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DIRECT ESTIMATES OF CERTAIN MIHEŞAN-DURRMEYER TYPE OPERATORS

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Abstract. In this note, we consider a Durrmeyer type operator having the basis functions in summation and integration due to Miheşan [Creative Math. Inf. 17 (2008), 466–472.] and Păltănea [Carpathian J. Math. 24 (2008), no. 3, 378–385.] that preserve the linear functions. We present a Voronovskaja type theorem, local approximation theorem by means of second order modulus of continuity and weighted approximation for these operators. In the last section of the paper, we obtain the rate of approximation for absolutely continuous functions having a derivative equivalent with a function of bounded variation.

1. Introduction

Miheşan [19] constructed an important generalization of the well-known Szász operators depending on $\alpha \in \mathbb{R}$ as

$$\mathcal{G}_n^{(\alpha)}(f;x) = \sum_{k=0}^{\infty} m_{n,k}^{(\alpha)}(x) f\left(\frac{k}{n}\right),\ x \in [0,\infty), \tag{1.1}$$

where $m_{n,k}^{(\alpha)}(x) = \dfrac{(\alpha)_k}{k!} \dfrac{\left(\frac{nx}{\alpha}\right)^k}{\left(1+\frac{nx}{\alpha}\right)^{\alpha+k}}$, $(\alpha)_k = \alpha(\alpha+1)\cdots(\alpha+k-1), (\alpha)_0 = 1$, is the rising factorial and $\alpha + nx > 0$. The operator $\mathcal{G}_n^{(\alpha)}$ preserve the linear polynomials, and for special values of α, one can obtain some well-known operators.

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Gupta and Noor [12] introduced certain mixed Szász-beta operators and studied some direct results and an error estimate in simultaneous approximation. Păltănea [20] presented a modification of the Phillips operators by considering the generalized basis functions under integration based on certain parameter $\rho > 0$. Gupta and Rassias [14] defined a Durrmeyer modification of certain Szász type operators and established some approximation properties e.g. weighted approximation, asymptotic formula and error estimation in terms of modulus of smoothness, in [24] Sucu and Varma proposed a Stancu type modification of Jakimovski-Levitan type Szász operators involving Sheffer polynomials and derived rate of convergence for these operators with the help of Korovkin theorem. In 2015, Acar [1] constructed the general Szász-Mirakyan operators and obtained quantitative Grüss type Voronovskaya theorems by with the help of weighted modulus of continuity. Goyal et al. [10] presented a one parameter family of mixed Durrmeyer type operators. They established some quantitative estimates for the rate of convergence. Gupta [11] constructed a sequence of hybrid operators with weights of the Păltănea basis function and studied some approximation properties of these operators. Acu and Gupta [2] proposed mixed hybrid operators involving two parameters. They proved the Korovkin type approximation theorem and the rate of approximation for unbounded functions with derivatives of bounded variation. Very recently, Kajla et al. [17] introduced the Baskakov-Szász type operators involving inverse Pólya-Eggenberger distribution and investigated their direct results. Erençin and Raşa [6] constructed and established a modification of Gamma operators. In the literature, many researchers have studied the approximation properties of different summation-integral type operators (cf. [3], [7], [9], [13], [16], [23], [26] etc.)

2. Construction of Operators and Basic Results

Let $\rho > 0$. For $\gamma > 0$ and $C_\gamma[0,\infty) := \Big\{ f \in C[0,\infty) : |f(t)| \le N_f e^{\gamma t},$ for some $N_f > 0 \Big\}$, we construct a new sequence of summation-integral type operators as follows:

$$\mathcal{P}_{n,\rho}^{(\alpha)}(f;x) = \sum_{k=1}^{\infty} m_{n,k}^{(\alpha)}(x) \int_0^\infty s_{n,k}^{\rho}(t) f(t)dt + m_{n,0}^{(\alpha)}(x) f(0), \tag{2.1}$$

where $s_{n,k}^{\rho}(t) = n\rho e^{-n\rho t} \dfrac{(n\rho t)^{k\rho-1}}{\Gamma(k\rho)}$ and $m_{n,k}^{(\alpha)}(x)$ is defined as above. It is important to note here that this form of operators (2.1) preserve constant as well as linear functions. Also, these operators in limiting case include to the discrete operators due to Miheşan [19].

Special cases:

(1) If $\alpha \to \infty$, we get the operators due to Păltănea [20]. Also, for this case if $\rho = 1$, we get the Phillips operators [22].

(2) If $\alpha = n$ and $\rho \to \infty$, we get the classical Baskakov operators [4].
(3) Similarly, if $\alpha = -n$ and $\rho \to \infty$, these operators include Bernstein operators [5].
(4) If $\alpha = nx$ and $\rho \to \infty$, we obtain Lupaş operator [18].

The present article deals with some direct results of the operators $\mathcal{P}_{n,\rho}^{(\alpha)}$. Here we obtain a Voronovskaja asymptotic formula, direct estimate by means of smoothness, Lipschitz type maximal space, weighted approximation and the rate of approximation for functions having derivatives of bounded variation.

Let $e_i(t) = t^i, i = \overline{0,6}$.

Lemma 2.1. *For the operators $\mathcal{G}_n^{(\alpha)}(f;x)$, we have*

(i) $\mathcal{G}_n^{(\alpha)}(e_0;x) = 1$,
(ii) $\mathcal{G}_n^{(\alpha)}(e_1;x) = x$,
(iii) $\mathcal{G}_n^{(\alpha)}(e_2;x) = x^2 + \frac{x(nx+\alpha)}{n\alpha}$,
(iv) $\mathcal{G}_n^{(\alpha)}(e_3;x) = \frac{x^3(1+\alpha)(2+\alpha)}{\alpha^2} + \frac{3x^2(1+\alpha)}{n\alpha} + \frac{x}{n^2}$,
(v) $\mathcal{G}_n^{(\alpha)}(e_4;x) = \frac{x^4(1+\alpha)(2+\alpha)(3+\alpha)}{\alpha^3} + \frac{6x^3(1+\alpha)(2+\alpha)}{n\alpha^2} + \frac{7x^2(1+\alpha)}{n^2\alpha} + \frac{x}{n^3}$,
(vi) $\mathcal{G}_n^{(\alpha)}(e_5;x) = \frac{x^5(1+\alpha)(2+\alpha)(3+\alpha)(4+\alpha)}{\alpha^4} + \frac{10x^4(1+\alpha)(2+\alpha)(3+\alpha)}{n\alpha^3} + \frac{25x^3(1+\alpha)(2+\alpha)}{n^2\alpha^2} + \frac{15x^2(1+\alpha)}{n^3\alpha} + \frac{x}{n^4}$,
(vii) $\mathcal{G}_n^{(\alpha)}(e_6;x) = \frac{x^6(1+\alpha)(2+\alpha)(3+\alpha)(4+\alpha)(5+\alpha)}{\alpha^5} + \frac{15x^5(1+\alpha)(2+\alpha)(3+\alpha)(4+\alpha)}{n\alpha^4} + \frac{65x^4(1+\alpha)(2+\alpha)(3+\alpha)}{n^2\alpha^3} + \frac{90x^3(1+\alpha)(2+\alpha)}{n^3\alpha^2} + \frac{31x^2(1+\alpha)}{n^4\alpha} + \frac{x}{n^5}$.

Proof. The proofs of the parts $(i)-(iii)$ are given in ([19], Lemma 4.1). The proof of $(iv)-(vii)$ can be computed following the same idea of proof of ([19], Lemma 4.1). □

Lemma 2.2. *For the operators $\mathcal{P}_{n,\rho}^{(\alpha)}(f;x)$, we have*

(i) $\mathcal{P}_{n,\rho}^{(\alpha)}(e_0;x) = 1$,
(ii) $\mathcal{P}_{n,\rho}^{(\alpha)}(e_1;x) = x$,
(iii) $\mathcal{P}_{n,\rho}^{(\alpha)}(e_2;x) = \frac{x^2(1+\alpha)}{\alpha} + \frac{x(1+\rho)}{n\rho}$,
(iv) $\mathcal{P}_{n,\rho}^{(\alpha)}(e_3;x) = \frac{x^3(1+\alpha)(2+\alpha)}{\alpha^2} + \frac{3x^2(1+\alpha)(1+\rho)}{n\alpha\rho} + \frac{x(1+\rho)(2+\rho)}{n^2\rho^2}$,
(v) $\mathcal{P}_{n,\rho}^{(\alpha)}(e_4;x) = \frac{x^4(1+\alpha)(2+\alpha)(3+\alpha)}{\alpha^3} + \frac{6x^3(1+\alpha)(2+\alpha)(1+\rho)}{n\alpha^2\rho} + \frac{x^2(1+\alpha)(1+\rho)(11+7\rho)}{n^2\alpha\rho^2} + \frac{x(1+\rho)(2+\rho)(3+\rho)}{n^3\rho^3}$,
(vi) $\mathcal{P}_{n,\rho}^{(\alpha)}(e_5;x) = \frac{x^5(1+\alpha)(2+\alpha)(3+\alpha)(4+\alpha)}{\alpha^4} + \frac{10x^4(1+\alpha)(2+\alpha)(3+\alpha)(1+\rho)}{n\rho\alpha^3} + \frac{5x^3(1+\alpha)(2+\alpha)(1+\rho)(7+5\rho)}{n^2\rho^2\alpha^2} + \frac{5x^2(1+\alpha)(1+\rho)(2+\rho)(5+3\rho)}{n^3\rho^3\alpha} + \frac{x(1+\rho)(2+\rho)(3+\rho)(4+\rho)}{n^4\rho^4}$,
(vii) $\mathcal{P}_{n,\rho}^{(\alpha)}(e_6;x) = \frac{x^6(1+\alpha)(2+\alpha)(3+\alpha)(4+\alpha)(5+\alpha)}{\alpha^5} + \frac{15x^5(1+\alpha)(2+\alpha)(3+\alpha)(4+\alpha)(1+\rho)}{n\rho\alpha^4} + \frac{5x^4(1+\alpha)(2+\alpha)(3+\alpha)(1+\rho)(17+13\rho)}{n^2\rho^2\alpha^3} + \frac{15x^3(1+\alpha)(2+\alpha)(1+\rho)(3+2\rho)(5+3\rho)}{n^3\rho^3\alpha^2} + \frac{x^2(1+\alpha)(1+\rho)(2+\rho)(137+\rho(132+31\rho))}{n^4\rho^4\alpha} + \frac{x(1+\rho)(2+\rho)(3+\rho)(4+\rho)(5+\rho)}{n^5\rho^5}$.

Proof. This lemma follows easily using Lemma 2.1 and the definition of Gamma function,

$$\int_0^\infty s_{n,k}^{\rho}(t)t^l dt = \int_0^\infty n\rho e^{-n\rho t}\frac{(n\rho t)^{k\rho-1}}{\Gamma(k\rho)}t^l = \frac{\Gamma(k\rho+l)}{\Gamma(k\rho)(n\rho)^l}, l=\overline{0,6}.$$

Hence the details are omitted. ☐

Lemma 2.3. *From Lemma 2.2, we obtain*

(i) $\mathcal{P}_{n,\rho}^{(\alpha)}(t-x;x)=0,$

(ii) $\mathcal{P}_{n,\rho}^{(\alpha)}((t-x)^2;x)=\frac{x^2}{\alpha}+\frac{x(1+\rho)}{n\rho},$

(iii) $\mathcal{P}_{n,\rho}^{(\alpha)}((t-x)^4;x)=\frac{3x^4(2+\alpha)}{\alpha^3}+\frac{6x^3(2+\alpha)(1+\rho)}{n\alpha^2\rho}+\frac{x^2(1+\rho)(11+7\rho+3\alpha(1+\rho))}{n^2\alpha\rho^2}$
$+\frac{x(1+\rho)(2+\rho)(3+\rho)}{n^3\rho^3},$

(iv) $\mathcal{P}_{n,\rho}^{(\alpha)}((t-x)^6;x)=\frac{5x^6(24+\alpha(26+3\alpha))}{\alpha^5}+\frac{15x^5(24+\alpha(26+3\alpha))(1+\rho)}{n\alpha^4\rho}$
$+\frac{5x^4(1+\rho)(102+78\rho+\alpha(103+83\rho+9\alpha(1+\rho)))}{n^2\alpha^3\rho^2}$
$+\frac{15x^3(1+\rho)\left(\alpha^2(1+\rho)^2+2(3+2\rho)(5+3\rho)+\alpha(5+3\rho)(5+4\rho)\right)}{n^3\alpha^2\rho^3}$
$+\frac{x^2(1+\rho)(2+\rho)(137+5\alpha(1+\rho)(13+5\rho)+\rho(132+31\rho))}{n^4\rho^4\alpha}+\frac{x(1+\rho)(2+\rho)(3+\rho)(4+\rho)(5+\rho)}{n^5\rho^5}.$

Remark 2.1. *If* $\alpha=\alpha(n)\to\infty$, *as* $n\to\infty$ *and* $\lim_{n\to\infty}\frac{n}{\alpha(n)}=c\in\mathbb{R}$, *then*

$$\lim_{n\to\infty} n\ \Theta_{n,\rho}^{\alpha,1}(x) = 0,$$
$$\lim_{n\to\infty} n\ \Theta_{n,\rho}^{\alpha,2}(x) = cx^2+x\left(1+\frac{1}{\rho}\right),$$
$$\lim_{n\to\infty} n^2\ \Theta_{n,\rho}^{\alpha,4}(x) = 3cx^4+6x^3\left(c+\frac{1}{\rho}\right)+3x^2\left(1+\frac{1}{\rho}\right)^2,$$

where $\Theta_{n,\rho}^{\alpha,m}:=\mathcal{P}_{n,\rho}^{(\alpha)}((t-x)^m;x), m=1,2,4.$

3. Direct Results

Theorem 3.1. *Let* $f\in C_\gamma[0,\infty)$ *and* $\alpha=\alpha(n)\to\infty$ *as* $n\to\infty$. *Then* $\lim_{n\to\infty}\mathcal{P}_{n,\rho}^{(\alpha)}(f;x)=f(x)$, *uniformly in each compact subset of* $[0,\infty)$.

Proof. From Lemma 2.2, $\mathcal{P}_{n,\rho}^{(\alpha)}(e_0;x)=1$, $\mathcal{P}_{n,\rho}^{(\alpha)}(e_1;x)\to x$, $\mathcal{P}_{n,\rho}^{(\alpha)}(e_2;x)\to x^2$, as $n\to\infty$ uniformly in each compact subset of $[0,\infty)$. By Bohman-Korovkin Theorem, it follows that $\mathcal{P}_{n,\rho}^{(\alpha)}(f;x)\to f(x)$, as $n\to\infty$ uniformly in each compact subset of $[0,\infty)$. ☐

3.1. **Voronovskaja type theorem.**

In this section, we establish Voronvoskaja type result for the $\mathcal{P}_{n,\rho}^{(\alpha)}$ operators.

Theorem 3.2. *Let* $f\in C_\gamma[0,\infty)$ *and* $\alpha=\alpha(n)\to\infty$, *as* $n\to\infty$. *If* f'' *exists at a point* $x\in[0,\infty)$ *and* $\lim_{n\to\infty}\frac{n}{\alpha(n)}=c\in\mathbb{R}$, *then we have*

$$\lim_{n\to\infty} n\left[\mathcal{P}_{n,\rho}^{(\alpha)}(f;x)-f(x)\right]=\frac{1}{2}\left[cx^2+x\left(1+\frac{1}{\rho}\right)\right]f''(x).$$

Proof. Applying Taylor's expansion, we can write

$$f(t) = f(x) + f'(x)(t-x) + \frac{1}{2}f''(x)(t-x)^2 + \varepsilon(t,x)(t-x)^2, \tag{3.1}$$

where $\lim_{t\to x}\varepsilon(t,x) = 0$. By using the linearity of the operator $\mathcal{P}_{n,\rho}^{(\alpha)}$, we get

$$\mathcal{P}_{n,\rho}^{(\alpha)}(f;x) - f(x) = \mathcal{P}_{n,\rho}^{(\alpha)}((t-x);x)f'(x) + \frac{1}{2}\mathcal{P}_{n,\rho}^{(\alpha)}((t-x)^2;x)f''(x) + \mathcal{P}_{n,\rho}^{(\alpha)}(\varepsilon(t,x)(t-x)^2;x).$$

Applying the Cauchy-Schwarz inequality, we obtain

$$n\mathcal{P}_{n,\rho}^{(\alpha)}(\varepsilon(t,x)(t-x)^2;x) \leq \sqrt{\mathcal{P}_{n,\rho}^{(\alpha)}(\varepsilon^2(t,x);x)}\sqrt{n^2\mathcal{P}_{n,\rho}^{(\alpha)}((t-x)^4;x)}. \tag{3.2}$$

In view of Theorem 3.1, we get

$$\lim_{n\to\infty}\mathcal{P}_{n,\rho}^{(\alpha)}(\varepsilon^2(t,x);x) = \varepsilon^2(x,x) = 0. \tag{3.3}$$

Combining (3.2)-(3.3) and Remark 2.1, we have

$$\lim_{n\to\infty} n\mathcal{P}_{n,\rho}^{(\alpha)}(\varepsilon(t,x)(t-x)^2;x) = 0. \tag{3.4}$$

Hence

$$\lim_{n\to\infty} n\left[\mathcal{P}_{n,\rho}^{(\alpha)}(f;x) - f(x)\right] = \frac{1}{2}\left[cx^2 + x\left(1+\frac{1}{\rho}\right)\right]f''(x).$$

Thus the theorem follows immediately by using Remark 2.1. □

3.2. **Local approximation.** Let $\tilde{C_B}[0,\infty)$ be the space of all real valued bounded and uniformly continuous functions f on $[0,\infty)$, endowed with the norm

$$||f||_{\tilde{C_B}[0,\infty)} = \sup_{x\in[0,\infty)}|f(x)|.$$

For $f \in \tilde{C_B}[0,\infty)$, the Steklov mean is defined as

$$f_h(x) = \frac{4}{h^2}\int_0^{\frac{h}{2}}\int_0^{\frac{h}{2}}\left[2f(x+u+v) - f(x+2(u+v))\right]du\, dv. \tag{3.5}$$

By simple computation, it is observed that

a) $\|f_h - f\|_{\tilde{C_B}[0,\infty)} \leq \omega_2(f,h)$.

b) $f_h', f_h'' \in \tilde{C_B}[0,\infty)$ and $\|f_h'\|_{\tilde{C_B}[0,\infty)} \leq \frac{5}{h}\omega(f,h)$, $\quad \|f_h''\|_{\tilde{C_B}[0,\infty)} \leq \frac{9}{h^2}\omega_2(f,h)$,

where the second order modulus of continuity is defined as

$$\omega_2(f,\delta) = \sup_{x,u,v\geq 0}\ \sup_{|u-v|\leq\delta}|f(x+2u) - 2f(x+u+v) + f(x+2v)|,\ \ \delta \geq 0.$$

The usual modulus of continuity of $f \in \tilde{C_B}[0,\infty)$ is given by

$$\omega(f,\delta) = \sup_{x,u,v\geq 0}\ \sup_{|u-v|\leq\delta}|f(x+u) - f(x+v)|.$$

Theorem 3.3. *Let $f \in \tilde{C_B}[0,\infty)$. Then for every $x \geq 0$, the following inequality holds*

$$\left|\mathcal{P}_{n,\rho}^{(\alpha)}(f;x) - f(x)\right| \leq 5\omega\left(f, \sqrt{\Theta_{n,\rho}^{\alpha,2}(x)}\right) + \frac{13}{2}\omega_2\left(f, \sqrt{\Theta_{n,\rho}^{\alpha,2}(x)}\right).$$

Proof. For $x \geq 0$, and applying the Steklov mean f_h that is given by (3.5), we can write

$$\left|\mathcal{P}_{n,\rho}^{(\alpha)}(f;x) - f(x)\right| \leq \mathcal{P}_{n,\rho}^{(\alpha)}\left(|f-f_h|;x\right) + \left|\mathcal{P}_{n,\rho}^{(\alpha)}\left(f_h - f_h(x);x\right)\right| + |f_h(x) - f(x)|. \tag{3.6}$$

From (2.1), for every $f \in \tilde{C}_B[0,\infty)$ we have

$$\left|\mathcal{P}_{n,\rho}^{(\alpha)}(f;x)\right| \leq ||f||_{\tilde{C_B}[0,\infty)}. \tag{3.7}$$

Using property (a) of Steklov mean and (3.7), we get

$$\mathcal{P}_{n,\rho}^{(\alpha)}\left(|f-f_h|;x\right) \leq \|\mathcal{P}_{n,\rho}^{(\alpha)}\left(f-f_h\right)\|_{\tilde{C_B}[0,\infty)} \leq \|f-f_h\|_{\tilde{C_B}[0,\infty)} \leq \omega_2(f,h).$$

By Taylor's expansion and Cauchy-Schwarz inequality, we have

$$\begin{aligned}\left|\mathcal{P}_{n,\rho}^{(\alpha)}\left(f_h - f_h(x);x\right)\right| &\leq \|f_h'\|_{\tilde{C_B}[0,\infty)}\sqrt{\mathcal{P}_{n,\rho}^{(\alpha)}\left((t-x)^2;x\right)} \\ &\quad + \frac{1}{2}\|f_h''\|_{\tilde{C_B}[0,\infty)}\mathcal{P}_{n,\rho}^{(\alpha)}\left((t-x)^2;x\right).\end{aligned}$$

By Lemma 2.3 and property (b) of Steklov mean, we obtain

$$\left|\mathcal{P}_{n,\rho}^{(\alpha)}\left(f_h - f_h(x);x\right)\right| \leq \frac{5}{h}\omega(f,h)\sqrt{\Theta_{n,\rho}^{\alpha,2}(x)} + \frac{9}{2h^2}\omega_2(f,h)\Theta_{n,\rho}^{\alpha,2}(x).$$

Choosing $h = \sqrt{\Theta_{n,\rho}^{\alpha,2}(x)}$, and substituting the values of the above estimates in (3.6), we get the desired relation. □

Let $\mu_1 \geq 0, \mu_2 > 0$ be fixed. We consider the following Lipschitz-type space (see [21]):

$$Lip_M^{(\mu_1,\mu_2)}(r) := \left\{f \in C[0,\infty) : |f(t)-f(x)| \leq M\frac{|t-x|^r}{(t+\mu_1x^2+\mu_2x)^{\frac{r}{2}}};\ x,t \in (0,\infty)\right\},$$

where $0 < r \leq 1$.

Theorem 3.4. *Let $f \in Lip_M^{(\mu_1,\mu_2)}(r)$ and $r \in (0,1]$. Then, for all $x \in (0,\infty)$, we have*

$$\left|\ \mathcal{P}_{n,\rho}^{(\alpha)}(f;x) - f(x)\ \right| \leq M\left(\frac{\Theta_{n,\rho}^{\alpha,2}(x)}{\mu_1x^2+\mu_2x}\right)^{\frac{r}{2}}.$$

Proof. Using Hölder's inequality with $p=\frac{2}{r}, q=\frac{2}{2-r}$, we obtain
$| \mathcal{P}_{n,\rho}^{(\alpha)}(f;x)-f(x) |$

$$\begin{aligned}
&= \sum_{k=1}^{\infty} m_{n,k}^{(\alpha)}(x)\int_0^{\infty} s_{n,k}^{\rho}(t)|f(t)-f(x)|dt + m_{n,0}^{(\alpha)}(x)|f(0)-f(x)| \\
&\leq \sum_{k=1}^{\infty} m_{n,k}^{(\alpha)}(x)\left(\int_0^{\infty} s_{n,k}^{\rho}(t)|f(t)-f(x)|^{\frac{2}{r}}dt\right)^{\frac{r}{2}} + m_{n,0}^{(\alpha)}(x)|f(0)-f(x)| \\
&\leq \left\{\sum_{k=1}^{\infty} m_{n,k}^{(\alpha)}(x)\int_0^{\infty} s_{n,k}^{\rho}(t)|f(t)-f(x)|^{\frac{2}{r}}dt + m_{n,0}^{(\alpha)}(x)|f(0)-f(x)|^{\frac{2}{r}}\right\}^{\frac{r}{2}} \\
&\quad \times\left(\sum_{k=0}^{\infty} m_{n,k}^{(\alpha)}(x)\right)^{\frac{2-r}{2}} \\
&= \left\{\sum_{k=1}^{\infty} m_{n,k}^{(\alpha)}\int_0^{\infty} s_{n,k}^{\rho}(t)|f(t)-f(x)|^{\frac{2}{r}}dt + m_{n,0}^{(\alpha)}(x)|f(0)-f(x)|^{\frac{2}{r}}\right\}^{\frac{r}{2}} \\
&\leq M\left(\sum_{k=1}^{\infty} m_{n,k}^{(\alpha)}(x)\int_0^{\infty} s_{n,k}^{\rho}(t)\frac{(t-x)^2}{(t+\mu_1x^2+\mu_2x)}dt + m_{n,0}^{(\alpha)}(x)\frac{x^2}{(\mu_1x^2+\mu_2x)}\right)^{\frac{r}{2}} \\
&\leq \frac{M}{(\mu_1x^2+\mu_2x)^{\frac{r}{2}}}\left(\sum_{k=1}^{\infty} m_{n,k}^{(\alpha)}(x)\int_0^{\infty} s_{n,k}^{\rho}(t)(t-x)^2dt + x^2m_{n,0}^{(\alpha)}(x)\right)^{\frac{r}{2}} \\
&= \frac{M}{(\mu_1x^2+\mu_2x)^{\frac{r}{2}}}\left(\mathcal{P}_{n,\rho}^{(\alpha)}((t-x)^2;x)\right)^{\frac{r}{2}} = \frac{M}{(\mu_1x^2+\mu_2x)^{\frac{r}{2}}}(\Theta_{n,\rho}^{\alpha,2}(x))^{\frac{r}{2}}.
\end{aligned}$$

Thus, the proof is completed. □

Theorem 3.5. *For any $f\in \tilde{C_B^1}[0,\infty)$ and $x\in[0,\infty)$, we have*

$$\left|\mathcal{P}_{n,\rho}^{(\alpha)}(f;x)-f(x)\right| \leq 2\sqrt{\Theta_{n,\rho}^{\alpha,2}(x)}\ \omega\left(f', \sqrt{\Theta_{n,\rho}^{\alpha,2}(x)}\ \right). \tag{3.8}$$

Proof. Let $f\in \tilde{C_B^1}[0,\infty)$. For any $t,x\in[0,\infty)$, we have

$$f(t)-f(x)=f'(x)(t-x)+\int_x^t (f'(u)-f'(x))\,du.$$

Applying $\mathcal{P}_{n,\rho}^{(\alpha)}(\cdot;x)$ on both sides of the above relation, we get

$$\mathcal{P}_{n,\rho}^{(\alpha)}(f(t)-f(x);x)=f'(x)\mathcal{P}_{n,\rho}^{(\alpha)}(t-x;x)+\mathcal{P}_{n,\rho}^{(\alpha)}\left(\int_x^t (f'(u)-f'(x))\,du;x\right).$$

Using the well known property of modulus of continuity

$$|f(t)-f(x)|\leq \omega(f,\delta)\left(\frac{|t-x|}{\delta}+1\right), \delta>0,$$

we obtain

$$\left|\int_x^t (f'(u)-f'(x))\,du\right| \le \omega(f',\delta)\left(\frac{(t-x)^2}{\delta}+|t-x|\right).$$

Therefore, it follows

$$\begin{aligned}\left|\mathcal{P}_{n,\rho}^{(\alpha)}(f;x)-f(x)\right| &\le |f'(x)|\,|\mathcal{P}_{n,\rho}^{(\alpha)}(t-x;x)| \\ &\quad +\omega(f',\delta)\left\{\frac{1}{\delta}\mathcal{P}_{n,\rho}^{(\alpha)}((t-x)^2;x)+\mathcal{P}_{n,\rho}^{(\alpha)}(|t-x|;x)\right\}.\end{aligned}$$

Using Cauchy-Schwarz inequality, we have

$$\begin{aligned}\left|\mathcal{P}_{n,\rho}^{(\alpha)}(f;x)-f(x)\right| &\le |f'(x)|\,|\mathcal{P}_{n,\rho}^{(\alpha)}(t-x;x)| \\ &\quad +\omega(f',\delta)\left\{\frac{1}{\delta}\sqrt{\mathcal{P}_{n,\rho}^{(\alpha)}((t-x)^2;x)}+1\right\}\sqrt{\mathcal{P}_{n,\rho}^{(\alpha)}((t-x)^2;x)}.\end{aligned}$$

Choosing $\delta=\sqrt{\Theta_{n,\rho}^{\alpha,2}(x)}$, the required result follows. □

Let $H_\xi[0,\infty)$ be the space of all real valued functions on $[0,\infty)$ satisfying the condition $|f(x)|\le N_f\xi(x)$, where N_f is a positive constant depending only on f and $\xi(x)=1+x^2$ is a weight function. Let $C_\xi[0,\infty)$ be the space of all continuous functions in $H_\xi[0,\infty)$ endowed with the norm

$$\|f\|_\xi := \sup_{x\in[0,\infty)}\frac{|f(x)|}{\xi(x)}$$

and

$$C_\xi^0[0,\infty) := \left\{f\in C_\xi[0,\infty) : \lim_{x\to\infty}\frac{|f(x)|}{\xi(x)} \quad \text{exists and is finite}\right\}.$$

The usual modulus of continuity of f on $[0,b]$ is defined as

$$\omega_b(f,\delta)=\sup_{0<|t-x|\le\delta}\ \sup_{x,t\in[0,b]}|f(t)-f(x)|.$$

Theorem 3.6. *Let $f\in C_\xi[0,\infty)$. Then, we have*

$$|\mathcal{P}_{n,\rho}^{(\alpha)}(f;x)-f(x)|\le 4N_f(1+x^2)\Theta_{n,\rho}^{\alpha,2}(x)+2\omega_{b+1}\left(f,\sqrt{\Theta_{n,\rho}^{\alpha,2}(x)}\right). \tag{3.9}$$

Proof. From [15], for $x\in[0,b]$ and $t\ge 0$, we have

$$|f(t)-f(x)|\le 4N_f(1+x^2)(t-x)^2+\left(1+\frac{|t-x|}{\delta}\right)\omega_{b+1}(f,\delta),\ \ \delta>0.$$

By Cauchy-Schwarz inequality, we may write

$$\begin{aligned}|\mathcal{P}_{n,\rho}^{(\alpha)}(f;x)-f(x)| &\le 4N_f(1+x^2)\mathcal{P}_{n,\rho}^{(\alpha)}((t-x)^2;x)+\omega_{b+1}(f,\delta)\left(1+\frac{1}{\delta}\mathcal{P}_{n,\rho}^{(\alpha)}(|t-x|;x)\right) \\ &\le 4N_f(1+x^2)\Theta_{n,\rho}^{\alpha,2}(x)+\omega_{b+1}(f,\delta)\left(1+\frac{1}{\delta}\sqrt{\Theta_{n,\rho}^{\alpha,2}(x)}\right).\end{aligned}$$

Now, choosing $\delta=\sqrt{\Theta_{n,\rho}^{\alpha;2}(x)}$, we obtain (3.9). □

4. Weighted approximation

Theorem 4.1. *Let $f \in C_\xi^0[0,\infty)$ and $\alpha = \alpha(n) \to \infty$ as $n \to \infty$. Then, we have*

$$\lim_{n\to\infty} \|\mathcal{P}_{n,\rho}^{(\alpha)}(f) - f\|_\xi = 0. \tag{4.1}$$

Proof. In order to prove this result it is sufficient to verify the following three relations (see [8])

$$\lim_{n\to\infty} \|\mathcal{P}_{n,\rho}^{(\alpha)}(t^m;x) - x^m\|_\xi = 0,\ m = 0,1,2. \tag{4.2}$$

Since $\mathcal{P}_{n,\rho}^{(\alpha)}(1;x) = 1$, the condition in (4.2) holds true for $m = 0$.
By Lemma 2.2, we have

$$\|\mathcal{P}_{n,\rho}^{(\alpha)}(t;x) - x\|_\xi = \sup_{x\geq 0} \frac{1}{1+x^2}|x-x| = 0. \tag{4.3}$$

Thus, $\lim_{n\to\infty} \|\mathcal{P}_{n,\rho}^{(\alpha)}(t;x) - x\|_\xi = 0$.
Finally, we obtain

$$\begin{aligned} \|\mathcal{P}_{n,\rho}^{(\alpha)}(t^2;x) - x^2\|_\xi &= \sup_{x\geq 0} \frac{1}{1+x^2}\left| x^2 + \frac{x(nx+\alpha)}{n\alpha} - x^2 \right| \\ &\leq \sup_{x\geq 0} \frac{x^2}{1+x^2}\frac{1}{|\alpha|} + \sup_{x\geq 0}\frac{x}{1+x^2}\frac{1}{n}, \end{aligned} \tag{4.4}$$

which implies that $\lim_{n\to\infty} \|\mathcal{P}_{n,\rho}^{(\alpha)}(t^2;x) - x^2\|_\xi = 0$. □

Lemma 4.1. [26] *Let $f \in C_\xi^0[0,\infty)$, then:*

i) *$\Omega(f;\delta)$ is a monotone increasing function of δ;*
ii) $\lim_{\delta\to 0^+} \Omega(f;\delta) = 0$;
iii) *for each $m \in \mathbb{N}, \Omega(f,m\delta) \leq m\Omega(f;\delta)$;*
iv) *for each $\lambda \in [0,\infty), \Omega(f;\lambda\delta) \leq (1+\lambda)\Omega(f;\delta)$.*

Theorem 4.2. *Let $f \in C_\xi^0[0,\infty)$. If $\alpha = \alpha(n) \to \infty$, as $n \to \infty$ and $\lim_{n\to\infty} \frac{n}{\alpha(n)} = c \in \mathbb{R}$, then there exists $\tilde{n} \in \mathbb{N}$ and a positive constant $M(\rho,c)$ depending on ρ and c such that*

$$\sup_{x\in(0,\infty)} \frac{|\mathcal{P}_{n,\rho}^{(\alpha)}(f;x) - f(x)|}{(1+x^2)^{\frac{5}{2}}} \leq M(\rho,c)\Omega\left(f;n^{-1/2}\right),\ \textit{for } n > \tilde{n}. \tag{4.5}$$

Proof. For $t > 0, x \in (0,\infty)$ and $\delta > 0$, by the definition of $\Omega(f;\delta)$ and Lemma 4.1, we may write

$$\begin{aligned} |f(t) - f(x)| &\leq (1+(x+|x-t|)^2)\Omega(f;|t-x|) \\ &\leq 2(1+x^2)(1+(t-x)^2)\left(1+\frac{|t-x|}{\delta}\right)\Omega(f;\delta). \end{aligned}$$

Since the operator $\mathcal{P}_{n,\rho}^{(\alpha)}$ is linear and positive, we get

$$|\mathcal{P}_{n,\rho}^{(\alpha)}(f;x)-f(x)| \leq 2(1+x^2)\Omega(f;\delta)\left\{1+\mathcal{P}_{n,\rho}^{(\alpha)}((t-x)^2;x)\right.$$
$$\left.+\mathcal{P}_{n,\rho}^{(\alpha)}\left((1+(t-x)^2)\frac{|t-x|}{\delta};x\right)\right\}. \tag{4.6}$$

From the Remark 2.1 it follows that there is $n_1 \in \mathbb{N}$ such that

$$\mathcal{P}_{n,\rho}^{(\alpha)}((t-x)^2;x) \leq M_1(\rho,c)\frac{(1+x^2)}{n}, \text{ for } n > n_1, \tag{4.7}$$

where $M_1(\rho,c)$ is a positive constant depending on ρ and c. Using Cauchy-Schwarz inequality, we can write

$$\mathcal{P}_{n,\rho}^{(\alpha)}\left((1+(t-x)^2)\frac{|t-x|}{\delta};x\right) \leq \frac{1}{\delta}\sqrt{\mathcal{P}_{n,\rho}^{(\alpha)}((t-x)^2;x)} \tag{4.8}$$
$$+\frac{1}{\delta}\sqrt{\mathcal{P}_{n,\rho}^{(\alpha)}((t-x)^4;x)}\sqrt{\mathcal{P}_{n,\rho}^{(\alpha)}((t-x)^2;x)}.$$

From the Remark 2.1 it follows that there is $n_2 \in \mathbb{N}$ such that

$$\sqrt{\mathcal{P}_{n,\rho}^{(\alpha)}((t-x)^4;x)} \leq M_2(\rho,c)\frac{(1+x^2)}{n}, \text{ for } n > n_2, \tag{4.9}$$

where $M_2(\rho,c)$ is a positive constant depending on ρ and c.

Let $\tilde{n} = \max\{n_1, n_2\}$. Collecting the estimates (4.6)-(4.9) and choosing

$$M(\rho,c) = 2\left(1+M_1(\rho,c)+\sqrt{M_1(\rho,c)}+M_2(\rho,c)\sqrt{M_1(\rho,c)}\right),$$

$\delta = \frac{1}{\sqrt{n}}$, for $n > \tilde{n}$, we get the required result (4.5). □

5. Rate of convergence

In this section, we discuss the approximation of functions with a derivative of bounded variation.

Let $DBV[0,\infty)$ be the class of all functions $f \in H_\xi[0,\infty)$, having a derivative of bounded variation on every finite subinterval of $[0,\infty)$. The function $f \in DBV[0,\infty)$ has the following representation

$$f(x) = \int_0^x g(t) + f(0),$$

where g is a function of bounded variation on each finite subinterval of $[0,\infty)$.

In order to study the convergence of the operators $\mathcal{P}_{n,\rho}^{(\alpha)}$ for functions having a derivative of bounded variation, we rewrite the operators (2.1) as follows

$$\mathcal{P}_{n,\rho}^{(\alpha)}(f;x) = \int_0^\infty \mathcal{S}_{n,\rho}^{(\alpha)}(x,t)f(t)dt, \tag{5.1}$$

$$\mathcal{S}_{n,\rho}^{(\alpha)}(x,t) = \sum_{k=1}^{\infty} m_{n,k}^{(\alpha)}(x)s_{n,k}^{\rho}(t) + m_{n,0}^{(\alpha)}(x)\delta(t),$$

$\delta(t)$ being the Dirac-delta function.

Lemma 5.1. *Let $\alpha = \alpha(n) \to \infty$, as $n \to \infty$ and $\lim_{n\to\infty} \frac{n}{\alpha(n)} = c \in \mathbb{R}$. For all $x \in (0,\infty)$ and sufficiently large n, we have*

$$\text{i) } \lambda_{n,\rho}^{(\alpha)}(x,t) = \int_0^t \mathcal{S}_{n,\rho}^{(\alpha)}(x,u)du \le \frac{M(\rho,c)}{(x-t)^2}\frac{1+x^2}{n},\ 0 \le t < x,$$

$$\text{ii) } 1-\lambda_{n,\rho}^{(\alpha)}(x,t) = \int_t^\infty \mathcal{S}_{n,\rho}^{(\alpha)}(x,u)du \le \frac{M(\rho,c)}{(t-x)^2}\frac{1+x^2}{n},\ x \le t < \infty,$$

where $M(\rho,c)$ is a positive constant depending on ρ and c.

Proof. For sufficiently large n it follows Remark 2.1 that

$$\mathcal{P}_{n,\rho}^{(\alpha)}((u-x)^2;x) < M(\rho,c)\frac{1+x^2}{n}. \tag{5.2}$$

Applying Lemma 2.3, we have

$$\begin{aligned}\lambda_{n,\rho}^{(\alpha)}(x,t) &= \int_0^t \mathcal{S}_{n,\rho}^{(\alpha)}(x,u)du \le \int_0^t \left(\frac{x-u}{x-t}\right)^2 \mathcal{S}_{n,\rho}^{(\alpha)}(x,u)du \\ &\le \frac{1}{(x-t)^2}\mathcal{P}_{n,\rho}^{(\alpha)}\left((u-x)^2;x\right) \le \frac{M(\rho,c)}{(x-t)^2}\frac{1+x^2}{n}.\end{aligned}$$

The proof of ii) is similar hence the details are omitted. □

Theorem 5.1. *Let $f \in DBV[0,\infty)$, $\alpha = \alpha(n) \to \infty$, as $n \to \infty$ and $\lim_{n\to\infty} \frac{n}{\alpha(n)} = c \in \mathbb{R}$. Then, for every $x \in (0,\infty)$ and sufficiently large n, we have*
$|\mathcal{P}_{n,\rho}^{(\alpha)}(f;x) - f(x)|$

$$\begin{aligned}&\le \sqrt{M(\rho,c)\frac{1+x^2}{n}}\left|\frac{f'(x+)-f'(x-))}{2}\right| + M(\rho,c)\frac{1+x^2}{nx}\sum_{k=1}^{[\sqrt{n}]}\left(\bigvee_{x-\frac{x}{k}}^{x} f'_x\right) \\ &+\frac{x}{\sqrt{n}}\left(\bigvee_{x-\frac{x}{\sqrt{n}}}^{x} f'_x\right) + \left(4N_f + \frac{N_f+|f(x)|}{x^2}\right)M(\rho,c)\frac{1+x^2}{n} \\ &+|f'(x+)|\sqrt{M(\rho,c)\frac{1+x^2}{n}} + M(\rho,c)\frac{1+x^2}{nx^2}|f(2x)-f(x)-xf'(x+)| \\ &+\frac{x}{\sqrt{n}}\bigvee_{x}^{x+\frac{x}{\sqrt{n}}} f'_x + M(\rho,c)\frac{1+x^2}{nx}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x}^{x+\frac{x}{k}} f'_x,\end{aligned}$$

where $M(\rho,c)$ is a positive constant depending on ρ and c, $\bigvee_a^b f$ denotes the total variation of f on $[a,b]$ and f'_x is defined by

$$f'_x(t) = \begin{cases} f'(t)-f'(x-), & 0 \le t < x, \\ 0, & t = x, \\ f'(t)-f'(x+), & x < t < \infty. \end{cases} \tag{5.3}$$

Proof. For any $f \in DBV[0,\infty)$, from (5.3) we can write

$$f'(u) = \frac{1}{2}\left(f'(x+) + f'(x-)\right) + f'_x(u) + \frac{1}{2}\left(f'(x+) - f'(x-)\right) sgn(u-x) \\ + \delta_x(u)\left(f'(u) - \frac{1}{2}\left(f'(x+) + f'(x-)\right)\right), \tag{5.4}$$

where

$$\delta_x(u) = \begin{cases} 1, & u = x, \\ 0, & u \neq x. \end{cases}$$

Since $\mathcal{P}_{n,\rho}^{(\alpha)}(e_0;x) = 1$, using (5.1), for every $x \in (0,\infty)$ we get

$$\begin{aligned} \mathcal{P}_{n,\rho}^{(\alpha)}(f;x) - f(x) &= \int_0^\infty \mathcal{S}_{n,\rho}^{(\alpha)}(x,t)(f(t)-f(x))dt \qquad (5.5)\\ &= \int_0^\infty \mathcal{S}_{n,\rho}^{(\alpha)}(x,t) \int_x^t f'(u)du dt \\ &= -\int_0^x \left(\int_t^x f'(u)du\right) \mathcal{S}_{n,\rho}^{(\alpha)}(x,t)dt \\ &+ \int_x^\infty \left(\int_x^t f'(u)du\right) \mathcal{S}_{n,\rho}^{(\alpha)}(x,t)dt. \end{aligned}$$

Denote

$$I_1 := \int_0^x \left(\int_t^x f'(u)du\right) \mathcal{S}_{n,\rho}^{(\alpha)}(x,t)dt, \quad I_2 := \int_x^\infty \left(\int_x^t f'(u)du\right) \mathcal{S}_{n,\rho}^{(\alpha)}(x,t)dt.$$

Since $\int_x^t \delta_x(u)du = 0$, and using relation (5.4), we get

$$\begin{aligned} I_1 = \int_0^x \Bigg\{\int_t^x & \frac{1}{2}\left(f'(x+) + f'(x-)\right) + f'_x(u) \\ &+ \frac{1}{2}\left(f'(x+) - f'(x-)\right) sgn(u-x)du\Bigg\} \mathcal{S}_{n,\rho}^{(\alpha)}(x,t)dt \end{aligned}$$

$$= \frac{1}{2}\left(f'(x+) + f'(x-)\right)\int_0^x (x-t)\mathcal{S}_{n,\rho}^{(\alpha)}(x,t)dt + \int_0^x \left(\int_t^x f'_x(u)du\right) \mathcal{S}_{n,\rho}^{(\alpha)}(x,t)dt \\ - \frac{1}{2}\left(f'(x+) - f'(x-)\right)\int_0^x (x-t)\mathcal{S}_{n,\rho}^{(\alpha)}(x,t)dt. \tag{5.6}$$

In a similar way we find

$$
\begin{aligned}
I_2 &= \int_x^\infty \left\{\int_x^t \frac{1}{2}\left(f'(x+)+f'(x-)\right)+f'_x(u)\right.\\
&\quad \left.+\frac{1}{2}\left(f'(x+)-f'(x-)\right) sgn(u-x)du\right\}\mathcal{S}_{n,\rho}^{(\alpha)}(x,t)dt\\
&= \frac{1}{2}\left(f'(x+)+f'(x-)\right)\int_x^\infty (t-x)\mathcal{S}_{n,\rho}^{(\alpha)}(x,t)dt\\
&\quad +\int_x^\infty\left(\int_x^t f'_x(u)du\right)\mathcal{S}_{n,\rho}^{(\alpha)}(x,t)dt\\
&\quad +\frac{1}{2}\left(f'(x+)-f'(x-)\right)\int_x^\infty (t-x)\mathcal{S}_{n,\rho}^{(\alpha)}(x,t)dt. \qquad (5.7)
\end{aligned}
$$

Combining the relations (5.5)-(5.7), we get

$$
\begin{aligned}
&\mathcal{P}_{n,\rho}^{(\alpha)}(f;x)-f(x) = \frac{1}{2}\left(f'(x+)+f'(x-)\right)\int_0^\infty (t-x)\mathcal{S}_{n,\rho}^{(\alpha)}(x,t)dt\\
&+\frac{1}{2}\left(f'(x+)-f'(x-)\right)\int_0^\infty |t-x|\mathcal{S}_{n,\rho}^{(\alpha)}(x,t)dt\\
&-\int_0^x\left(\int_t^x f'_x(u)du\right)\mathcal{S}_{n,\rho}^{(\alpha)}(x,t)dt+\int_x^\infty\left(\int_x^t f'_x(u)du\right)\mathcal{S}_{n,\rho}^{(\alpha)}(x,t)dt.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\left|\mathcal{P}_{n,\rho}^{(\alpha)}(f;x)-f(x)\right| = \left|\frac{f'(x+)+f'(x-)}{2}\right|\left|\mathcal{P}_{n,\rho}^{(\alpha)}(t-x;x)\right|\\
&+\left|\frac{f'(x+)-f'(x-)}{2}\right|\mathcal{P}_{n,\rho}^{(\alpha)}(|t-x|;x)\\
&+\left|\int_0^x\left(\int_t^x f'_x(u)du\right)\mathcal{S}_{n,\rho}^{(\alpha)}(x,t)dt\right|+\left|\int_x^\infty\left(\int_x^t f'_x(u)du\right)\mathcal{S}_{n,\rho}^{(\alpha)}(x,t)dt\right|. \qquad (5.8)
\end{aligned}
$$

Now, let

$$E_{n,\rho}^{(\alpha)}(f'_x,x)=\int_0^x\left(\int_t^x f'_x(u)du\right)\mathcal{S}_{n,\rho}^{(\alpha)}(x,t)dt,$$

and

$$F_{n,\rho}^{(\alpha)}(f'_x,x)=\int_x^\infty\left(\int_x^t f'_x(u)du\right)\mathcal{S}_{n,\rho}^{(\alpha)}(x,t)dt.$$

Our problem is reduced to calculate the estimates of the terms $E_{n,\rho}^{(\alpha)}(f'_x,x)$ and $F_{n,\rho}^{(\alpha)}(f'_x,x)$. From the definition of $\lambda_{n,\rho}^{(\alpha)}$ given in Lemma 5.1, applying the integration by parts, we can write

$$E_{n,\rho}^{(\alpha)}(f'_x,x)=\int_0^x\left(\int_t^x f'_x(u)du\right)\frac{\partial}{\partial t}\lambda_{n,\rho}^{(\alpha)}(x,t)dt=\int_0^x f'_x(t)\lambda_{n,\rho}^{(\alpha)}(x,t)dt.$$

Thus,

$$\left|E_{n,\rho}^{(\alpha)}(f'_x,x)\right|\le\int_0^x |f'_x(t)|\lambda_{n,\rho}^{(\alpha)}(x,t)dt$$

$$\leq \int_0^{x-\frac{x}{\sqrt{n}}} |f'_x(t)|\lambda_{n,\rho}^{(\alpha)}(x,t)dt + \int_{x-\frac{x}{\sqrt{n}}}^{x} |f'_x(t)|\lambda_{n,\rho}^{(\alpha)}(x,t)dt.$$

Since $f'_x(x) = 0$ and $\lambda_{n,\rho}^{(\alpha)}(x,t) \leq 1$, we get

$$\int_{x-\frac{x}{\sqrt{n}}}^{x} |f'_x(t)|\lambda_{n,\rho}^{(\alpha)}(x,t)dt = \int_{x-\frac{x}{\sqrt{n}}}^{x} |f'_x(t) - f'_x(x)|\lambda_{n,\rho}^{(\alpha)}(x,t)dt$$

$$\leq \int_{x-\frac{x}{\sqrt{n}}}^{x} \bigvee_{t}^{x} f'_x dt \leq \bigvee_{x-\frac{x}{\sqrt{n}}}^{x} f'_x \int_{x-\frac{x}{\sqrt{n}}}^{x} dt = \frac{x}{\sqrt{n}} \bigvee_{x-\frac{x}{\sqrt{n}}}^{x} f'_x.$$

By applying Lemma 5.1 and considering $t = x - \frac{x}{u}$, we have

$$\int_0^{x-\frac{x}{\sqrt{n}}} |f'_x(t)|\lambda_{n,\rho}^{(\alpha)}(x,t)dt \leq M(\rho,c)\frac{1+x^2}{n} \int_0^{x-\frac{x}{\sqrt{n}}} |f'_x(t)|\frac{dt}{(x-t)^2}$$

$$\leq M(\rho,c)\frac{1+x^2}{n} \int_0^{x-\frac{x}{\sqrt{n}}} \left(\bigvee_{t}^{x} f'_x\right) \frac{dt}{(x-t)^2}$$

$$= M(\rho,c)\frac{1+x^2}{nx} \int_1^{\sqrt{n}} \left(\bigvee_{x-\frac{x}{u}}^{x} f'_x\right) du \leq M(\rho,c)\frac{1+x^2}{nx} \sum_{k=1}^{[\sqrt{n}]} \left(\bigvee_{x-\frac{x}{k}}^{x} f'_x\right).$$

Therefore,

$$|E_{n,\rho}^{(\alpha)}(f'_x,x)| \leq M(\rho,c)\frac{1+x^2}{nx} \sum_{k=1}^{[\sqrt{n}]} \left(\bigvee_{x-\frac{x}{k}}^{x} f'_x\right) + \frac{x}{\sqrt{n}} \left(\bigvee_{x-\frac{x}{\sqrt{n}}}^{x} f'_x\right). \tag{5.9}$$

Also, using integration by parts in $F_{n,\rho}^{(\alpha)}(f'_x,x)$ and applying Lemma 5.1, we have

$$\begin{aligned}|F_{n,\rho}^{(\alpha)}(f'_x,x) &\leq \left|\int_x^{2x} \left(\int_x^t f'_x(u)du\right) \frac{\partial}{\partial t}\left(1-\lambda_{n,\rho}^{(\alpha)}(x,t)\right) dt\right| \\ &+ \left|\int_{2x}^{\infty} \left(\int_x^t f'_x(u)du\right) \mathcal{S}_{n,\rho}^{(\alpha)}(x,t)dt\right| \\ &\leq \left|\int_x^{2x} f'_x(u)du\right| \left|1-\lambda_{n,\rho}^{(\alpha)}(x,2x)\right| + \int_x^{2x} |f'_x(t)| \left(1-\lambda_{n,\rho}^{(\alpha)}(x,t)\right) dt \\ &+ \left|\int_{2x}^{\infty} (f(t)-f(x))\mathcal{S}_{n,\rho}^{(\alpha)}(x,t)dt\right| + |f'(x+)| \left|\int_{2x}^{\infty} (t-x)\mathcal{S}_{n,\rho}^{(\alpha)}(x,t)dt\right|.\end{aligned}$$

We have

$$\int_x^{2x} |f'_x(t)| \left(1-\lambda_{n,\rho}^{(\alpha)}(x,t)\right) dt = \int_x^{x+\frac{x}{\sqrt{n}}} |f'_x(t)| \left(1-\lambda_{n,\rho}^{(\alpha)}(x,t)\right) dt$$

$$+ \int_{x+\frac{x}{\sqrt{n}}}^{2x} |f'_x(t)| \left(1-\lambda_{n,\rho}^{(\alpha)}(x,t)\right) dt = J_1 + J_2 \text{ (say)}. \tag{5.10}$$

Since $f_x'(x)=0$ and $1-\lambda_{n,\rho}^{(\alpha)}\le 1$, we get

$$\begin{aligned}J_1&=\int_x^{x+\frac{x}{\sqrt{n}}}|f_x'(t)-f_x'(x)|\left(1-\lambda_{n,\rho}^{(\alpha)}(x,t)\right)dt\\&\le\int_x^{x+\frac{x}{\sqrt{n}}}\left(\bigvee_x^{x+\frac{x}{\sqrt{n}}}f_x'\right)dt=\frac{x}{\sqrt{n}}\bigvee_x^{x+\frac{x}{\sqrt{n}}}f_x'.\end{aligned}$$

Applying Lemma 5.1 and considering $t=x+\dfrac{x}{u}$, we obtain

$$\begin{aligned}J_2&\le M(\rho,c)\frac{1+x^2}{n}\int_{x+\frac{x}{\sqrt{n}}}^{2x}\frac{1}{(t-x)^2}|f_x'(t)-f_x'(x)|dt\\&\le M(\rho,c)\frac{1+x^2}{n}\int_{x+\frac{x}{\sqrt{n}}}^{2x}\frac{1}{(t-x)^2}\left(\bigvee_x^{t}f_x'\right)dt=M(\rho,c)\frac{1+x^2}{nx}\int_1^{\sqrt{n}}\bigvee_x^{x+\frac{x}{u}}f_x'du\\&\le M(\rho,c)\frac{1+x^2}{nx}\sum_{k=1}^{[\sqrt{n}]}\int_k^{k+1}\left(\bigvee_x^{x+\frac{x}{u}}f_x'\right)du\le M(\rho,c)\frac{1+x^2}{nx}\sum_{k=1}^{[\sqrt{n}]}\left(\bigvee_x^{x+\frac{x}{k}}f_x'\right).\end{aligned}$$

Putting the values of J_1 and J_2 in (5.10), we get

$$\int_x^{2x}|f_x'(t)|\left(1-\lambda_{n,\rho}^{(\alpha)}(x,t)\right)dt\le\frac{x}{\sqrt{n}}\bigvee_x^{x+\frac{x}{\sqrt{n}}}f_x'+M(\rho,c)\frac{1+x^2}{nx}\sum_{k=1}^{[\sqrt{n}]}\left(\bigvee_x^{x+\frac{x}{k}}f_x'\right).$$

Therefore,

$$\begin{aligned}&|F_{n,\rho}^{(\alpha)}(f_x',x)|\\&\quad\le N_f\int_{2x}^{\infty}(t^2+1)\mathcal{S}_{n,\rho}^{(\alpha)}(x,t)dt+|f(x)|\int_{2x}^{\infty}\mathcal{S}_{n,\rho}^{(\alpha)}(x,t)dt\\&\quad+|f'(x+)|\sqrt{M(\rho,c)\frac{1+x^2}{n}}+M(\rho,c)\frac{1+x^2}{nx^2}|f(2x)-f(x)-xf'(x+)|\\&\quad+\frac{x}{\sqrt{n}}\bigvee_x^{x+\frac{x}{\sqrt{n}}}f_x'+M(\rho,c)\frac{1+x^2}{nx}\sum_{k=1}^{[\sqrt{n}]}\left(\bigvee_x^{x+\frac{x}{k}}f_x'\right).\end{aligned}\tag{5.11}$$

Since $t\le 2(t-x)$ and $x\le t-x$ when $t\ge 2x$, we obtain

$$\begin{aligned}&N_f\int_{2x}^{\infty}(t^2+1)\mathcal{S}_{n,\rho}^{(\alpha)}(x,t)dt+|f(x)|\int_{2x}^{\infty}\mathcal{S}_{n,\rho}^{(\alpha)}(x,t)dt\\&\le(N_f+|f(x)|)\int_{2x}^{\infty}\mathcal{S}_{n,\rho}^{(\alpha)}(x,t)dt+4N_f\int_{2x}^{\infty}(t-x)^2\mathcal{S}_{n,\rho}^{(\alpha)}(x,t)dt\\&\le\frac{N_f+|f(x)|}{x^2}\int_0^{\infty}(t-x)^2\mathcal{S}_{n,\rho}^{(\alpha)}(x,t)dt+4N_f\int_0^{\infty}(t-x)^2\mathcal{S}_{n,\rho}^{(\alpha)}(x,t)dt\\&\le\left(4N_f+\frac{N_f+|f(x)|}{x^2}\right)M(\rho,c)\frac{1+x^2}{n}.\end{aligned}\tag{5.12}$$

Using the inequality (5.12), it follows

$$\begin{aligned}|F_{n,\rho}^{(\alpha)}(f'_x,x)| &\leq \left(4N_f+\frac{N_f+|f(x)|}{x^2}\right)M(\rho,c)\frac{1+x^2}{n}\\&+|f'(x+)|\sqrt{M(\rho,c)\frac{1+x^2}{n}}\\&+M(\rho,c)\frac{1+x^2}{nx^2}|f(2x)-f(x)-xf'(x+)|\\&+\frac{x}{\sqrt{n}}\bigvee_{x}^{x+\frac{x}{\sqrt{n}}}f'_x+M(\rho,c)\frac{1+x^2}{nx}\sum_{k=1}^{[\sqrt{n}]}\left(\bigvee_{x}^{x+\frac{x}{k}}f'_x\right). \qquad (5.13)\end{aligned}$$

From (5.8), (5.9) and (5.13), we get the required result. ☐

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