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NORM INEQUALITIES FOR ELEMENTARY OPERATORS RELATED TO CONTRACTIONS AND OPERATORS WITH SPECTRA CONTAINED IN THE UNIT DISK IN NORM IDEALS

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Abstract. If $A, B \in \mathcal{B}(\mathcal{H})$ are normal contractions, then for every $X \in \mathcal{C}_{|||\cdot|||}(\mathcal{H})$ and $\alpha > 0$ holds

$$\left|\left|\left|\left(I - A^*A\right)^{\frac{\alpha}{2}} X \left(I - B^*B\right)^{\frac{\alpha}{2}}\right|\right|\right| \leqslant \left|\left|\left|\sum_{n=0}^{\infty} (-1)^n \binom{\alpha}{n} A^n X B^n\right|\right|\right|,$$

which generalizes a result of D.R. Jocić [Proc. Amer. Math. Soc. 126 (1998), no. 9, 2705–2713] for α not being an integer. Similar inequalities in the Schatten p-norms, for non-normal A, B and in the Q-norms if one of A or B is normal, are also given.

1. Introduction

Let $\mathcal{B}(\mathcal{H})$ and $\mathcal{C}_\infty(\mathcal{H})$ denote respectively spaces of all bounded and all compact linear operators acting on a separable, infinite-dimensional, complex Hilbert space $\mathcal{H}$. Each "symmetric gauge (s.g.) function" (also known as symmetric norming functions) Φ on sequences gives rise to a symmetric norm or a unitarily invariant (u.i.) norm on operators defined by $\|X\|_\Phi \stackrel{def}{=} \Phi(\{s_n(X)\}_{n=1}^\infty)$, with $s_1(X) \geqslant s_2(X) \geqslant \cdots$ being the singular values of X. We will denote by the symbol $|||\cdot|||$ any such norm, which is therefore defined on a naturally associated norm ideal

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$\mathfrak{C}_{|||\cdot|||}(\mathcal{H})$ of $\mathfrak{C}_\infty(\mathcal{H})$ and satisfies the invariance property $|||UXV||| = |||X|||$ for all $X \in \mathfrak{C}_{|||\cdot|||}(\mathcal{H})$ and for all unitary operators U, V. Even more, $|||AXB||| \leqslant |||CXD|||$ whenever $A^*A \leqslant C^*C$ and $BB^* \leqslant DD^*$. This is the consequence of Ky-Fan dominance property, which says that $|||X||| \leqslant |||Y|||$ if and only if $\sum_{k=1}^n s_k(X) \leqslant \sum_{k=1}^n s_k(Y)$ for all $n \in \mathbb{N}$, and the monotonicity of eigenvalues $\{\lambda_n\}_{n=1}^\infty$ of compact self-adjoint operators, which gives that

$$\begin{aligned} s_n(AXB) = \lambda_n^{\frac{1}{2}}(B^*X^*A^*AXB) &\leqslant \lambda_n^{\frac{1}{2}}(B^*X^*C^*CXB) \\ = \lambda_n^{\frac{1}{2}}(CXBB^*X^*C^*) &\leqslant \lambda_n^{\frac{1}{2}}(CXDD^*X^*C^*) = s_n(CXD) \quad (1.1) \end{aligned}$$

for all $n \in \mathbb{N}$, because $B^*X^*A^*AXB \leqslant B^*X^*C^*CXB$ implies $\lambda_n(B^*X^*A^*AXB) \leqslant \lambda_n(B^*X^*C^*CXB)$ and similarly $CXBB^*X^*C^* \leqslant CXDD^*X^*C^*$ implies that $\lambda_n(CXBB^*X^*C^*) \leqslant \lambda_n(CXDD^*X^*C^*)$. More details about the theory of symmetric normed ideals can be found in [2] and [7].

Each unitarily invariant norm is lower semi-continuous, i.e., $\left|\left|\left| w-\lim_{n\to\infty} X_n \right|\right|\right| \leqslant \liminf_{n\to\infty} |||X_n|||$. This follows from the well known representation formula $|||X||| = \sup\left\{\frac{|\mathrm{tr}(XY)|}{|||Y|||_*} : Y \text{ is finite dimensional}\right\}$, where $|||\cdot|||_*$ stands for the dual norm of $|||\cdot|||$ (see [7, Theorem 2.7 (d)]).

One way for creating new s.g functions, is to introduce their p (degree) modifications, for $p \geqslant 1$, $\Phi^{(p)}$ as a new s.g. function, defined by

$$\Phi^{(p)}\left((z_n)_{n=1}^\infty\right) \stackrel{def}{=} \sqrt[p]{\Phi\left((|z_n|^p)_{n=1}^\infty\right)},$$

which will be defined on its natural domain consisting of all complex sequences $z = (z_n)_{n=1}^\infty$ satisfying $(|z_n|^p)_{n=1}^\infty \in \ell_\Phi$. A simple proof that $\Phi^{(p)}$ is a s.g. function can be found in [5]. Specially Schatten p-norms, for $1 < p < \infty$, are exactly p modifications of the nuclear norm. When $p = 2$, then 2 - (degree) modifications are traditionally called Q-norms, so they can written in the form

$$|||X|||_{(2)} = |||X^*X|||^{\frac{1}{2}}.$$

Specially, Schatten p-norms are Q-norms if and only if $p \geqslant 2$.

Also we use some facts from the theory of holomorphic functional calculus on Banach algebras, specially the formula $fg(A) = f(A)g(A)$, which proof can be found in [1].

Throughout the text, for the operator $A \in \mathfrak{B}(\mathcal{H})$ we denote its spectral radius $r(A)$, defined by $r(A) = \inf_{n\in\mathbb{N}} \|A^n\|^{\frac{1}{n}}$.

First we define the analogue of the defect operator, we will need in the sequel.

Definition 1.1. *For $A \in \mathfrak{B}(\mathcal{H})$ such that $r(A) \leqslant 1$ and $\alpha > 0$, we define its defect operator as*

$$\Delta_{A,\alpha} \stackrel{def}{=} \lim_{\rho\to 1-}\left(\sum_{n=0}^\infty \rho^{2n}\frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)}A^{*n}A^n\right)^{-\frac{1}{2}}. \quad (1.2)$$

Note that if the operator $\sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)} A^{*n} A^n$ is bounded, which is by Banach–Steinhaus theorem equivalent to $\sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)} \|A^n f\|^2 < +\infty$ for every $f \in \mathcal{H}$, then it is equal to $\Delta_{A,\alpha}^{-2}$. Specially, this happens when, for example, A is a strict contraction, or more generally, if $r(A) < 1$. When it is applied to ρA instead of A, we have that $\Delta_{\rho A,\alpha} = \Big(\sum\limits_{n=0}^{\infty} \rho^{2n} \frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)} A^{*n} A^n\Big)^{-\frac{1}{2}}$, i.e. that $\Delta_{A,\alpha} = \lim_{\rho\to 1-} \Delta_{\rho A,\alpha}$. We also see that the definition is correct, since the operators $\Delta_{\rho A,\alpha}$ are positive contractions, and are decreasing in ρ, so they converge strongly to $\Delta_{A,\alpha}$.

Remark 1.2. In the case of Dirac measure, operator $\Delta_{A,\alpha}$ generalizes the operator $\Delta_{\mathscr{A}}$ from [6] for $\alpha \neq 1$. Specially, we have $\Delta_{A,1} = \Delta_A$

Since the coefficients $\frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)}$ in (1.2) are actually equal to $(-1)^n\binom{-\alpha}{n}$, for $A \in \mathcal{B}(\mathcal{H})$ normal, the operator $\Delta_{A,\alpha}$ can be given explicitly:

Lemma 1.3. *If $A \in \mathcal{B}(\mathcal{H})$ is a normal contraction, then $\Delta_{A,\alpha} = \Big(I - A^*A\Big)^{\frac{\alpha}{2}}$.*

Proof: Since A is normal contraction, for $\rho < 1$ we have

$$\sum_{n=0}^{\infty} \rho^{2n} \frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)} A^{*n} A^n = \sum_{n=0}^{\infty} \rho^{2n} (-1)^n \binom{-\alpha}{n} (A^*A)^n = \Big(I - \rho^2 A^*A\Big)^{-\alpha},$$

where the last equality follows from the spectral theorem for normal operators, as the series $\sum\limits_{n=0}^{\infty} \rho^{2n}(-1)^n\binom{-\alpha}{n} x^{2n}$ converges uniformly on $[0,1]$. Therefore we have

$$\Delta_{A,\alpha} = \lim_{\rho\to 1-} \Big(\sum_{n=0}^{\infty} \rho^{2n} \frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)} A^{*n} A^n\Big)^{-\frac{1}{2}} = \lim_{\rho\to 1-} \Big(I - \rho^2 A^*A\Big)^{\frac{\alpha}{2}} = \Big(I - A^*A\Big)^{\frac{\alpha}{2}},$$

where the last equality follows by the use of the spectral theorem, as proclaimed. $\square$

The next lemma provides a technical result which we will need in the sequel.

Lemma 1.4. *If $\alpha > 0$ and $A \in \mathcal{B}(\mathcal{H})$ such that $r(A) \leqslant 1$, then*

$$\Delta_{A,\alpha} \cdot \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)} A^{*n} A^n \cdot \Delta_{A,\alpha} \leqslant I.$$

Proof: Due to the operator monotonicity of function $t \mapsto t^{-1}$ and the very definition of the $\Delta_{A,\alpha}$, we have

$$\Delta_{\rho A,\alpha}^2 \leqslant \left(\sum_{n=0}^{N} \rho^{2n} \frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)} A^{*n} A^n\right)^{-1} \tag{1.5}$$

for every $N \in \mathbb{N}$ and for every $\rho \in [0,1)$. As we already noted that $\Delta_{\rho A,\alpha}$ converges strongly to $\Delta_{A,\alpha}$, with the same argument providing that $\Big(\sum\limits_{n=0}^{N} \rho^{2n} \frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)} A^{*n} A^n\Big)^{-1}$

converges strongly to $\left(\sum\limits_{n=0}^{N} \frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)} A^{*n}A^n\right)^{-1}$, by letting $\rho \to 1-$ in (1.5) we get

$$\Delta_{A,\alpha}^2 \leqslant \left(\sum_{n=0}^{N} \frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)} A^{*n}A^n\right)^{-1}$$

for every $N \in \mathbb{N}$. Therefore

$$\left(\sum_{n=0}^{N} \frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)} A^{*n}A^n\right)^{\frac{1}{2}} \Delta_{A,\alpha}^2 \left(\sum_{n=0}^{N} \frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)} A^{*n}A^n\right)^{\frac{1}{2}} \leqslant I,$$

which is equivalent to

$$\Delta_{A,\alpha} \cdot \sum_{n=0}^{N} \frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)} A^{*n}A^n \cdot \Delta_{A,\alpha} \leqslant I.$$

Hence, for every $N \in \mathbb{N}$ and every $f \in \mathcal{H}$ we have

$$\sum_{n=0}^{N} \frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)} \|A^n \Delta_{A,\alpha} f\|^2 = \left\langle \sum_{n=0}^{N} \frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)} \Delta_{A,\alpha} A^{*n}A^n \Delta_{A,\alpha} f, f \right\rangle \leqslant \langle f, f\rangle = \|f\|^2, \quad (1.9)$$

so by the Banach–Steinhaus theorem the operator $\Delta_{A,\alpha} \sum\limits_{n=0}^{\infty} \frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)} A^{*n}A^n \Delta_{A,\alpha}$ is bounded, and moreover $\left\langle \sum\limits_{n=0}^{\infty} \frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)} \Delta_{A,\alpha} A^{*n}A^n \Delta_{A,\alpha} f, f \right\rangle \leqslant \langle f, f\rangle$, which completes the proof. □

2. Main results

The first result reads as follows.

Theorem 2.1. *Let $A, B, X \in \mathcal{B}(\mathcal{H})$, such that A and B are normal contractions, $X \in \mathcal{C}_{|||\cdot|||}(\mathcal{H})$ and $\alpha > 0$, then*

$$\left|\left|\left| \left(I - A^*A\right)^{\frac{\alpha}{2}} \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)}{n!} A^n X B^n \left(I - B^*B\right)^{\frac{\alpha}{2}} \right|\right|\right| \leqslant \Gamma(\alpha) |||X||| \quad (2.1)$$

and

$$\left|\left|\left| \left(I - A^*A\right)^{\frac{\alpha}{2}} X \left(I - B^*B\right)^{\frac{\alpha}{2}} \right|\right|\right| \leqslant \left|\left|\left| \sum_{n=0}^{\infty} (-1)^n \binom{\alpha}{n} A^n X B^n \right|\right|\right|. \quad (2.2)$$

Proof: As

$$\left|\left|\left|\left(I-A^*A\right)^{\frac{\alpha}{2}}\sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha)}{n!}A^nXB^n\left(I-B^*B\right)^{\frac{\alpha}{2}}\right|\right|\right|$$

$$=\Gamma(\alpha)\left|\left|\left|\sum_{n=0}^{\infty}\sqrt{(-1)^n\binom{-\alpha}{n}}\left(I-A^*A\right)^{\frac{\alpha}{2}}A^nXB^n\sqrt{(-1)^n\binom{-\alpha}{n}}\left(I-B^*B\right)^{\frac{\alpha}{2}}\right|\right|\right|, \tag{2.3}$$

due to the $(-1)^n\binom{-\alpha}{n}=\frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)}$. Let us denote $C_n\overset{def}{=}\sqrt{(-1)^n\binom{-\alpha}{n}}\left(I-A^*A\right)^{\frac{\alpha}{2}}A^n$, and $D_n\overset{def}{=}B^n\sqrt{(-1)^n\binom{-\alpha}{n}}\left(I-B^*B\right)^{\frac{\alpha}{2}}$. Since A is normal, both A and A^* commute with $\left(I-A^*A\right)^{\frac{\alpha}{2}}$, then we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\|C_nf\|^2&=\left\langle\sum_{n=0}^{\infty}(-1)^n\binom{-\alpha}{n}\left(I-A^*A\right)^{\alpha}(A^*A)^nf,f\right\rangle\\&=\int\limits_{\sigma(A^*A)}\sum_{n=0}^{\infty}(-1)^n\binom{-\alpha}{n}x^n(1-x)^{\alpha}d\mu_f\\&=\int\limits_{\sigma(A^*A)}\chi_{[0,1)}d\mu_f=\langle P_{N(I-A^*A)^{\perp}}f,f\rangle=\left\|f-P_{N(I-A^*A)}f\right\|^2\leqslant\|f\|^2,\end{aligned} \tag{2.4}$$

where $d\mu_f=\langle Ef,f\rangle$ and E is spectral measure associated to the operator A^*A. Now, we have that $\sum\limits_{n=0}^{\infty}\|C_nf\|^2$ converges for every $f\in\mathcal{H}$, and similarly $\sum\limits_{n=0}^{\infty}\|D_nf\|^2$ converges for every $f\in\mathcal{H}$. Therefore we can apply [3, Theorem 2.2] to (2.3)

$$\Gamma(\alpha)\left|\left|\left|\sum_{n=0}^{\infty}C_nXD_n\right|\right|\right|\leqslant\Gamma(\alpha)\left|\left|\left|\sqrt{\sum_{n=0}^{\infty}C_n^*C_n}\,X\sqrt{\sum_{n=0}^{\infty}D_n^*D_n}\right|\right|\right|\leqslant\Gamma(\alpha)|||X|||, \tag{2.5}$$

where the last inequality holds because $\sum\limits_{n=0}^{\infty}C_n^*C_n$ and $\sum\limits_{n=0}^{\infty}D_n^*D_n$ are projectors by (2.4), which proves (2.1).

To prove (2.2), first we note that by (2.1) we have

$$\left|\left|\left|\left(I-\rho^2A^*A\right)^{\frac{\alpha}{2}}\sum_{n=0}^{\infty}(-1)^n\binom{-\alpha}{n}\rho^{2n}A^nXB^n\left(I-\rho^2B^*B\right)^{\frac{\alpha}{2}}\right|\right|\right|\leqslant|||X|||, \tag{2.6}$$

for all $0<\rho<1$. Let us denote $Z\overset{def}{=}\mathcal{J}_\rho X=\sum\limits_{n=0}^{\infty}(-1)^n\binom{-\alpha}{n}\rho^{2n}A^nXB^n$, and operator $A\otimes B$, defined by $A\otimes B\colon\mathfrak{B}(\mathcal{H})\to\mathfrak{B}(\mathcal{H}):X\mapsto AXB$. Since

A and B are contractions, we also have $\|A\otimes B\|\leqslant 1$. We see that $\mathcal{J}_\rho = \sum_{n=0}^{\infty}(-1)^n\binom{-\alpha}{n}\rho^{2n}(A\otimes B)^n$, is equal to $\Big(I-\rho^2 A\otimes B\Big)^{-\alpha}$ in the latest notation, in the sense of holomorphic functional calculus on $\mathfrak{B}(\mathcal{H})$. Therefore $X = \mathcal{J}_\rho{}^{-1}Z = \Big(I-\rho^2 A\otimes B\Big)^{\alpha}Z = \sum_{n=0}^{\infty}(-1)^n\binom{\alpha}{n}\rho^2 A^nZB^n$, so that (2.6) is equivalent to

$$\left|\left|\left|\Big(I-\rho^2A^*A\Big)^{\frac{\alpha}{2}}Z\Big(I-\rho^2B^*B\Big)^{\frac{\alpha}{2}}\right|\right|\right| \leqslant \left|\left|\left|\sum_{n=0}^{\infty}(-1)^n\binom{\alpha}{n}\rho^{2n}A^nZB^n\right|\right|\right|, \tag{2.7}$$

where, based on [4, Lem. 3.1], Z also belongs to $\mathfrak{C}_{|||\cdot|||}(\mathcal{H})$. Since $\Big(I-\rho^2A^*A\Big)^{\frac{\alpha}{2}}$ and $\Big(I-\rho^2B^*B\Big)^{\frac{\alpha}{2}}$ strongly converge to $\Big(I-A^*A\Big)^{\frac{\alpha}{2}}$ and $\Big(I-B^*B\Big)^{\frac{\alpha}{2}}$ respectively, the left hand side of (2.7) weakly converges to $\Big(I-A^*A\Big)^{\frac{\alpha}{2}}Z\Big(I-B^*B\Big)^{\frac{\alpha}{2}}$. On the right hand side of (2.7) as the series $\sum_{n=0}^{\infty}(-1)^n\binom{\alpha}{n}\rho^{2n}$ converges absolutely for $\rho=1$, based on the fact that it converges to $(1-1)^\alpha=0$, and that summands $(-1)^n\binom{\alpha}{n}$ have a constant sign for $n>\alpha$. Therefore

$$\begin{aligned}
&\left|\left|\left|\sum_{n=0}^{\infty}(-1)^n\binom{\alpha}{n}\rho^{2n}A^nZB^n - \sum_{n=0}^{\infty}(-1)^n\binom{\alpha}{n}A^nZB^n\right|\right|\right| \\
&\qquad\leqslant \sum_{n=0}^{\infty}\left|(-1)^n\binom{\alpha}{n}(1-\rho^{2n})\right|\,\|A^n\|\,\|B^n\|\,|||Z||| \\
&\qquad\leqslant \sum_{n=0}^{N}\left|(-1)^n\binom{\alpha}{n}(1-\rho^{2n})\right|\,|||Z||| + \sum_{n=N+1}^{\infty}2\left|(-1)^n\binom{\alpha}{n}\right|\,|||Z|||\,,
\end{aligned} \tag{2.8}$$

as A and B are contractions. The second term in (2.8) can be made arbitrarily small due to the argument used above, while the first term in (2.8) tends to 0, as $\rho\to 1-$. Now, from the lower semicontinuity of $|||\cdot|||$, we finally have

$$\begin{aligned}
&\left|\left|\left|\Big(I-A^*A\Big)^{\frac{\alpha}{2}}Z\Big(I-B^*B\Big)^{\frac{\alpha}{2}}\right|\right|\right| = \left|\left|\left|w-\lim_{\rho\to 1-}\Big(I-\rho^2A^*A\Big)^{\frac{\alpha}{2}}Z\Big(I-\rho^2B^*B\Big)^{\frac{\alpha}{2}}\right|\right|\right| \\
&\leqslant \liminf_{\rho\to 1-}\left|\left|\left|\sum_{n=0}^{\infty}(-1)^n\binom{\alpha}{n}\rho^{2n}A^nZB^n\right|\right|\right| \\
&= \lim_{\rho\to 1-}\left|\left|\left|\sum_{n=0}^{\infty}(-1)^n\binom{\alpha}{n}\rho^{2n}A^nZB^n\right|\right|\right| = \left|\left|\left|\sum_{n=0}^{\infty}(-1)^n\binom{\alpha}{n}A^nZB^n\right|\right|\right|,
\end{aligned} \tag{2.9}$$

which completes the proof of (2.2). □

Remark 2.2. Previous theorem generalizes mentioned [3, Theorem 2.3], where it is proven for $\alpha=1$. Let us also note that for $\alpha\in\mathbb{N}$ it could be proven by iterated application of [3, Theorem 2.3] to operators $(I-A^*A)^{\frac{\alpha-1}{2}}X(I-B^*B)^{\frac{\alpha-1}{2}}, (I-$

$A^*A)^{\frac{\alpha-2}{2}}(X-AXB)(I-B^*B)^{\frac{\alpha-2}{2}}, (I-A^*A)^{\frac{\alpha-3}{2}}(X-AXB-A(X-AXB)B)(I-B^*B)^{\frac{\alpha-3}{2}}$, *etc.* instead of X. For example, when $\alpha = 2$, we have

$$\left|\left|\left|(I-A^*A)\,X\,(I-B^*B)\right|\right|\right| = \left|\left|\left|\sqrt{I-A^*A}\sqrt{I-A^*A}\,X\sqrt{I-B^*B}\sqrt{I-B^*B}\right|\right|\right|$$

$$\leqslant \left|\left|\left|\sqrt{I-A^*A}\,X\sqrt{I-B^*B} - A\sqrt{I-A^*A}\,X\sqrt{I-B^*B}B\right|\right|\right| \tag{2.10}$$

$$= \left|\left|\left|\sqrt{I-A^*A}\,(X-AXB)\sqrt{I-B^*B}\right|\right|\right| \tag{2.11}$$

$$\leqslant \left|\left|\left|X-AXB-A(X-AXB)B\right|\right|\right| = \left|\left|\left|X-2AXB-A^2XB^2\right|\right|\right|, \tag{2.12}$$

where (2.10) and (2.12) follows by the use of [3, Theorem 2.3], applied to operators $\sqrt{I-A^*A}X\sqrt{I-B^*B}$ and $X-AXB$ respectively instead of X, while (2.11) follows from the commutativity of A and $\sqrt{I-A^*A}$. We see that the right hand side of (2.12) is equal to the (2.2) for $\alpha = 2$.

For operators with their spectra contained in the closed unit disk, no normality is needed for Schatten class norm inequalities. For the sake of completeness, we restate here the following direct consequence of the [4, Theorem 3.3].

Lemma 2.3. *Let $\alpha > 0$, $A, B \in \mathfrak{B}(\mathcal{H})$ and $X \in \mathfrak{C}_p(\mathcal{H})$, such that $\sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)}(\|A^nf\|^2+\|A^{*n}f\|^2+\|B^nf\|^2+\|B^{*n}f\|^2) < +\infty$, for every $f \in \mathcal{H}$. Then:*

$$\left\|\sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)}A^{*n}XB^n\right\|_p \leqslant \left\|\left(\sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)}A^n\left(\sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)}A^{*n}A^n\right)^{q-1}A^{*n}\right)^{\frac{1}{2q}}\right.$$

$$\left.\cdot X\left(\sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)}B^n\left(\sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)}B^{*n}B^n\right)^{r-1}B^{*n}\right)^{\frac{1}{2r}}\right\|_p, \tag{2.13}$$

where $p, q, r \geqslant 1$ are satisfying $\frac{2}{p} = \frac{1}{q} + \frac{1}{r}$.

With the use of the defect operator $\Delta_{A,\alpha}$, we have similar inequalities in a more compact form.

Theorem 2.4. *Let $\alpha > 0$ and $A, B \in \mathfrak{B}(\mathcal{H})$ and $X \in \mathfrak{C}_p(\mathcal{H})$, $r(A) \leqslant 1$, $r(B) \leqslant 1$ and $r(A)r(B) < 1$, and let we additionally assume*

$$\sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)}\left(\|A^{*n}f\|^2+\|B^{*n}f\|^2\right) < +\infty, \qquad \text{for every} \qquad f \in \mathcal{H}. \tag{2.14}$$

Then

$$\left\|\Delta_{A,\alpha}^{1-\frac{1}{q}}\sum_{n=1}^{\infty}\frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)}A^{*n}XB^n\Delta_{B,\alpha}^{1-\frac{1}{r}}\right\|_p \leqslant \left\|\Delta_{A^*,\alpha}^{-\frac{1}{q}}X\Delta_{B^*,\alpha}^{-\frac{1}{r}}\right\|_p \tag{2.15}$$

and

$$\left\|\Delta_{A,\alpha}^{1-\frac{1}{q}}X\Delta_{B,\alpha}^{1-\frac{1}{r}}\right\|_p \leqslant \left\|\Delta_{A^*,\alpha}^{-\frac{1}{q}}\sum_{n=0}^{\infty}(-1)^n\binom{\alpha}{n}A^{*n}XB^n\Delta_{B^*,\alpha}^{-\frac{1}{r}}\right\|_p, \tag{2.16}$$

where $p, q, r \geqslant 1$ *are satisfying* $\frac{2}{p} = \frac{1}{q} + \frac{1}{r}$.

Remark 2.5. We note that condition $\sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)} \|A^{*n} f\|^2 < +\infty$ is fulfilled in the case $r(A) < 1$, so it is relevant only in the case $r(A) = 1$, and similarly for B.

Proof: We will first prove the inequality (2.15). Since $r(A^*) = r(A) \leqslant 1$, we note that $\Delta_{A^*,\alpha}$ is well defined, while from the condition (2.14) we see that is actually equal to $\left(\sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)} A^n A^{*n}\right)^{-\frac{1}{2}}$, and the same reasoning can be applied for $\Delta_{B^*,\alpha}$. By the [4, Theorem 3.3], for $\rho \in [0,1)$ we get

$$\left\|\Delta_{\rho A,\alpha}^{1-\frac{1}{q}} \sum_{n=0}^{\infty} \rho^{2n} \frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)} A^{*n} X B^n \Delta_{\rho B,\alpha}^{1-\frac{1}{r}}\right\|_p = \left\|\sum_{n=0}^{\infty} A_n X B_n\right\|_p$$
$$\leqslant \left\|\left(\sum_{n=0}^{\infty} A_n^* \Big(\sum_{n=0}^{\infty} A_n A_n^*\Big)^{q-1} A_n\right)^{\frac{1}{2q}} X \left(\sum_{n=0}^{\infty} B_n \Big(\sum_{n=0}^{\infty} B_n^* B_n\Big)^{r-1} B_n^*\right)^{\frac{1}{2r}}\right\|_p, \quad (2.17)$$

where $A_n \overset{def}{=} \rho^n \sqrt{\frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)}} \Delta_{\rho A,\alpha}^{1-\frac{1}{q}} A^{*n}$, and $B_n \overset{def}{=} \rho^n \sqrt{\frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)}} B^n \Delta_{\rho B,\alpha}^{1-\frac{1}{r}}$. Since $r(A^*) \leqslant 1$, for every $\rho < 1$ we have that $\rho \sqrt[n]{\|A^{*n}\|} \leqslant \frac{1+\rho}{2} < 1$ for n being large enough, so the conditions of [4, Theorem 3.3] are fulfilled. Therefore

$$\sum_{n=0}^{\infty} A_n A_n^* = \Delta_{\rho A,\alpha}^{1-\frac{1}{q}} \sum_{n=0}^{\infty} \rho^{2n} \frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)} A^{*n} A^n \Delta_{\rho A,\alpha}^{1-\frac{1}{q}} = \Delta_{\rho A,\alpha}^{-\frac{2}{q}}, \quad (2.18)$$

and hence

$$\left(\sum_{n=0}^{\infty} A_n^* \Big(\sum_{n=0}^{\infty} A_n A_n^*\Big)^{q-1} A_n\right)^{\frac{1}{2q}} = \left(\sum_{n=0}^{\infty} \rho^{2n} \frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)} A^n \Delta_{\rho A,\alpha}^{1-\frac{1}{q}} \Delta_{\rho A,\alpha}^{\frac{2-2q}{q}} \Delta_{\rho A,\alpha}^{1-\frac{1}{q}} A^{*n}\right)^{\frac{1}{2q}}$$
$$= \left(\sum_{n=0}^{\infty} \rho^{2n} \frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)} A^n A^{*n}\right)^{\frac{1}{2q}} = \Delta_{\rho A^*,\alpha}^{-\frac{1}{q}}. \quad (2.19)$$

Similarly

$$\left(\sum_{n=0}^{\infty} B_n \Big(\sum_{n=0}^{\infty} B_n^* B_n\Big)^{r-1} B_n^*\right)^{\frac{1}{2r}} = \Delta_{\rho B^*,\alpha}^{-\frac{1}{r}}, \quad (2.20)$$

so that (2.17) actually becomes

$$\left\|\Delta_{\rho A,\alpha}^{1-\frac{1}{q}} \sum_{n=0}^{\infty} \rho^{2n} \frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)} A^{*n} X B^n \Delta_{\rho B,\alpha}^{1-\frac{1}{r}}\right\|_p \leqslant \left\|\Delta_{\rho A^*,\alpha}^{-\frac{1}{q}} X \Delta_{\rho B^*,\alpha}^{-\frac{1}{r}}\right\|_p. \quad (2.21)$$

As we already noted, operators $\sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)} A^n A^{*n}$ and $\sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)} B^n B^{*n}$ are bounded, and they are equal to $\Delta_{A^*,\alpha}^{-2}$ and $\Delta_{B^*,\alpha}^{-2}$ respectively. Since all coefficients appearing in the definition (1.2) are positive, we have $\Delta_{\rho A^*,\alpha}^{-2} \leqslant \Delta_{A^*,\alpha}^{-2}$ and $\Delta_{\rho B^*,\alpha}^{-2} \leqslant \Delta_{B^*,\alpha}^{-2}$. As the functions $t \mapsto t^{\frac{1}{q}}$ and $t \mapsto t^{\frac{1}{r}}$ are operator monotone (since

$r, q \geqslant 1$), we also have $\Delta_{\rho A^*,\alpha}^{-\frac{2}{q}} \leqslant \Delta_{A^*,\alpha}^{-\frac{2}{q}}$ and $\Delta_{\rho B^*,\alpha}^{-\frac{2}{r}} \leqslant \Delta_{B^*,\alpha}^{-\frac{2}{r}}$, and due to (1.1) we therefore have

$$\left\|\Delta_{\rho A^*,\alpha}^{-\frac{1}{q}} X \Delta_{\rho B^*,\alpha}^{-\frac{1}{r}}\right\|_p \leqslant \left\|\Delta_{A^*,\alpha}^{-\frac{1}{q}} X \Delta_{B^*,\alpha}^{-\frac{1}{r}}\right\|_p. \tag{2.22}$$

Once again, by the monotonicity of $\Delta_{\rho A,\alpha}^{-2}$ in ρ, and operator monotonicity of the function $t \mapsto t^{-1}$, we also have $\Delta_{A,\alpha}^2 \leqslant \Delta_{\rho A,\alpha}^2$, and because $0 \leqslant 1 - \frac{1}{q} \leqslant 1$, the function $t \mapsto t^{1-\frac{1}{q}}$ is operator monotone as well, therefore $\Delta_{A,\alpha}^{2-\frac{2}{q}} \leqslant \Delta_{\rho A,\alpha}^{2-\frac{2}{q}}$ and $\Delta_{B,\alpha}^{2-\frac{2}{r}} \leqslant \Delta_{\rho B,\alpha}^{2-\frac{2}{r}}$. Now, by the argument similar to (2.22) we get

$$\begin{aligned}\left\|\Delta_{A,\alpha}^{1-\frac{1}{q}} \sum_{n=0}^{\infty} \rho^{2n} \frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)} A^{*n} X B^n \Delta_{B,\alpha}^{1-\frac{1}{r}}\right\|_p \\ \leqslant \left\|\Delta_{\rho A,\alpha}^{1-\frac{1}{q}} \sum_{n=0}^{\infty} \rho^{2n} \frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)} A^{*n} X B^n \Delta_{\rho B,\alpha}^{1-\frac{1}{r}}\right\|_p\end{aligned} \tag{2.23}$$

for every $\rho \in (0,1)$. To finish the proof of (2.15), all we need is to let $\rho \to 1-$. Since $r(A^*)r(B) = r(A)r(B) < 1$, for sufficiently large n we have $\sqrt[n]{\|A^{*n}\|\,\|B^n\|} \leqslant c < 1$, and therefore $\|A^{*n} X B^n\|_p \leqslant c^n \|X\|_p$, hence the series $\sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)} A^{*n} X B^n$ converges in the Schatten p-norm. With the argument similar to that already used in the proof of (2.8), we can prove that $\sum_{n=0}^{\infty} \rho^{2n} \frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)} A^{*n} X B^n$ converges to $\sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)} A^{*n} X B^n$ in the Schatten p-norm as $\rho \to 1-$. As this implies weak convergence of $\sum_{n=0}^{\infty} \rho^{2n} \frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)} A^{*n} X B^n$ as $\rho \to 1-$, from the lower semicontinuity of the Schatten p-norms, (2.21), (2.22) and (2.23), we have

$$\begin{aligned}&\left\|\Delta_{A,\alpha}^{1-\frac{1}{q}} \sum_{n=1}^{\infty} \frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)} A^{*n} X B^n \Delta_{B,\alpha}^{1-\frac{1}{r}}\right\|_p \\ &= \left\| w - \lim_{\rho\to 1-} \Delta_{A,\alpha}^{1-\frac{1}{q}} \sum_{n=1}^{\infty} \rho^{2n} \frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)} A^{*n} X B^n \Delta_{B,\alpha}^{1-\frac{1}{r}}\right\|_p \\ &\leqslant \liminf_{\rho\to 1-} \left\|\Delta_{A,\alpha}^{1-\frac{1}{q}} \sum_{n=1}^{\infty} \rho^{2n} \frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)} A^{*n} X B^n \Delta_{B,\alpha}^{1-\frac{1}{r}}\right\|_p \\ &\leqslant \liminf_{\rho\to 1-} \left\|\Delta_{\rho A,\alpha}^{1-\frac{1}{q}} \sum_{n=0}^{\infty} \rho^{2n} \frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)} A^{*n} X B^n \Delta_{\rho B,\alpha}^{1-\frac{1}{r}}\right\|_p \\ &\leqslant \liminf_{\rho\to 1-} \left\|\Delta_{\rho A^*,\alpha}^{-\frac{1}{q}} X \Delta_{\rho B^*,\alpha}^{-\frac{1}{r}}\right\|_p \leqslant \liminf_{\rho\to 1-} \left\|\Delta_{A^*,\alpha}^{-\frac{1}{q}} X \Delta_{B^*,\alpha}^{-\frac{1}{r}}\right\|_p = \left\|\Delta_{A^*,\alpha}^{-\frac{1}{q}} X \Delta_{B^*,\alpha}^{-\frac{1}{r}}\right\|_p\end{aligned} \tag{2.24}$$

which concludes the proof of (2.15).

To prove (2.16), first we note that the operator $\mathcal{J}_\rho = \sum_{n=0}^{\infty}(-1)^n\binom{-\alpha}{n}\rho^{2n}(A^*\otimes B)^n$ is defined since $r(\rho A^*\otimes\rho B) \leqslant r(\rho A^*)r(\rho B) \leqslant \rho^2 < 1$ and by the similar argument used in the proof of (2.2), from (2.21) we get

$$\left\|\Delta_{\rho A,\alpha}^{1-\frac{1}{q}} X \Delta_{\rho B,\alpha}^{1-\frac{1}{r}}\right\|_p \leqslant \left\|\Delta_{\rho A^*,\alpha}^{-\frac{1}{q}} \sum_{n=0}^{\infty}(-1)^n\binom{\alpha}{n}\rho^{2n}A^{*n}XB^n\Delta_{\rho B^*,\alpha}^{-\frac{1}{r}}\right\|_p. \tag{2.25}$$

Similarly to the proof of (2.22) and (2.23), it can be shown

$$\left\|\Delta_{\rho A^*,\alpha}^{-\frac{1}{q}} \sum_{n=0}^{\infty}(-1)^n\binom{\alpha}{n}\rho^{2n}A^{*n}XB^n\Delta_{\rho B^*,\alpha}^{-\frac{1}{r}}\right\|_p \leqslant \left\|\Delta_{A^*,\alpha}^{-\frac{1}{q}} \sum_{n=0}^{\infty}(-1)^n\binom{\alpha}{n}\rho^{2n}A^{*n}XB^n\Delta_{B^*,\alpha}^{-\frac{1}{r}}\right\|_p, \tag{2.26}$$

as well as

$$\left\|\Delta_{A,\alpha}^{1-\frac{1}{q}} X \Delta_{B,\alpha}^{1-\frac{1}{r}}\right\|_p \leqslant \left\|\Delta_{\rho A,\alpha}^{1-\frac{1}{q}} X \Delta_{\rho B,\alpha}^{1-\frac{1}{r}}\right\|_p, \tag{2.27}$$

so that

$$\left\|\Delta_{A,\alpha}^{1-\frac{1}{q}} X \Delta_{B,\alpha}^{1-\frac{1}{r}}\right\|_p \leqslant \left\|\Delta_{A^*,\alpha}^{-\frac{1}{q}} \sum_{n=0}^{\infty}(-1)^n\binom{\alpha}{n}\rho^{2n}A^{*n}XB^n\Delta_{B^*,\alpha}^{-\frac{1}{r}}\right\|_p. \tag{2.28}$$

If $r(A^*)r(B) < 1$, then $\|A^{*n}\|\,\|B^n\| < c^n$ for some $0 < c < 1$ and sufficiently large n, so the series on the right hand side converges absolutely in the Schatten p-norm for $\rho = 1$. By letting $\rho \to 1-$ in (2.28) we get the desired inequality. □

For the case $r(A) = r(B) = 1$, we provide some sufficient conditions to threat (2.15) and (2.16).

Theorem 2.6. *Let $\alpha > 0$, $A, B \in \mathfrak{B}(\mathcal{H})$ and $X \in \mathfrak{C}_p(\mathcal{H})$, $r(A) \leqslant 1, r(B) \leqslant 1$, and*

$$\sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)}\left(\|A^{*n}f\|^2 + \|B^{*n}f\|^2\right) < +\infty, \qquad \textit{for every} \qquad f \in \mathcal{H}. \tag{2.29}$$

a) *If*

$$\sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)}\left(\|A^{n}f\|^2 + \|B^{n}f\|^2\right) < +\infty, \qquad \textit{for every} \qquad f \in \mathcal{H}, \tag{2.30}$$

then

$$\left\|\Delta_{A,\alpha}^{1-\frac{1}{q}} \sum_{n=1}^{\infty}\frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)}A^{*n}XB^n\Delta_{B,\alpha}^{1-\frac{1}{r}}\right\|_p \leqslant \left\|\Delta_{A^*,\alpha}^{-\frac{1}{q}} X \Delta_{B^*,\alpha}^{-\frac{1}{r}}\right\|_p. \tag{2.31}$$

b) *If A and B are contractions, then*

$$\left\|\Delta_{A,\alpha}^{1-\frac{1}{q}} X \Delta_{B,\alpha}^{1-\frac{1}{r}}\right\|_p \leqslant \left\|\Delta_{A^*,\alpha}^{-\frac{1}{q}} \sum_{n=0}^{\infty}(-1)^n\binom{\alpha}{n}A^{*n}XB^n\Delta_{B^*,\alpha}^{-\frac{1}{r}}\right\|_p, \tag{2.32}$$

where $p, q, r \geqslant 1$ are satisfying $\frac{2}{p} = \frac{1}{q} + \frac{1}{r}$.

Proof: As it can be seen from the proof of Th. 2.4, that to obtain inequality

$$\left\|\Delta_{\rho A,\alpha}^{1-\frac{1}{q}}\sum_{n=0}^{\infty}\rho^{2n}\frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)}A^{*n}XB^{n}\Delta_{\rho B,\alpha}^{1-\frac{1}{r}}\right\|_p \leqslant \left\|\Delta_{\rho A^*,\alpha}^{-\frac{1}{q}}X\Delta_{\rho B^*,\alpha}^{-\frac{1}{r}}\right\|_p, \tag{2.33}$$

for every $\rho\in[0,1)$, we only needed $r(A^*)\leqslant 1$ and $r(B)\leqslant 1$, which is assumed in the statement of the Th. 2.6. Similarly, from (2.29), operators $\sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)}A^nA^{*n}$ and $\sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)}B^nB^{*n}$ are bounded, and equal to $\Delta_{A^*,\alpha}^{-2}$ and $\Delta_{B^*,\alpha}^{-2}$ respectively, having also

$$\left\|\Delta_{\rho A^*,\alpha}^{-\frac{1}{q}}X\Delta_{\rho B^*,\alpha}^{-\frac{1}{r}}\right\|_p \leqslant \left\|\Delta_{A^*,\alpha}^{-\frac{1}{q}}X\Delta_{B^*,\alpha}^{-\frac{1}{r}}\right\|_p, \tag{2.34}$$

and

$$\left\|\Delta_{A,\alpha}^{1-\frac{1}{q}}\sum_{n=0}^{\infty}\rho^{2n}\frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)}A^{*n}XB^{n}\Delta_{B,\alpha}^{1-\frac{1}{r}}\right\|_p \leqslant \left\|\Delta_{\rho A,\alpha}^{1-\frac{1}{q}}\sum_{n=0}^{\infty}\rho^{2n}\frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)}A^{*n}XB^{n}\Delta_{\rho B,\alpha}^{1-\frac{1}{r}}\right\|_p. \tag{2.35}$$

To finish the proof of (2.31), all we have to do is to show that $\sum_{n=0}^{\infty}\rho^{2n}\frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)}A^{*n}XB^n$ converges to $\sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)}A^{*n}XB^n$ weakly. Indeed, for every $f,g\in\mathcal{H}$ we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}(1-\rho^{2n})\frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)}\langle A^{*n}XB^nf,g\rangle = \sum_{n=0}^{\infty}(1-\rho^{2n})\frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)}\langle XB^nf,A^ng\rangle\\
&\leqslant \sum_{n=0}^{\infty}(1-\rho^{2n})\frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)}\|X\|_p\|B^nf\|\,\|A^ng\|\\
&\leqslant \|X\|_p\left(\sum_{n=0}^{\infty}(1-\rho^{2n})\frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)}\|B^nf\|^2\right)^{\frac{1}{2}}\left(\sum_{n=0}^{\infty}(1-\rho^{2n})\frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)}\|A^ng\|^2\right)^{\frac{1}{2}},
\end{aligned} \tag{2.36}$$

and now, we have that the right hand side of (2.36) converges to 0 as $\rho\to 1-$, based on (2.30). Rest of the proof parallels the proof of (2.15).

To prove (2.32), similarly to the proof of (2.16) we have

$$\left\|\Delta_{A,\alpha}^{1-\frac{1}{q}}X\Delta_{B,\alpha}^{1-\frac{1}{r}}\right\|_p \leqslant \left\|\Delta_{A^*,\alpha}^{-\frac{1}{q}}\sum_{n=0}^{\infty}(-1)^n\binom{\alpha}{n}\rho^{2n}A^{*n}XB^n\Delta_{B^*,\alpha}^{-\frac{1}{r}}\right\|_p. \tag{2.37}$$

Since $\|A\|=\|B\|=1$, as already shown in (2.8), we have that the sum on the right hand side converges in Schatten p-norm to $\Delta_{A^*,\alpha}^{-\frac{1}{q}}\sum_{n=0}^{\infty}(-1)^n\binom{\alpha}{n}A^{*n}XB^n\Delta_{B^*,\alpha}^{-\frac{1}{r}}$ as $\rho\to 1-$, which completes the proof. □

We can prove analogous inequalities for Q-norms, whenever at least one of operators A or B is normal. First we rephrase [6, Lem. 3.4] for $\Omega = \mathbb{N}$ and μ being a counting measure.

Lemma 2.7. *Let $A_n, B_n \in \mathcal{B}(\mathcal{H})$ such that $\sum_{n=1}^{\infty}\Big(\|A_n f\|^2 + \|B_n f\|^2\Big) < +\infty$ for every $f \in \mathcal{H}$. If A_n consists of commuting normal operators then for every Q-norm $|||\cdot|||_{(2)}$, and every $X \in \mathcal{C}_{|||\cdot|||_{(2)}}(\mathcal{H})$ then*

$$\left|\left|\left|\sum_{n=1}^{\infty} A_n^* X B_n\right|\right|\right|_{(2)} \leqslant \left|\left|\left|\sqrt{\sum_{n=1}^{\infty} A_n^* A_n}\, X\right|\right|\right|_{(2)} \left\|\sum_{n=1}^{\infty} B_n^* B_n\right\|^{\frac{1}{2}}, \tag{2.38}$$

and similarly, if B_n consists of commuting normal operators, then

$$\left|\left|\left|\sum_{n=1}^{\infty} A_n^* X B_n\right|\right|\right|_{(2)} \leqslant \left\|\sum_{n=1}^{\infty} A_n^* A_n\right\|^{\frac{1}{2}} \cdot \left|\left|\left| X\sqrt{\sum_{n=1}^{\infty} B_n^* B_n}\right|\right|\right|_{(2)}. \tag{2.39}$$

Lemma 2.7 enables us that for Q-norms, which includes Schatten p-norms for $p \geqslant 2$, to partially drop the assumption of normality in 2.1, and enhance and simplify 2.4.

Theorem 2.8. *Let $\alpha > 0$, $A, B \in \mathcal{B}(\mathcal{H})$ such that $X^*X \in \mathcal{C}_{|||\cdot|||}(\mathcal{H})$, $r(A) \leqslant 1$ and $r(B) \leqslant 1$. If A is normal, then the following inequality holds*

$$\left|\left|\left|\Big(I - A^*A\Big)^{\frac{\alpha}{2}} \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)} A^n X B^n \Delta_{B,\alpha}\right|\right|\right|_{(2)} \leqslant |||X|||_{(2)}\,, \tag{2.40}$$

and if B is additionally a contraction or $r(A)r(B) < 1$, we also have

$$\left|\left|\left|\Big(I - A^*A\Big)^{\frac{\alpha}{2}} X \Delta_{B,\alpha}\right|\right|\right|_{(2)} \leqslant \left|\left|\left|\sum_{n=0}^{\infty} (-1)^n \binom{\alpha}{n} A^n X B^n\right|\right|\right|_{(2)}. \tag{2.41}$$

If B is normal, then

$$\left|\left|\left|\Delta_{A,\alpha} \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)} A^n X B^n \Big(I - B^*B\Big)^{\frac{\alpha}{2}}\right|\right|\right|_{(2)} \leqslant |||X|||_{(2)}\,, \tag{2.42}$$

and if A is additionally a contraction or $r(A)r(B) < 1$, we also have

$$\left|\left|\left|\Delta_{A,\alpha} X \Big(I - B^*B\Big)^{\frac{\alpha}{2}}\right|\right|\right|_{(2)} \leqslant \left|\left|\left|\sum_{n=0}^{\infty} (-1)^n \binom{\alpha}{n} A^n X B^n\right|\right|\right|_{(2)}. \tag{2.43}$$

Proof: To prove (2.40), all we have to do is to apply (2.38) to the left hand side of (2.40)

$$\begin{aligned} &\left|\left|\left|\Big(I-A^*A\Big)^{\frac{\alpha}{2}}\sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)}A^nXB^n\Delta_{B,\alpha}\right|\right|\right|_{(2)} \\ \leqslant &\left|\left|\left|\sqrt{\Big(I-A^*A\Big)^{\frac{\alpha}{2}}\sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)}A^{*n}A^n\Big(I-A^*A\Big)^{\frac{\alpha}{2}}}X\right|\right|\right|_{(2)} \\ \times &\left\|\sqrt{\Delta_{B,\alpha}\sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)}B^{*n}B^n\Delta_{B,\alpha}}\right\| \leqslant |||X||| \end{aligned} \tag{2.44}$$

where $A_n \stackrel{def}{=} \sqrt{\frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)}}\Big(I-A^*A\Big)^{\frac{\alpha}{2}}A^n$ and $B_n \stackrel{def}{=} \sqrt{\frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)}}B^n\Delta_{B,\alpha}$, in the notation of Lemma 2.7. Let us note that $\sum\limits_{n=0}^{\infty}\|A_nf\|^2 < +\infty$ was proved in (2.4), while $\sum\limits_{n=0}^{\infty}\|B_nf\|^2 < +\infty$, as well as $\left\|\sum_{n=0}^{\infty}B_n^*B_n\right\| \leqslant 1$ follows from the Lemma 1.4. Proof of (2.41) now follows from the (2.40) like in the Theorems 2.1 and 2.4. Proof of (2.42) follows similarly, by the use of (2.39) instead of (2.38). □

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References

1. J. L. Daleckii and M. G. Krein, *Stability of Solutions of Differential Equations in Banach Space*, American Mathematical Society, Providence, R.I., 2002.
2. I. C. Gohberg and M.G. Krein, *Introduction to the Theory of Linear Non-selfadjoint Operators*, American Mathematical Society, Providence, R.I., 1969.
3. D. R. Jocić ,*Cauchy-Schwarz and means inequalities for elementary operators in into norm ideals*, Proc. Amer. Math. Soc. **126** (1998), no. 9, 2705–2713.
4. D. R. Jocić, *Cauchy-Schwarz norm inequalities for weak*-integrals of operator valued functions*, J. Funct. Anal. **218** (2005), no. 2, 318–346.
5. D. R. Jocić, *Multipliers of elementary operators and comparison of row and column space Schatten p norms*, Linear Algebra Appl. **413** (2009), no. 11 2062–2070.
6. D. R. Jocić, S. Milošević, and V. Đurić, *Norm inequalities for elementary operators and other inner product type integral transformers with the spectra contained in the unit disc*, Filomat, (to appear).
7. B. Simon, *Trace Ideals and Their Applications*, American Mathematical Society, Mat. Surveys Monogr. **120**, Providence, R.I., 2005.

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