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REFINEMENTS OF HÖLDER–MCCARTHY INEQUALITY AND YOUNG INEQUALITY

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This paper is dedicated to the memory of Professor Takayuki Furuta

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Abstract. We refine the Hölder–McCarthy inequality. The point is the convexity of the function induced by Hölder–McCarthy inequality. Also we discuss the equivalent between refined Hölder–McCarthy inequality and refined Young inequality with type of Kittaneh and Manasrah.

1. Introduction

Throughout this note, a capital letter means a (bounded linear) operator acting on a Hilbert space $\mathcal{H}$. An operator A is said to be positive, denoted by $A \geq 0$, if $(Ax, x) \geq 0$ for all $x \in \mathcal{H}$.

McCarthy [5] proved the following inequalities: Let A be positive operator acting on a Hilbert space $\mathcal{H}$. Then

(i) $(A^\mu x, x) \leq (Ax, x)^\mu \|x\|^{2(1-\mu)}$ for $\mu \in [0, 1]$ and $x \in \mathcal{H}$.

(ii) $(A^\mu x, x) \geq (Ax, x)^\mu \|x\|^{2(1-\mu)}$ for $\mu > 1$ and $x \in \mathcal{H}$.

Moreover (i) and (ii) are simplified to the following (iii) and (iv), respectively:

(iii) $(A^\mu x, x) \leq (Ax, x)^\mu$ for $\mu \in [0, 1]$ and $\|x\| = 1$.

(iv) $(A^\mu x, x) \geq (Ax, x)^\mu$ for $\mu > 1$ and $\|x\| = 1$.

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The inequalities (i) and (ii) are proved by using the integral representation of A and the Hölder inequality. Hence they are called the Hölder–McCarthy inequality.

On the other hand, the following inequality is named as the Young inequality, cf. [2]: For $A, B \geq 0$,

$$\mu A + (1-\mu)B \geq B \#_\mu A \quad \text{for } 0 \leq \mu \leq 1,$$

where $B \#_\mu A = B^{\frac{1}{2}}(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})^\mu B^{\frac{1}{2}}$ is the μ-operator geometric mean. Its simplified form is as follows: For $A \geq 0$,

$$\mu A + 1 - \mu \geq A^\mu \quad \text{for } 0 \leq \mu \leq 1.$$

It is known that the Hölder–McCarthy inequality (iii) and the Young inequality are equivalent, e.g. [2, §3.1.3].

As a refinement of the Young inequality, Kittaneh and Manasrah [3] proposed that

$$(1-\mu)a + \mu b \geq a^{1-\mu}b^\mu + \min\{\mu, 1-\mu\}(\sqrt{a} - \sqrt{b})^2$$

for all positive numbers a, b and $\mu \in [0, 1]$. It is simplified as follows:

$$\mu a + 1 - \mu - a^\mu \geq \min\{\mu, 1-\mu\}(1 + a - 2\sqrt{a})$$

for all positive numbers a and $\mu \in [0, 1]$. We now understand it as the inequality

$$\mu A + 1 - \mu - A^\mu \geq \min\{\frac{1-\mu}{1-\nu}, \frac{\mu}{\nu}\}(\nu A + 1 - \nu - A^\nu).$$

As a matter of fact, if we take $\nu = \frac{1}{2}$ and $A = aI$, where I is the identity operator, then we easily obtain the simplified inequality mentioned above. In succession, Manasrah and Kittaneh generalized refined Young inequalities in [4].

Based on recent results on refinements of Young inequality, Alzer et al. proposed the following estimation [1: Theorem2.1]: If $0 < \mu < \nu < 1$, $\lambda \geq 1$ and $a, b > 0$, then

$$\left(\frac{1-\nu}{1-\mu}\right)^\lambda < \frac{A_\nu^\lambda - G_\nu^\lambda}{A_\mu^\lambda - G_\mu^\lambda} < \left(\frac{\nu}{\mu}\right)^\lambda$$

holds, where $A_\tau = (1-\tau)a + \tau b$ and $G_\tau = a^{1-\tau}b^\tau$.

In this paper, we improve the Hölder–McCarthy inequality, whose point is the convexity of the function $f(\mu) = \frac{(A^\mu x, x)}{(Ax, x)^\mu}$. Moreover we point out that the improved Hölder–McCarthy inequality is equivalent to an improved Young inequality in the sense of Kittaneh and Manasrah.

2. Hölder–McCarthy inequality

As an approach to the Hölder–McCarthy inequality, we consider the function defined by the ratio; $f(\mu) = \frac{(A^\mu x, x)}{(Ax, x)^\mu}$. We first show the convexity of the function.

Theorem 2.1. *Let A be a positive operator on $\mathcal{H}$ and $x \in \mathcal{H}$ with $Ax \neq 0$. If $f(\mu) = \frac{(A^\mu x, x)}{(Ax, x)^\mu}$, then $f(\mu)$ is a convex function on $[0, \infty)$. Moreover if A is invertible, then $f(\mu)$ is a convex function on $(-\infty, \infty)$.*

Proof. First of all, we note that $(A^\mu x, x)$ is log-convex, i.e.,

$$(A^{\frac{\mu+\nu}{2}}x, x) \leq (A^\mu x, x)^{\frac{1}{2}}(A^\nu x, x)^{\frac{1}{2}}.$$

It is easily checked as follows:

$$(A^{\frac{\mu+\nu}{2}}x, x) \leq \|A^{\frac{\mu}{2}}x\|\|A^{\frac{\nu}{2}}x\| = (A^\mu x, x)^{\frac{1}{2}}(A^\nu x, x)^{\frac{1}{2}}.$$

By this and the arithmetic-geometric mean inequality, we have

$$\frac{1}{2}\left(\frac{(A^\mu x, x)}{(Ax, x)^\mu} + \frac{(A^\nu x, x)}{(Ax, x)^\nu}\right) \geq \frac{(A^\mu x, x)^{\frac{1}{2}}(A^\nu x, x)^{\frac{1}{2}}}{(Ax, x)^{\frac{\mu+\nu}{2}}} \geq \frac{(A^{\frac{\mu+\nu}{2}}x, x)}{(Ax, x)^{\frac{\mu+\nu}{2}}},$$

that is, $f(\frac{\mu+\nu}{2}) \leq \frac{1}{2}(f(\mu) + f(\nu))$. □

Remark 2.2. It is remarkable that the convexity of $f(\mu)$ implies the Hölder–McCarthy inequality. As a matter of fact, if $x \in \mathcal{H}$ is unit vector, then $f(\mu)$ defined in above satisfies $f(0) = f(1) = 1$. Hence the convexity of it implies the Hölder–McCarthy inequality (iii) and (iv).

Next we propose a refinement of the Hölder–McCarthy inequality:

Theorem 2.3. *Let $A \geq 0$, $\|x\| = 1$ and $\lambda \geq 1$. Then*

$$m(\mu,\nu)\left(1 - \left(\frac{(A^\nu x, x)}{(Ax, x)^\nu}\right)^\lambda\right) \leq 1 - \left(\frac{(A^\mu x, x)}{(Ax, x)^\mu}\right)^\lambda \leq M(\mu,\nu)\left(1 - \left(\frac{(A^\nu x, x)}{(Ax, x)^\nu}\right)^\lambda\right)$$

hold for $\mu, \nu \in (0,1)$, where $m(\mu,\nu) = \min\{\frac{1-\mu}{1-\nu}, \frac{\mu}{\nu}\}$ and $M(\mu,\nu) = \max\{\frac{1-\mu}{1-\nu}, \frac{\mu}{\nu}\}$. Moreover two inequalities in above are equivalent.

Proof. It follows from the preceding theorem that $f^\lambda(\mu)$ is a convex function by $\lambda \geq 1$.

If $\nu \geq \mu$, then we have

$$\frac{f^\lambda(\mu) - f^\lambda(0)}{\mu - 0} \leq \frac{f^\lambda(\nu) - f^\lambda(0)}{\nu - 0},$$

that is,

$$1 - f^\lambda(\mu) \geq \frac{\mu}{\nu}(1 - f^\lambda(\nu)).$$

Next, if $\mu \geq \nu$, then we have

$$\frac{f^\lambda(1) - f^\lambda(\mu)}{1 - \mu} \geq \frac{f^\lambda(1) - f^\lambda(\nu)}{1 - \nu},$$

that is,

$$1 - f^\lambda(\mu) \geq \frac{1-\mu}{1-\nu}(1 - f^\lambda(\nu)).$$

Hence the first inequality is proved. Finally, the equivalence between two inequalities is ensured by permuting μ and ν. Actually, if we do in the first inequality, then we have the second one by $\max\{a, b\} = [\min\{\frac{1}{a}, \frac{1}{b}\}]^{-1}$ for $a, b > 0$; the converse is shown by the same way. □

We here discuss the previous result under the case $\lambda \in (0, 1]$.

Theorem 2.4. *Let $A \geq 0$, $\|x\| = 1$ and $0 < \lambda \leq 1$. If $1 \geq \nu \geq \mu > 0$, then*

$$1 - \left(\frac{(A^\mu x, x)}{(Ax, x)^\mu} \right)^\lambda \geq \frac{\mu}{\nu} \left(1 - \left(\frac{(A^\nu x, x)}{(Ax, x)^\nu} \right)^\lambda \right).$$

Proof. It follows from the arithmetic-geometric mean inequality that

$$1 - \frac{\mu}{\nu} + \frac{\mu}{\nu} \left(\frac{(A^\nu x, x)}{(Ax, x)^\nu} \right)^\lambda \geq \left(\frac{(A^\nu x, x)}{(Ax, x)^\nu} \right)^{\lambda \cdot \frac{\mu}{\nu}} = \left(\frac{(A^\nu x, x)^{\frac{\mu}{\nu}}}{(Ax, x)^{\nu \frac{\mu}{\nu}}} \right)^\lambda \geq \left(\frac{(A^\mu x, x)}{(Ax, x)^\mu} \right)^\lambda$$

by $\frac{\mu}{\nu} \in (0, 1)$. □

3. Hölder–McCarthy inequality and Young inequality

We first give an elementary proof to the following known refinement of the Young inequality

Theorem 3.1. *Let $A \geq 0$ and $0 \leq \mu, \nu \leq 1$, and $m(\mu, \nu)$ and $M(\mu, \nu)$ be as in Theorem 2.3. Then*

$$m(\mu, \nu)(\nu A + 1 - \nu - A^\nu) \leq \mu A + 1 - \mu - A^\mu \leq M(\mu, \nu)(\nu A + 1 - \nu - A^\nu).$$

Moreover, two inequalities in above are equivalent.

Proof. It is sufficient to prove the numerical case for the left hand side, i.e.,

$$\mu a + 1 - \mu - a^\mu \geq m(\mu, \nu)(\nu a + 1 - \nu - a^\nu) \quad \text{for } a > 0.$$

If $\mu \geq \nu$, then $\frac{1-\mu}{1-\nu} \leq 1$ and $\frac{\mu-\nu}{1-\nu} + \frac{1-\mu}{1-\nu} = 1$ and so

$$\begin{aligned} \mu a + 1 - \mu &- \frac{1-\mu}{1-\nu}(\nu a + 1 - \nu - a^\nu) \\ &= \mu a - \frac{\nu(1-\mu)}{1-\nu} a + \frac{1-\mu}{1-\nu} a^\nu \\ &= \frac{\mu-\nu}{1-\nu} a + \frac{1-\mu}{1-\nu} a^\nu \\ &\geq a^{\frac{\mu-\nu}{1-\nu}} a^{\frac{\nu(1-\mu)}{1-\nu}} = a^\mu. \end{aligned}$$

If $\nu \geq \mu$, then

$$\mu a + 1 - \mu - \frac{\mu}{\nu}(\nu a + 1 - \nu - a^\nu) = 1 - \frac{\mu}{\nu} + \frac{\mu}{\nu} a^\nu \geq a^\mu.$$

Hence we have the first inequality.

The second inequality and the equivalence between two inequalities are obtained by $\max\{a, b\} = [\min\{\frac{1}{a}, \frac{1}{b}\}]^{-1}$ for $a, b > 0$, as in the proof of Theorem 2.3. □

Finally, we discuss the equivalence between refined Hölder–McCarthy inequality and refined Young inequality.

Theorem 3.2. *Refined Hölder–McCarthy inequality and refined Young inequality are equivalent, i.e.,*

$$1 - \frac{(A^\mu x, x)}{(Ax, x)^\mu} \geq m(\mu, \nu) \left(1 - \frac{(A^\nu x, x)}{(Ax, x)^\nu} \right) \quad \textit{for unit vectors } x, \tag{1}$$

$$\mu A + 1 - \mu - A^{\mu} \geq m(\mu,\nu)(\nu A + 1 - \nu - A^{\nu}) \tag{2}$$

are equivalent for given $\mu, \nu \in (0,1)$, where $m(\mu,\nu)$ is as in Theorem 2.3.

Proof. Assume that (1) holds and x is a unit vector. If $\nu \geq \mu$, then we have

$$\begin{aligned} &\mu(Ax,x) + 1 - \mu - \frac{\mu}{\nu}(\nu(Ax,x) + 1 - \nu - (A^{\nu}x,x)) \\ &= \frac{\nu-\mu}{\nu} + \frac{\mu}{\nu}(A^{\nu}x,x) \geq (A^{\nu}x,x)^{\frac{\mu}{\nu}} \geq (A^{\mu}x,x) \end{aligned}$$

by the (classical) Young inequality and Hölder–McCarthy inequality.

If $\mu \geq \nu$, then

$$\begin{aligned} &\mu(Ax,x) + 1 - \mu - \frac{1-\mu}{1-\nu}(\nu(Ax,x) + 1 - \nu - (A^{\nu}x,x)) \\ &= \left(\left(\frac{\mu-\nu}{1-\nu}A + \frac{1-\mu}{1-\nu}A^{\nu}\right)x,x\right) \geq (A^{\frac{\mu-\nu}{1-\nu}}A^{\frac{\nu(1-\mu)}{1-\nu}}x,x) = (A^{\mu}x,x). \end{aligned}$$

For the reverse implication (2) $\Rightarrow$ (1), we replace A by kA in (2) where $k = (Ax,x)^{-1}$. Thus we have

$$\begin{aligned} &\mu(Ax,x)^{-1}(Ax,x) + 1 - \mu - (Ax,x)^{-\mu}(A^{\mu}x,x) \\ &\geq m(\mu,\nu)(\nu(Ax,x)^{-1}(Ax,x) + 1 - \nu - (Ax,x)^{-\nu}(A^{\nu}x,x)), \end{aligned}$$

which is just arranged as (1), i.e.,

$$1 - \frac{(A^{\mu}x,x)}{(Ax,x)^{\mu}} \geq m(\mu,\nu)\left(1 - \frac{(A^{\nu}x,x)}{(Ax,x)^{\nu}}\right).$$

□

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