3.5). □

Corollary 3.2. *Let X be a Minkowski plane admitting a nonzero linear operator reversing orthogonality. Then X is strictly convex and smooth or neither strictly convex nor smooth.*

It is an open problem whether the above necessary conditions are also sufficient for the existence of an operator reversing orthogonality.

3.2. **Antinorm.** The *antinorm* $\|\cdot\|_a$ corresponding to a given norm $\|\cdot\|$ in a two-dimensional space X is defined as

$$\|x\|_a := \|\Phi x\|^* = \sup\{[y, x] : \ \|y\| = 1\}$$

where $\Phi\colon X \to X^*$ is an isomorphism corresponding to a non-degenerate bilinear symplectic form $[\cdot, \cdot]$; namely $\Phi x(y) := [y, x]$. The form is *non-degenerate* if and only if $[x, u] = 0$ for all $x \in X$ implies $u = 0$, and it is *symplectic* iff $[x, x] = 0$ for all $x \in X$ (or, equivalently, $[y, x] = -[x, y]$ for all $x, y \in X$). On a two-dimensional vector space such a non-degenerate bilinear symplectic form is practically unique (up to a nonzero scalar multiple). For the details we refer to [11]. Let us denote by $\perp^a$ the Birkhoff orthogonality relation corresponding to the antinorm $\|\cdot\|_a$.

It is due to Busemann [4] (see also [11, Theorem 1]) that the antinorm reverses Birkhoff orthogonality, i.e.,

$$x \perp y \iff y \perp^a x, \qquad x, y \in X \tag{3.1}$$

(and it is a unique norm with this property).

Theorem 3.3. *Let X be a Minkowski plane and let $T\colon X \to X$ be a nontrivial linear operator. Then T reverses orthogonality (satisfies (2.1)) if and only if, with some positive constant γ, one of the following, equivalent, conditions holds true:*

$$\|Tx\|_a = \gamma\|x\|, \qquad x \in X; \tag{3.2}$$

$$\|Tx\| = \gamma\|x\|_a, \qquad x \in X; \tag{3.3}$$

$$\|Tu\|_a = \gamma, \qquad u \in S; \tag{3.4}$$

$$\|Tv\| = \gamma, \qquad v \in S_a, \tag{3.5}$$

where S and S_a denote unit spheres with respect to the norm $\|\cdot\|$ and the antinorm $\|\cdot\|_a$, respectively.

Proof. Since the antinorm reverses Birkhoff orthogonality (cf. (3.1)) we have that (2.1) is equivalent to

$$x \perp y \implies Tx \perp^a Ty, \qquad x, y \in X$$

and to

$$x \perp^a y \implies Tx \perp Ty, \qquad x, y \in X.$$

Thus T, as a linear operator from $(X, \|\cdot\|)$ into $(X, \|\cdot\|_a)$ or from $(X, \|\cdot\|_a)$ into $(X, \|\cdot\|)$, respectively, is an orthogonality preserving operator. By the result of Blanco and Turnšek (cf. Theorem 1.3) we get (3.2) and (3.3), respectively. And obviously (3.4) is equivalent to (3.2) and (3.5) is equivalent to (3.3). $\square$

Remark 3.4. Now, we can give a short explanation why the mapping T from Example 2.3 satisfies (2.1). The l_1 norm ($\|x\|_1 = \|(x_1, x_2)\|_1 = |x_1| + |x_2|$) is the antinorm for the maximum (l_∞) norm in $\mathbb{R}^2$. We have

$$\|Tx\|_1 = |x_1 - x_2| + |x_1 + x_2| = 2\max\{|x_1|, |x_2|\} = 2\|x\|_\infty.$$

Thus the operator T satisfies (3.2), which is equivalent to (2.1).

Remark 3.5. Using the notion of antinorm we can give another explanation of Theorem 3.1. As we know, (2.1) is equivalent to (3.2). The antinorm is defined as a norm in the dual space X^*. Since X is not strictly convex, X^* is not smooth. But then, due to (3.2), X could not be smooth — a contradiction with assumptions upon X. Analogously, if X is not smooth, then X^* is not strictly convex and thus X is not strictly convex as well. Hence in a normed space X which is smooth but not strictly convex or strictly convex but not smooth the condition (3.2), whence also (2.1), cannot be satisfied.

Suppose now, that the only nontrivial linear operators satisfying (2.1) are linear similarities. Thus, for some $\gamma_1 > 0$ we have

$$\|Tx\| = \gamma_1 \|x\|, \qquad x \in X.$$

On the other hand, from Theorem 3.3, we have for some $\gamma_2 > 0$

$$\|Tx\| = \gamma_2 \|x\|_a, \qquad x \in X.$$

Thus for $\gamma := \gamma_1/\gamma_2$ we have

$$\|x\|_a = \gamma \|x\|, \qquad x \in X,$$

which means (cf. [4] or [11, Corollary 1]) that $\|\cdot\|$ is a Radon norm. Thus we arrive at the following result.

Theorem 3.6. *Let X be a Minkowski plane. The class of nontrivial linear operators reversing orthogonality coincides with the class of all linear similarities if and only if X is a Radon plane.*

4. Normed spaces with the dimension greater than two

As it was said, if the dimension of X is greater than two, then Birkhoff orthogonality is symmetric if and only if the norm comes from an inner product. If X is an inner product space, then (2.1) is equivalent to (1.1). Conversely, if (2.1)$\Leftrightarrow$(1.1), then we have also (2.1)$\Leftrightarrow$(1.3), and therefore for any linear similarity $T\colon X \to X$ and for arbitrary $x, y \in X$ we have

$$x \perp y \quad\Longrightarrow\quad Ty \perp Tx \quad\Longrightarrow\quad y \perp x,$$

i.e., $\perp$ is symmetric, hence X is an inner product space.

Theorem 4.1. *Let X be a smooth normed space with $\dim X \geq 3$. Then there exists a nonzero, linear operator $T\colon X \to X$ satisfying (2.1) (reversing orthogonality) if and only if X is an inner product space.*

Proof. Obviously, if X is an inner product space, then (2.1) is equivalent to (1.1) and (1.2). Hence the class of solutions of (2.1) is nonempty and consists of all linear similarities.

For the proof of the reverse, assume that there exists a nonzero, linear operator $T\colon X \to X$ satisfying (2.1). Let $x, y, z \in X$ be such that $x \perp z$ and $y \perp z$. Thus we have $Tz \perp T(x+y)$ (since X is smooth) and then $T^2(x+y) \perp T^2 z$ which, in turn, give us $x + y \perp z$. So the orthogonality is additive on the left which, since $\dim X \geq 3$, is possible only if X is an inner product space ([9], cf. also [1, Theorem 4.18 (ii)]). □

Thus any smooth normed space (of dimension greater than 2) which is not an inner product space does not admit a nontrivial linear mapping reversing orthogonality. The question is whether the assumption of smoothness is necessary.

References

1. J. Alonso, H. Martini, and S. Wu, *On Birkhoff orthogonality and isosceles orthogonality in normed linear spaces*, Aequationes Math. **83** (2012), no. 1-2, 153–189.
2. G. Birkhoff, *Orthogonality in linear metric spaces*, Duke Math. J. **1** (1935), 169–172.
3. A. Blanco, A. Turnšek, *On maps that preserve orthogonality in normed spaces*, Proc. Roy. Soc. Edinburgh Sect. A **136** (2016), 709–716.
4. H. Busemann, *The isoperimetric problem in the Minkowski plane*, Amer. J. Math. **69** (1947), 863–871.
5. J. Chmieliński, *Linear mappings approximately preserving orthogonality*, J. Math. Anal. Appl. **304** (2005), 158–169.
6. M. M. Day, *Some characterizations of inner-product spaces*, Trans. Amer. Math. Soc. 62 (1947), 320–337.
7. R. C. James, *Orthogonality in normed linear linear spaces*, Duke Math. J. **12** (1945), 291–301.
8. R. C. James, *Orthogonality and linear functionals in normed linear spaces*, Trans. Amer. Math. Soc. **61** (1947), 265–292.
9. R. C. James, *Inner products in normed linear spaces*, Bull. Amer. Math. Soc. **53** (1947), 559–566.
10. A. Koldobsky, *Operators preserving orthogonality are isometries*, Proc. Roy. Soc. Edinburgh Sect. A **123** (1993), 835–837.
11. H. Martini and K.J. Swanepoel, *Antinorms and Radon curves*, Aequationes Math. **72** (2006), no. 1-2, 110–138.
12. J. Radon, *Über eine besondere Art ebener konvexer Kurven*, Leipz. Ber. **68** (1916), 123–128.

Department of Mathematics, Pedagogical University of Cracow, Podchorążych 2, 30-084 Kraków, Poland

E-mail address: jacek@up.krakow.pl