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COMPARISON RESULTS FOR PROPER MULTISPLITTINGS OF RECTANGULAR MATRICES

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ABSTRACT. The least square solution of minimum norm of a rectangular linear system of equations can be found out iteratively by using matrix splittings. However, the convergence of such an iteration scheme arising out of a matrix splitting is practically very slow in many cases. Thus, works on improving the speed of the iteration scheme have attracted great interest. In this direction, comparison of the rate of convergence of the iteration schemes produced by two matrix splittings is very useful. But, in the case of matrices having many matrix splittings, this process is time-consuming. The main goal of the current article is to provide a solution to the above issue by using proper multisplittings. To this end, we propose a few comparison theorems for proper weak regular splittings and proper nonnegative splittings first. We then derive convergence and comparison theorems for proper multisplittings with the help of the theory of proper weak regular splittings.

1. INTRODUCTION

Let us consider a rectangular system of linear equations of the form

$$Ax = b, \tag{1}$$

where A is a real, large and sparse matrix of order $m \times n$, x is an unknown real n-vector, and b is a given real m-vector. If (1) is inconsistent, then one usually seeks the least square solution of minimum norm. This solution vector

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x is then computed by $x = A^{\dagger}b$, where $A^{\dagger}$ is the Moore–Penrose inverse of A (see Section 2, for its definition). In a wide variety of such problems, including the Neumann problem and those for elastic bodies with free surfaces, the finite difference formulations lead to a singular, consistent linear system of the form (1), where A is large and sparse. In these situations, one can opt for an iterative method for finding the least square solution of minimum norm. Such a method where A is rectangular or (1) is inconsistent, is studied in [4]. In particular, the authors of [4] have introduced the following iteration scheme to find the least square solution of minimum norm of the system (1)

$$x^{i+1} = U^{\dagger}Vx^{i} + U^{\dagger}b, \quad i = 0, 1, 2, \ldots, \tag{2}$$

where $A = U - V$ is a proper splitting. A splitting $A = U - V$ of $A \in \mathbb{R}^{m\times n}$ (the set of all real $m \times n$ matrices) is called a *proper splitting* [4] if $R(U) = R(A)$ and $N(U) = N(A)$, where $R(A)$ and $N(A)$ denote the range space and the null space of A, respectively. The iteration scheme (2) is said to be *convergent* if the spectral radius of the iteration matrix $U^{\dagger}V$ is less than 1. For the proper splitting $A = U - V$, the same authors [4] proved that the iteration scheme (2) converges to $x = A^{\dagger}b$, the least squares solution of minimum norm, for any initial vector x^0 if and only if the iteration scheme (2) is convergent (see [4], Corollary 1). The advantage of the iterative method for solving the rectangular system of linear equations (1) is that it avoids the use of the normal system $A^TAx = A^Tb$, where A^TA is frequently ill-conditioned and influenced greatly by roundoff errors (see [12]). (Here A^T stands for the transpose of a matrix A.)

Berman and Plemmons [4] also proved a few convergence results for different classes of proper splittings without calling them by any name. Later on, Climent *et al.* [6], Climent and Perea [7] introduced different classes of proper splittings and studied its convergence theory. Subsequently, it is carried forward by Mishra and Sivakumar [16], Jena *et al.* [13], Mishra [15], Baliarsingh and Mishra [2], and Giri and Mishra [11], to name a few. Comparison theorems between the spectral radii of matrices are useful tools in the analysis of the rate of convergence of iterative methods or for judging the efficiency of pre-conditioners. A matrix A may have different matrix splittings (say $A = U_1 - V_1 = U_2 - V_2$). In practice, we seek such an U which not only makes the computation x^{i+1}(given x^i) simpler but also yields the spectral radius of $U^{\dagger}V$ (which is of course less than 1) as small as possible for the faster rate of convergence of the iteration scheme (2). An accepted rule for preferring one iteration scheme to another is to choose the iteration scheme having the smaller spectral radius. In this context, Jena *et al.* [13], Giri and Mishra [11], Mishra [14, 15] and Baliarsingh and Mishra [2] obtained various comparison results for different class of matrix splittings of rectangular matrices. In this article, we propose a few more comparison results.

But one of the drawbacks of the above-discussed theory is that this process needs more time when a matrix has many splittings as one can compare two matrix splittings at a time. A natural question arises at this level is "can we have a faster iteration scheme than (2)". This is answered by O'Leary and White

A splitting of a real rectangular matrix A is an expression of the form $A = U - V$, where U and V are matrices of the same order as in A.

[17] who have introduced the concept of the multisplitting method for obtaining the parallel solution of linear system of equations of the form (1), but in the square nonsingular matrix setting. A real $n \times n$ matrix A is called *monotone* (or a matrix of "*monotone kind*") if $Ax \geq 0 \Rightarrow x \geq 0$. This notion was introduced by Collatz, who has shown that A is monotone if and only if A^{-1} exists and $A^{-1} \geq 0$ (see Section 2, for meaning of $B \geq 0$). The book by Collatz [8] has details of how monotone matrices arise naturally in the study of finite difference approximation methods for certain elliptic partial differential equations. The problem of characterizing monotone (also referred as *inverse positive*) matrices in terms of matrix splittings has been extensively dealt with in the literature. The book by Berman and Plemmons [5] gives an excellent account of many of these characterizations and its extension to rectangular matrices.

O'Leary and White [17] have provided the convergence theory of multisplittings for the class of monotone matrices (see [8]). The triplet $(U_k, V_k, E_k)_{k=1}^{p}$ is called a *multisplitting* of $A \in \mathbb{R}^{n\times n}$ if
(i) $A = U_k - V_k,$ for each $k = 1, 2, \ldots, p,$
(ii) $E_k \geq 0$ is a non-zero and diagonal matrix, for each $k = 1, 2, \ldots, p,$
(iii) $\sum_{k=1}^{p} E_k = I,$ where I is the identity matrix.
Using the multisplitting $(U_k, V_k, E_k)_{k=1}^{p}$, the authors of [17] considered the following iteration scheme:

$$x^{i+1} = Hx^i + Gb, \quad i = 0, 1, 2, \ldots, \tag{3}$$

where $H = \sum_{k=1}^{p} E_k U_k^{-1} V_k$ and $G = \sum_{k=1}^{p} E_k U_k^{-1}$. The same authors [17] proved that if $A = U_k - V_k, \quad k = 1, 2, \ldots, p$ is a weak regular splitting of a monotone matrix A, then the iteration scheme (3) converges for any initial vector x^0. In contrast to the vast literature available on solving the square nonsingular system of linear equations, iteratively, the researches on solving the rectangular system of linear equations, iteratively are limited. In particular, the theory of multisplittings has not been studied much for rectangular matrices. Climent and Perea [7] first introduced the concept of a proper multisplitting. Thereafter, Baliarsingh and Jena [1] applied the same theory to solve the square singular system of linear equations. In this note, we revisit the same theory first and add a few more results to the existing theory with the objective to solve the rectangular linear systems. Some of the results obtained in this paper dealing with multisplittings theory are completely new even for square nonsingular matrices.

The contents of this paper are organized in the following order. Next Section includes some notation and fundamental concepts concerned in our study. In Section 3, we introduce our main results. Section 3 further divided into three subsections. In subsection 3.1, we establish a number of comparison results between two proper weak regular splittings of different types. This is a prelude to subsection 3.2, in which we study similar results as of subsection 3.1, but for proper nonnegative splittings of different types. Finally, subsection 3.3, is devoted to the study of multisplittings of a rectangular matrix.

2. Preliminaries

To present a reader-friendly convergence analysis of rectangular matrix splittings, we first explain some basic notation and definitions. In the subsequent sections, $\mathbb{R}^n$ means an n-dimensional Euclidean space. If $L \oplus M = \mathbb{R}^n$, then $P_{L,M}$ is referred as the projection onto L along M. So, $P_{L,M}A = A$ if and only if $R(A) \subseteq L$ and $AP_{L,M} = A$ if and only if $N(A) \supseteq M$. If $L \perp M$, then $P_{L,M}$ will be denoted by P_L. For $A \in \mathbb{R}^{m\times n}$, the unique matrix $X \in \mathbb{R}^{n\times m}$ is called the *Moore–Penrose inverse* of A if it satisfies the following four equations:

$$AXA = A, \quad XAX = X, \quad (AX)^T = AX \text{ and } (XA)^T = XA,$$

and is denoted by $A^\dagger$. It always exists, and $A^\dagger = A^{-1}$ in the case of a nonsingular matrix A. Properties of $A^\dagger$ which will be frequently used in this paper are: $R(A^\dagger) = R(A^T)$; $N(A^\dagger) = N(A^T)$; $AA^\dagger = P_{R(A)}$ and $A^\dagger A = P_{R(A^T)}$ (see [3] for more details).

A matrix $A \in \mathbb{R}^{m\times n}$ is called *non-negative* if $A \geq 0$ and $B \geq C$ if $B - C \geq 0$. Here $A \geq 0$ means all the entries of A are non-negative. Again, $B \gneq C$ means $B \geq C$ and $B \neq C$. Similarly, a matrix $A \in \mathbb{R}^{m\times n}$ is called *positive* if each element of A is positive, and is denoted by $A > 0$. We also use the above notation for vectors as vectors can be seen as $n \times 1$ matrices. A matrix $A \in \mathbb{R}^{m\times n}$ is called *semi-monotone* if $A^\dagger \geq 0$. For a matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$, the set of indices $i, j = 1, 2, \ldots, n$ will be denoted by S. A matrix A is *reducible* if there exists a non-void index set R, $R \subset S$ and $R \neq S$ such that $a_{ij} = 0$ for $i \in R$ and $j \in S - R$, otherwise the matrix A is *irreducible*. Clearly, each positive matrix is irreducible. The *spectral radius* of a matrix $A \in \mathbb{R}^{n\times n}$ is denoted by $\rho(A)$, and is equal to the maximum of the moduli of the eigenvalues of A. Let A and B be two matrices of appropriate order such that the products AB and BA are defined. Then $\rho(AB) = \rho(BA)$.

Before proceeding to our main results, we first revisit the theory of proper regular splittings and proper weak regular splittings, introduced by Jena *et al.* [13].

Definition 2.1. ([13], Definition 1.1)
A proper splitting $A = U - V$ of $A \in \mathbb{R}^{m\times n}$ is called a *proper regular splitting* if $U^\dagger \geq 0$ and $V \geq 0$.

The same authors [13] proved the following comparison theorem for proper regular splittings in order to improve the convergence speed of the iteration scheme (2).

Theorem 2.2. ([13], Theorem 3.3)
Let $A = U_1 - V_1 = U_2 - V_2$ be two proper regular splittings of a semimonotone matrix $A \in \mathbb{R}^{m\times n}$. If $U_1^\dagger \geq U_2^\dagger$, then

$$\rho(U_1^\dagger V_1) \leq \rho(U_2^\dagger V_2) < 1.$$

We next reproduce the definition of a larger class of matrices than the class of proper regular splittings.

Definition 2.3. ([13], Definition 1.2)
A proper splitting $A = U - V$ of $A \in \mathbb{R}^{m\times n}$ is called a *proper weak regular splitting* if $U^\dagger \geq 0$ and $U^\dagger V \geq 0$.

Berman and Plemmons [4] obtained the following convergence result for a proper weak regular splitting without specifying the name of this class.

Theorem 2.4. ([4], Corollary 4)
Let $A = U - V$ be a proper weak regular splitting of $A \in \mathbb{R}^{m\times n}$. Then $A^\dagger \geq 0$ if and only if $\rho(U^\dagger V) < 1$.

One can find that, there exists a convergent splitting which is not a proper weak regular splitting. To address convergence theory in this situation, we now have the following definition from [6], where the authors call it as a weak nonnegative splitting of second type. However, we call here as a proper weak regular splitting of type II.

Definition 2.5. ([6], Definition 2)
A proper splitting $A = U - V$ of $A \in \mathbb{R}^{m\times n}$ is called a *proper weak regular splitting of type II* if $U^\dagger \geq 0$ and $VU^\dagger \geq 0$.

Note that the proper weak regular splitting of type I is same as the proper weak regular splitting. Another remark drawn from the above definition is that it cannot be ensured convergence of all splittings by the known convergence results for the proper weak regular splitting of type I. To overcome this issue, Mishra and Sivakumar [16] proved the following convergence result for the proper weak regular splitting of type II. Note that the same authors call it as the weak pseudo regular splitting, but we call it here as the proper weak regular splitting of type II.

Theorem 2.6. ([16], Remark 3.5)
Let $A = U - V$ be a proper weak regular splitting of type II of $A \in \mathbb{R}^{m\times n}$. Then $A^\dagger \geq 0$ if and only if $\rho(U^\dagger V) < 1$.

Observe that Theorem 2.4 and Theorem 2.6 together extend [10], Theorem 3.4 (i) for rectangular matrices while the other part is extended in the next section. We next recall the definition of proper nonnegative splitting of type I (or proper nonnegative splitting) which is more general than proper weak regular splitting of type I.

Definition 2.7. ([15], Definition 3.1)
A proper splitting $A = U - V$ of $A \in \mathbb{R}^{m\times n}$ is called a *proper nonnegative splitting* if $U^\dagger V \geq 0$.

We remark that earlier, Climent *et al.* [6] also introduced the above definition but they call this as weak splitting. For later use, we record first the following convergence result.

Lemma 2.8. ([6], Theorem 2 & [15], Lemma 3.5)
Let $A = U - V$ be a proper nonnegative splitting of $A \in \mathbb{R}^{m\times n}$. Then $A^\dagger V \geq 0$ if and only if $\rho(U^\dagger V) = \dfrac{\rho(A^\dagger V)}{1 + \rho(A^\dagger V)} < 1$.

Next, we recollect the definition of a proper nonnegative splitting of type II proposed by Climent *et al.* [6]. Note that the proper nonnegative splitting of type I is same as the proper nonnegative splitting.

Definition 2.9. ([6], Definition 2 & [2], Definition 3.14)
A proper splitting $A = U - V$ of $A \in \mathbb{R}^{m\times n}$ is called a *proper nonnegative splitting of type II* if $VU^\dagger \geq 0$.

A convergence result for a proper nonnegative splitting of type II is stated next.

Lemma 2.10. ([6], Remark 2)
Let $A = U - V$ be a proper nonnegative splitting of type II of $A \in \mathbb{R}^{m\times n}$. Then $VA^\dagger \geq 0$ if and only if $\rho(VU^\dagger) = \dfrac{\rho(VA^\dagger)}{1+\rho(VA^\dagger)} < 1$.

3. Main Results

3.1. Proper Weak Regular Splitting of Different Types.

The first main result, presented below partially generalizes the other part of [10], Theorem 3.4.

Lemma 3.1. *Let $A = U - V$ be a proper weak regular splitting of type II of a semimonotone matrix $A \in \mathbb{R}^{m\times n}$. Suppose that $\rho(U^\dagger V) > 0$. Then there exists a vector $x \gneq 0$ such that $U^\dagger Vx = \rho(U^\dagger V)x$, $Ax \gneq 0$ and $Vx \gneq 0$.*

Proof. We have $VU^\dagger \geq 0$. By [20], Theorem 2.20, there exists an eigenvector $z \geq 0$ such that

$$VU^\dagger z = \rho(VU^\dagger)z. \tag{4}$$

Therefore, $z \in R(V) \subseteq R(U)$. Define $x = U^\dagger z$. Then $x \geq 0$. Pre-multiplying (4) by $U^\dagger$, we obtain

$$U^\dagger Vx = \rho(VU^\dagger)x. \tag{5}$$

Suppose that $x = 0$. Then $U^\dagger z = 0$ so that $z \in R(U) \cap N(U^T)$. Thus, $z = 0$, a contradiction. So $x \neq 0$. Now we prove the inequality $Ax \geq 0$. Theorem 2.6 and [6], Theorem 1 (4) yield

$$0 \leq (1-\rho(VU^\dagger))z = (I - VU^\dagger)z = (I - VU^\dagger)Ux = Ax.$$

Clearly, $Ax \neq 0$ otherwise $Ax = 0$ implies $x = 0$, a contradiction. From (4), we have $Vx \geq 0$. Pre-multiplying (5) by U, we get $Vx = \rho(U^\dagger V)Ux$, i.e., $Ux = \dfrac{Vx}{\rho(U^\dagger V)}$. Therefore, we get

$$0 \leq Ax = U(I - U^\dagger V)x = (1-\rho(U^\dagger V))Ux = \frac{(1-\rho(U^\dagger V))}{\rho(U^\dagger V)}Vx.$$

So $Vx \neq 0$. If $Vx = 0$, then $Ax = 0$, again a contradiction. □

Convergence of an iteration scheme is usually accelerated by a pre-conditioner. It is a square matrix Q of order m which on pre-multiplication makes the convergence of the iterative method for the system with the matrix QA faster than the original system with the matrix A. Hence, instead of solving (1), we solve

$$QAx = Qb, \text{i.e., } A_1x = c.$$

The method of finding an effective pre-conditioner Q for general problems is a mathematical challenge. Nevertheless, many specific problems are being successfully solved using preconditioned iterative solvers. But the problem is how to choose an effective pre-conditioner. This is settled next, with a comparison result of the rate of convergence of two different linear systems which is a generalization of the [10], Theorem 3.5 for rectangular matrices. However, the assumptions are not exactly the same.

Theorem 3.2. *Let $A_1, A_2 \in \mathbb{R}^{m\times n}$. Let $A_1 = U_1 - V$ and $A_2 = U_2 - V$ be two proper weak regular splittings of different types. Suppose that $\rho(U_1^\dagger V) > 0$ and $\rho(U_2^\dagger V) > 0$. If $V \neq 0$ and $A_2^\dagger > A_1^\dagger \geq 0$, then*

$$\rho(U_1^\dagger V) < \rho(U_2^\dagger V) < 1.$$

Proof. By Theorem 2.4 and Theorem 2.6, it follows that $\rho(U_i^\dagger V) < 1$ for each $i = 1, 2$. Define $G_1 = A_1^\dagger V,\ G_2 = A_2^\dagger V,\ \tilde{G}_1 = VA_1^\dagger$ and $\tilde{G}_2 = VA_2^\dagger$. Using [4], Theorem 1 (3) and [6], Theorem 1 (6), we have

$$G_i = A_i^\dagger V = (I - U_i^\dagger V)^{-1} U_i^\dagger V,\ i = 1, 2$$
$$\text{and } \tilde{G}_i = VA_i^\dagger = VU_i^\dagger (I - VU_i^\dagger)^{-1},\ i = 1, 2.$$

Let us first assume that $A_1 = U_1 - V$ is a proper weak regular splitting of type I and $A_2 = U_2 - V$ is a proper weak regular splitting of type II. Then G_1 and $\tilde{G}_2$ are non-negative matrices and

$$\rho(G_i) = \rho(\tilde{G}_i) = \frac{\rho(U_i^\dagger V)}{1 - \rho(U_i^\dagger V)} = \frac{\rho(VU_i^\dagger)}{1 - \rho(VU_i^\dagger)} \quad \text{for each } i = 1, 2.$$

We only need to show that $\rho(G_1) < \rho(G_2)$. By Lemma 3.1, there exists an eigenvector $x \geq 0$, such that $U_2^\dagger Vx = \rho(U_2^\dagger V)x$ and $Vx \geq 0$. Using $A_2^\dagger > A_1^\dagger \geq 0$, we get

$$\rho(G_2)x = G_2 x = A_2^\dagger Vx > A_1^\dagger Vx = G_1 x. \tag{6}$$

Hence, by [5], Theorem 2.1.11, the strict inequality $\rho(G_1) < \rho(G_2)$ follows directly. If $A_1 = U_1 - V$ is a proper weak regular splitting of type II and $A_2 = U_2 - V$ is a proper weak regular splitting of type I, then $\tilde{G}_1$ and G_2 are non-negative matrices. Again, by Lemma 3.1, there exists an eigenvector $z \geq 0$ such that $U_1^\dagger Vz = \rho(U_1^\dagger V)z$ and $Vz \geq 0$. Thus

$$G_2 z = A_2^\dagger Vz > A_1^\dagger Vz = G_1 z = \rho(G_1)z. \tag{7}$$

The strict inequality $\rho(G_1) < \rho(G_2)$ then follows from [5], Theorem 2.1.11 which yields the desired claim. □

In the above result, one cannot drop the assumption $A_2^\dagger > A_1^\dagger \geq 0$ which can be seen from the example illustrated next.

Example 3.3. Let $A_1 = \begin{pmatrix} 7 & -7/2 & 7 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 8 & -4 & 8 \\ 0 & 2 & 0 \end{pmatrix} - \begin{pmatrix} 1 & -1/2 & 1 \\ 0 & 1 & 0 \end{pmatrix} =$ $U_1 - V$ and $A_2 = \begin{pmatrix} 3 & -3/2 & 3 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 4 & -2 & 4 \\ 0 & 2 & 0 \end{pmatrix} - \begin{pmatrix} 1 & -1/2 & 1 \\ 0 & 1 & 0 \end{pmatrix} = U_2 - V$. Then $A_1 = U_1 - V$ is a proper weak regular splitting of type I and $A_2 = U_2 - V$ is a proper weak regular splitting of type II. We have $A_2^\dagger = \begin{pmatrix} 0.1667 & 0.2500 \\ 0 & 1 \\ 0.1667 & 0.2500 \end{pmatrix} \geq$ $A_1^\dagger = \begin{pmatrix} 0.0714 & 0.2500 \\ 0 & 1 \\ 0.0714 & 0.2500 \end{pmatrix} \geq 0$. But $\rho(U_1^\dagger V) = \rho(U_2^\dagger V) = 0.5$.

We conclude this section with another comparison theorem for two different linear systems having two different types of proper weak regular splittings.

Theorem 3.4. *Let $A_1, A_2 \in \mathbb{R}^{m\times n}$. Let $A_1 = U_1 - V_1$ and $A_2 = U_2 - V_2$ be two proper weak regular splittings of different types. Suppose that $\rho(U_1^\dagger V_1) > 0$ and $\rho(U_2^\dagger V_2) > 0$. Assume that $V_1 \neq 0$, $V_2 \neq 0$ and $A_2^\dagger > A_1^\dagger \geq 0$. If $V_1 \leq V_2$, then*

$$\rho(U_1^\dagger V_1) < \rho(U_2^\dagger V_2) < 1.$$

Proof. By Theorem 2.4 and Theorem 2.6, we obtain $\rho(U_i^\dagger V_i) < 1, \ i = 1, 2$. The remaining proof is similar to the proof of Theorem 3.2, with the exception that in place of (6) we have to use one additional inequality

$$\rho(G_2)x = G_2 x = A_2^\dagger V_2 x > A_1^\dagger V_1 x = G_1 x,$$

and in place of (7), we need $G_2 z = A_2^\dagger V_2 z > A_1^\dagger V_1 z = G_1 z = \rho(G_1) z$. □

Note that Theorem 3.2 is a special case of the above result as the assumption $V_1 \leq V_2$ is automatically fulfilled when $V_1 = V_2$.

The example given below demonstrates that the converse of the above theorem is not true.

Example 3.5. Let $A_1 = \begin{pmatrix} 2 & -2 & 4 \\ 2 & 4 & -2 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 1 & -2 & 3 \\ 1 & 3 & -2 \end{pmatrix}$. Then $A_2^\dagger = \begin{pmatrix} 0.3333 & 0.3333 \\ 0.0667 & 0.2667 \\ 0.2667 & 0.0667 \end{pmatrix} > A_1^\dagger = \begin{pmatrix} 0.1667 & 0.1667 \\ 0 & 0.1667 \\ 0.1667 & 0 \end{pmatrix} \geq 0$.
Let $U_1 = \begin{pmatrix} 3 & -3 & 6 \\ 2 & 4 & -2 \end{pmatrix}$ and $U_2 = \begin{pmatrix} 2 & -2 & 4 \\ 2 & 4 & -2 \end{pmatrix}$. Then $A_1 = U_1 - V_1$ is a proper weak regular splitting of type I and $A_2 = U_2 - V_2$ is a proper weak regular splittings of type II. We have $0.3 = \rho(U_1^\dagger V_1) < 0.5 = \rho(U_2^\dagger V_2) < 1$. But $V_1 = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \nleq V_2 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.

3.2. **Proper Nonnegative Splittings of Different Types.** The plan of this section is to obtain new comparison results for proper nonnegative splittings of different types in order to speed up the rate of convergence of the iteration scheme (2). The class of proper nonnegative splittings contains earlier two classes of splittings, and hence study of this class of matrices assumes significance. We now prove the following comparison result which partially extends [19], Theorem 2.11 to rectangular matrices.

Theorem 3.6. *Let $A = U_1 - V_1 = U_2 - V_2$ be two convergent proper nonnegative splittings of the same type of a semi-monotone matrix $A \in \mathbb{R}^{m\times n}$. If there exists α, $0 < \alpha \leq 1$, such that $V_1 \leq \alpha V_2$ and $\rho(A^\dagger V_i) > 0$, $i = 1$ or 2, then*

$$\rho(U_1^\dagger V_1) \leq \rho(U_2^\dagger V_2) < 1,$$

whenever $\alpha = 1$ and

$$\rho(U_1^\dagger V_1) < \rho(U_2^\dagger V_2) < 1,$$

whenever $0 < \alpha < 1$.

Proof. Assume that the given splittings are convergent proper nonnegative splittings of type I. So, we have $\rho(U_1^\dagger V_1) < 1$. By Lemma 2.8, we get $A^\dagger V_1 \geq 0$. The conditions $A^\dagger \geq 0$ and $V_1 \leq \alpha V_2$ together imply

$$0 \leq A^\dagger V_1 \leq \alpha A^\dagger V_2.$$

It then follows from [20], Theorem 2.21 that

$$\rho(A^\dagger V_1) \leq \alpha \rho(A^\dagger V_2). \tag{8}$$

Since $f(\eta) = \dfrac{\eta}{1+\eta}$ is a strictly increasing function for $\eta \geq 0$, so

$$\frac{\rho(A^\dagger V_1)}{1+\rho(A^\dagger V_1)} \leq \frac{\alpha\rho(A^\dagger V_2)}{1+\alpha\rho(A^\dagger V_2)}.$$

For $\alpha = 1$, the required result follows from [14], Lemma 2.8, since $\rho(U_i^\dagger V_i) = \dfrac{\rho(A^\dagger V_i)}{1+\rho(A^\dagger V_i)} > 0$ for $i = 1$ or 2. If $0 < \alpha < 1$, then from (8), we get

$$\rho(A^\dagger V_1) < \rho(A^\dagger V_2),$$

and proceeding as before, we get the desired result. The proof goes parallel in the case of proper nonnegative splitting of type II. □

Remark 3.7. The above theorem is also true if we replace the condition *the same type* by *different types.*

Another comparison result for proper nonnegative splittings of different types is established below which generalizes [19], Theorem 2.14.

Theorem 3.8. *Let $A = U_1 - V_1 = U_2 - V_2$ be two convergent proper nonnegative splittings of different types of a semi-monotone matrix $A \in \mathbb{R}^{m\times n}$. If there exists $0 < \alpha \leq 1$, such that $U_2^\dagger \leq \alpha U_1^\dagger$, then*

$$\rho(U_1^\dagger V_1) \leq \rho(U_2^\dagger V_2) < 1,$$

whenever $\alpha = 1$ and

$$\rho(U_1^\dagger V_1) < \rho(U_2^\dagger V_2) < 1,$$

whenever $0 < \alpha < 1$.

Proof. Assume that $A = U_1 - V_1$ is a convergent proper nonnegative splitting of type I and $A = U_2 - V_2$ is a convergent proper nonnegative splitting of type II. It then follows from [5], Lemma 6.2.1 that $(I - U_1^{\dagger}V_1)^{-1} \geq 0$ and $(I - V_2U_2^{\dagger})^{-1} \geq 0$, respectively. By using [6], Theorem 1 (6) and the given condition $U_2^{\dagger} \leq \alpha U_1^{\dagger}$, we have

$$A^{\dagger} = U_2^{\dagger}(I - V_2U_2^{\dagger})^{-1} \leq \alpha U_1^{\dagger}(I - V_2U_2^{\dagger})^{-1}. \tag{9}$$

Pre-multiplying (9) by $(I - U_1^{\dagger}V_1)^{-1}$, we get

$$(I - U_1^{\dagger}V_1)^{-1}A^{\dagger} \leq \alpha(I - U_1^{\dagger}V_1)^{-1}U_1^{\dagger}(I - V_2U_2^{\dagger}) = \alpha A^{\dagger}(I - V_2U_2^{\dagger})^{-1}. \tag{10}$$

Since $U_1^{\dagger}V_1 \geq 0$, there exists an eigenvector $x \geq 0$ such that

$$x^TU_1^{\dagger}V_1 = \rho(U_1^{\dagger}V_1)x^T.$$

So $x \in R(V_1^T) \subseteq R(A^T)$. Pre-multiplying (10) by x^T, we get

$$\frac{1}{1 - \rho(U_1^{\dagger}V_1)}x^TA^{\dagger} \leq \alpha x^TA^{\dagger}(I - V_2U_2^{\dagger})^{-1}.$$

By [5], Theorem 2.1.11, it then follows that

$$\frac{1}{1 - \rho(U_1^{\dagger}V_1)} \leq \frac{\alpha}{1 - \rho(V_2U_2^{\dagger})} = \frac{\alpha}{1 - \rho(U_2^{\dagger}V_2)},$$

i.e,

$$\rho(U_2^{\dagger}V_2) \geq (1 - \alpha) + \alpha\rho(U_1^{\dagger}V_1). \tag{11}$$

As $x^TA^{\dagger} \geq 0$ and $x^TA^{\dagger} \neq 0$. Suppose that $x^TA^{\dagger} = 0$, then $x^TA^{\dagger}A = 0$, i.e., $(A^{\dagger}A)^Tx = A^{\dagger}Ax = x = 0$, a contradiction. Hence $x^TA^{\dagger} \neq 0$. Now, the desired result follows immediately from (11).
In the case of $A = U_1 - V_1$ is a proper nonnegative splitting of type II and $A = U_2 - V_2$ is a proper nonnegative splitting of type I, the proof is analogous to the above proof. □

The next result addresses the question of existence of an α, which is an extension of [19], Corollary 2.15.

Theorem 3.9. *Let $A = U_1 - V_1 = U_2 - V_2$ be two convergent proper nonnegative splittings of different types of a semimonotone matrix $A \in \mathbb{R}^{m\times n}$. If $U_1^{\dagger} > U_2^{\dagger}$, then there exists α, $0 < \alpha < 1$, such that $U_2^{\dagger} \leq \alpha U_1^{\dagger}$ and $\rho(U_1^{\dagger}V_1) < \rho(U_2^{\dagger}V_2) < 1$.*

Proof. Denote

$$U_1^{\dagger} = (a_{ij}), \quad U_2^{\dagger} = (b_{ij}),\ i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m.$$

From $U_1^{\dagger} > U_2^{\dagger}$, we get

$$a_{ij} > b_{ij}, \quad i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m.$$

If there exists $b_{ij} > 0$ for some i, j, then let $\alpha = \max\limits_{\substack{0\leq i\leq n \\ 0\leq j\leq n}} \left\{ \frac{b_{ij}}{a_{ij}} |\ b_{ij} > 0 \right\}$, otherwise, $0 < \alpha < 1$ is arbitrary. Clearly, $0 < \alpha < 1$ and

$$b_{ij} \leq \alpha\ a_{ij}, \quad i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m,$$

i.e.,

$$U_2^\dagger \leq \alpha U_1^\dagger.$$

By Theorem 3.8, the inequality follows. □

The example given below demonstrates that the converse of Theorem 3.9 is not true.

Example 3.10. Let $A = \begin{pmatrix} 5 & -4 & 0 \\ -7 & 7 & 0 \end{pmatrix}$. Then $A^\dagger = \begin{pmatrix} 1 & 0.5714 \\ 1 & 0.7143 \\ 0 & 0 \end{pmatrix} \geq 0$. Let $U_1 = \begin{pmatrix} 5 & -1 & 0 \\ -7 & 7 & 0 \end{pmatrix}$ and $U_2 = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 8 & 0 \end{pmatrix}$. Then $A = U_1 - V_1$ is a proper nonnegative splitting of type I and $A = U_2 - V_2$ is a proper nonnegative splitting of type II. We have $0.7500 = \rho(U_1^\dagger V_1) < \rho(U_2^\dagger V_2) = 0.9015 < 1$, and for $\alpha = 0.8$, $U_2^\dagger = \begin{pmatrix} 0.2000 & 0 \\ 0 & 0.1250 \\ 0 & 0 \end{pmatrix} \leq \begin{pmatrix} 0.2000 & 0.0286 \\ 0.2000 & 0.1429 \\ 0 & 0 \end{pmatrix} = \alpha U_1^\dagger$. But $U_1^\dagger = \begin{pmatrix} 0.2500 & 0.0357 \\ 0.2500 & 0.1786 \\ 0 & 0 \end{pmatrix} \not\geq \begin{pmatrix} 0.2000 & 0 \\ 0 & 0.1250 \\ 0 & 0 \end{pmatrix} = U_2^\dagger$.

The following example shows that Theorem 3.8 and Theorem 3.9 do not valid, if we consider proper nonnegative splittings of same types instead of different types.

Example 3.11. Let $A = \begin{pmatrix} 3 & -2 & 3 \\ -2 & 3 & -2 \end{pmatrix}$. Then $A^\dagger = \begin{pmatrix} 3/10 & 1/5 \\ 2/5 & 3/5 \\ 3/10 & 1/5 \end{pmatrix} > 0$. Let $U_1 = \begin{pmatrix} 12 & -10 & 12 \\ -8 & 15 & -8 \end{pmatrix}$ and $U_2 = \begin{pmatrix} 25/2 & -10 & 25/2 \\ -8 & 15 & -8 \end{pmatrix}$. Then $A = U_1 - V_1 = U_2 - V_2$ are two convergent proper nonnegative splittings of type I. We have $U_1^\dagger = \begin{pmatrix} 0.0750 & 0.0500 \\ 0.0800 & 0.1200 \\ 0.0750 & 0.0500 \end{pmatrix} > U_2^\dagger = \begin{pmatrix} 0.0698 & 0.0465 \\ 0.0744 & 0.1163 \\ 0.0698 & 0.0465 \end{pmatrix}$, and for $\alpha = 0.9690 < 1$, $U_2^\dagger = \begin{pmatrix} 0.0698 & 0.0465 \\ 0.0744 & 0.1163 \\ 0.0698 & 0.0465 \end{pmatrix} \leq \begin{pmatrix} 0.0727 & 0.0484 \\ 0.0775 & 0.1163 \\ 0.0727 & 0.0484 \end{pmatrix} = \alpha U_1^\dagger$. But $\rho(U_1^\dagger V_1) = \rho(U_2^\dagger V_2) = 0.8$.

The condition $A^\dagger \geq 0$ in Theorem 3.8 and Theorem 3.9 is not redundant, and is illustrated hereunder by an example.

Example 3.12. Let $A = \begin{pmatrix} 2 & -7 & 2 \\ -8 & 5 & -8 \end{pmatrix}$. Then $A^\dagger = \begin{pmatrix} -0.0543 & -0.0761 \\ -0.1739 & -0.0435 \\ -0.0543 & -0.0761 \end{pmatrix} < 0$. Let $U_1 = \begin{pmatrix} 4 & -35 & 4 \\ -16 & 25 & -16 \end{pmatrix}$ and $U_2 = \begin{pmatrix} 3 & -21/2 & 3 \\ -12 & 15/2 & -12 \end{pmatrix}$. Then $A = U_1 - V_1$ is a proper nonnegative splitting of type I and $A = U_2 - V_2$ is a proper nonnegative splitting of type II. We have $0.3333 = \rho(U_2^\dagger V_2) < \rho(U_1^\dagger V_1) = 0.8$. But $U_2^\dagger = \begin{pmatrix} -0.0362 & -0.0507 \\ -0.1159 & -0.0290 \\ -0.0362 & -0.0507 \end{pmatrix} < \begin{pmatrix} -0.0272 & -0.0380 \\ -0.0348 & -0.0087 \\ -0.0272 & -0.0380 \end{pmatrix} = U_1^\dagger$.

The above example also motivates us to prove the following theorem which is a generalization of [21], Theorem 2.4 to rectangular matrices. However, we provide below a short new proof.

Theorem 3.13. *Let $A = U_1 - V_1 = U_2 - V_2$ be two convergent proper nonnegative splittings of different types of $A \in \mathbb{R}^{m\times n}$. If $A^\dagger \leq 0$ and $U_2^\dagger \geq U_1^\dagger$, then*

$$\rho(U_1^\dagger V_1) \leq \rho(U_2^\dagger V_2) < 1.$$

In particular, if $A^\dagger < 0$ and $U_2^\dagger > U_1^\dagger$, then

$$\rho(U_1^\dagger V_1) < \rho(U_2^\dagger V_2) < 1.$$

Proof. Assume that $A = U_1 - V_1$ is a proper nonnegative of type I and $A = U_2 - V_2$ is a proper nonnegative of type II. Then there exists an eigenvector $x \geq 0$ such that

$$x^T U_1^\dagger V_1 = \rho(U_1^\dagger V_1)x^T \tag{12}$$

Therefore, $x \in R(V_1^T) \subseteq R(U_1^T) = R(A^T)$. From the given condition $U_2^\dagger \geq U_1^\dagger$, we obtain the following inequality

$$A^\dagger = U_2^\dagger(I - V_2U_2^\dagger)^{-1} \geq U_1^\dagger(I - V_2U_2^\dagger)^{-1}. \tag{13}$$

Pre-multiplying (13) by $(I - U_1^\dagger V_1)^{-1}$, we obtain

$$(I - U_1^\dagger V_1)^{-1}A^\dagger \geq (I - U_1^\dagger V_1)^{-1}U_1^\dagger(I - V_2U_2^\dagger)^{-1} = A^\dagger(I - V_2U_2^\dagger)^{-1}. \tag{14}$$

Again, pre-multiplying (14) by x^T, we get

$$\frac{1}{1-\rho(U_1^\dagger V_1)}x^T A^\dagger \geq x^T A^\dagger(I - V_2U_2^\dagger)^{-1}. \tag{15}$$

Let $z = x^T A^\dagger$. Clearly, $z \leq 0$ and $z \neq 0$. Otherwise, $x \in R(A^T) \cap N(A)$, which is a contradiction. So, we get

$$\frac{1}{1-\rho(U_1^\dagger V_1)}(-z) \leq (-z)(I - V_2U_2^\dagger)^{-1}.$$

Now, the required result follows from [5], Theorem 2.1.11. The proof follows similarly when $A = U_1 - V_1$ is proper nonnegative of type II and $A = U_1 - V_1$ is proper nonnegative of type I. □

3.3. **Comparison of Proper Multisplittings.** Improving the rate of convergence of the iteration scheme (2) is a problem of interest for getting the solution faster. In this direction, Climent and Perea [7] proposed multisplitting theory for rectangular matrices while the authors of [17] studied the same problem in the nonsingular matrix setting. Here, we revisit the same theory proposed by Climent and Perea [7] first, and then produced a few new convergence and comparison theorems for proper multisplittings. In this context, the definition of a proper multisplitting is recalled below.

Definition 3.14. ([7], Definition 2)
The triplet $(U_k, V_k, E_k)_{k=1}^{p}$ is called a *proper multisplitting* of $A \in \mathbb{R}^{m\times n}$ if
(i) $A = U_k - V_k$ is a proper splitting, for each $k = 1, 2, \ldots, p$,
(ii) $E_k \geq 0$, for each $k = 1, 2, \ldots, p$ is a diagonal $n \times n$ matrix, and $\sum_{k=1}^{p} E_k = I$, where I is the $n \times n$ identity matrix.

Using the above definition, Climent and Perea [7] have considered the iteration scheme for solving (1) as follows:

$$x^{i+1} = Hx^i + Gb, \quad i = 0, 1, 2, \ldots, \tag{16}$$

where $H = \sum_{k=1}^{p} E_k U_k^{\dagger} V_k$ and $G = \sum_{k=1}^{p} E_k U_k^{\dagger}$. Here onwards, all H and G are defined as above unless stated otherwise.

A proper multisplitting is called a *proper regular multisplitting* or a *proper weak regular multisplitting*, if each one of the proper splitting is a proper regular splitting or a proper weak regular splitting, respectively. Climent and Perea [7] obtained the following results for a proper weak regular multisplitting.

Lemma 3.15. ([7], Lemma 1)
Let $(U_k, V_k, E_k)_{k=1}^{p}$ be a proper weak regular multisplitting of $A \in \mathbb{R}^{m\times n}$. Then
(i) $H \geq 0$ and therefore H^j for $j = 0, 1, \ldots$.
(ii) $\sum_{k=1}^{p} E_k U_k^{\dagger} A = (I - H)A^{\dagger}A$.
(iii) $(I + H + H^2 + \cdots + H^m)(I - H) = I - H^{m+1}$.

Theorem 3.16. ([7], Theorem 4)
Let $(U_k, V_k, E_k)_{k=1}^{p}$ be a proper weak regular multisplitting of a semi-monotone matrix $A \in \mathbb{R}^{m\times n}$. Then $\rho(H) < 1$.

It is of interest to know the type of splitting $B - C$ of A that yields the iteration scheme (16) which is restated as what can we say about the type of the induced splitting $A = B - C$ being induced by $H = \sum_{k=1}^{p} E_k U_k^{\dagger} V_k$. This problem in nonsingular matrix setting is also discussed by Elsner [9]. With an additional hypothesis $R(E_k) \subseteq R(A^T)$, for each $k = 1, 2, \ldots, p$, of a proper weak regular multisplitting, we establish the following new result which addresses the above issue partially.

Theorem 3.17. *Let $(U_k, V_k, E_k)_{k=1}^{p}$ be a proper weak regular multisplitting of a semi-monotone matrix $A \in \mathbb{R}^{m\times n}$. Then the unique splitting $A = B - C$ induced by H with $B = A(I - H)^{-1}$ is a convergent proper weak regular splitting if $R(E_k) \subseteq R(A^T)$, for each $k = 1, 2, \ldots, p$.*

Proof. By using the condition $R(E_k) \subseteq R(A^T)$, we have $A^\dagger A E_k = E_k$ and $E_k A^\dagger A = E_k$. Then

$$\begin{aligned} A^\dagger A H &= A^\dagger A \sum_{k=1}^{p} E_k U_k^\dagger V_k \\ &= \sum_{k=1}^{p} A^\dagger A E_k U_k^\dagger V_k \\ &= \sum_{k=1}^{p} E_k U_k^\dagger V_k \\ &= \sum_{k=1}^{p} E_k U_k^\dagger V_k A^\dagger A \\ &= H A^\dagger A \\ &= H. \end{aligned}$$

Now, post-multiplying Lemma 3.15 (ii) by $A^\dagger$, we get $G = (I-H)A^\dagger$. By Theorem 3.16, we obtain $\rho(H) < 1$ and so $(I-H)$ is invertible. From equation (16), we obtain $B^\dagger = G = (I-H)A^\dagger$. Let $X = A(I-H)^{-1}$. Then $XB^\dagger = AA^\dagger$ and $B^\dagger X = (I-H)A^\dagger A(I-H)^{-1} = (A^\dagger A - HA^\dagger A)(I-H)^{-1} = (A^\dagger A - A^\dagger AH)(I-H)^{-1} = A^\dagger A(I-H)(I-H)^{-1} = A^\dagger A$ which imply $XB^\dagger$ and $B^\dagger X$ are symmetric. Also, $XB^\dagger X = AA^\dagger A(I-H)^{-1} = A(I-H)^{-1} = X$ and $B^\dagger XB^\dagger = A^\dagger A(I-H)A^\dagger = (A^\dagger A - A^\dagger AH)A^\dagger = (A^\dagger A - HA^\dagger A)A^\dagger = (I-H)A^\dagger AA^\dagger = (I-H)A^\dagger = B^\dagger$. Therefore, $B = A(I-H)^{-1}$.

Clearly, $R(B) = R(A)$ as $B = A(I-H)^{-1}$. Next we prove that $N(B) = N(A)$. Let $x \in N(A)$. Then $0 = Ax = B(I-H)x = B(x-Hx) = B(x - \sum_{k=1}^{p} E_k U_k^\dagger V_k x) = Bx$, since $N(V_k) \supseteq N(A)$. So $N(A) \subseteq N(B)$. Again, let $y \in N(B)$. Then we get $By = A(I-H)^{-1}y = 0$. Pre-multiplying $A^\dagger$, we get $A^\dagger A(I-H)^{-1}y = 0$. Again, using the fact that $A^\dagger AH = HA^\dagger A$ and pre-multiplying A, we get $Ay = 0$. So $N(B) \subseteq N(A)$. Thus $N(B) = N(A)$. Next, we have to prove that $A = B - C$ is unique. Suppose that there exists another induced splitting $A = \tilde{B} - \tilde{C}$ such that $\tilde{B} = A(I-H)^{-1}$. Then $\tilde{B}^\dagger \tilde{C} = H$ and $\tilde{B}H = \tilde{B}\tilde{B}^\dagger \tilde{C} = \tilde{C} = \tilde{B} - A$. So $\tilde{B} = A + \tilde{B}H$, i.e., $\tilde{B}(I-H) = A$. This reveals that $\tilde{B} = A(I-H)^{-1} = B$ and therefore, H induces the unique proper splitting $A = B - C$. Finally, $B^\dagger = G \geq 0$ and $B^\dagger C = B^\dagger(B-A) = B^\dagger B - B^\dagger A = A^\dagger A - A^\dagger A(I-H) = A^\dagger AH = H \geq 0$. By Theorem 3.16, we get $\rho(B^\dagger C) = \rho(H) < 1$. □

The following example illustrates Theorem 3.17.

Example 3.18. Let $A = \begin{pmatrix} 3 & -2 \\ -2 & 3 \\ 3 & -2 \end{pmatrix}$.

Set $U_1 = \begin{pmatrix} 6 & -6 \\ -4 & 9 \\ 6 & -6 \end{pmatrix}$, $U_2 = \begin{pmatrix} 9 & -8 \\ -6 & 12 \\ 9 & -8 \end{pmatrix}$, $E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $E_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

Then $(U_k, V_k, E_k)_{k=1}^{2}$ is a proper weak regular multisplitting. Also, $R(E_k) \subseteq R(A^T)$, for each $k = 1, 2$. Thus, all the conditions of Theorem 3.17 are satisfied. We observe that $B = A(I-H)^{-1} = \begin{pmatrix} 6 & -8 \\ -4 & 12 \\ 6 & -8 \end{pmatrix}$ and the induced splitting $A = B - C$ is proper weak regular with $\rho(B^\dagger C) = 0.75 < 1$, as $R(B) = R(A)$, $N(B) = N(A)$, $B^\dagger = \begin{pmatrix} 3/20 & 1/5 & 3/20 \\ 1/20 & 3/20 & 1/20 \end{pmatrix} \geq 0$ and $B^\dagger C = \begin{pmatrix} 1/2 & 0 \\ 0 & 3/4 \end{pmatrix} \geq 0$.

Next result says that the induced splitting is also a proper regular splitting under the assumption of an extra condition $A \geq 0$.

Theorem 3.19. *Let $(U_k, V_k, E_k)_{k=1}^{p}$ be a proper weak regular multisplitting of a semi-monotone matrix $A \in \mathbb{R}^{m\times n}$. Then the splitting $A = B - C$ induced by H is a proper regular splitting if $A \geq 0$ and $R(E_k) \subseteq R(A^T)$, for each $k = 1, 2, \ldots, p$.*

Proof. By Theorem 3.17, the splitting $A = B - C$ induced by H is proper weak regular. Now we have to show that $C \geq 0$. So $C = B - A = A(I-H)^{-1} - A = A(I-H)^{-1}H \geq 0$, since $H \geq 0$ and $\rho(H) < 1$ by Theorem 3.16. □

We obtain the following corollary for a square nonsingular matrix A.

Corollary 3.20. *Let $(U_k, V_k, E_k)_{k=1}^{p}$ be a weak regular multisplitting of a monotone matrix $A \in \mathbb{R}^{n\times n}$. Then the splitting $A = B - C$ induced by H is a regular splitting if $A \geq 0$.*

Next theorem compares the spectral radii between a multisplitting and a splitting of a real rectangular matrix A.

Theorem 3.21. *Let $(U_k, V_k, E_k)_{k=1}^{p}$ be a proper weak regular multisplitting of a semi-monotone matrix $A \in \mathbb{R}^{m\times n}$ and $\underline{U}, \overline{U} \in \mathbb{R}^{m\times n}$ such that*

$$\overline{U}^\dagger \leq U_k^\dagger \leq \underline{U}^\dagger, \text{ for each } k = 1, 2, \ldots, p$$

and $R(E_k) \subseteq R(A^T)$, for each $k = 1, 2, \ldots, p$.

(i) If $A = \overline{U} - \overline{V}$ is a proper regular splitting and row sums of $\overline{U}^\dagger$ are positive, then

$$\rho(H) \leq \rho(\overline{U}^\dagger \overline{V}).$$

(ii) If $A = \underline{U} - \underline{V}$ is a proper regular splitting, then

$$\rho(\underline{U}^\dagger \underline{V}) \leq \rho(H).$$

Proof. (i) Let $\widetilde{U}_1 = B$, $\widetilde{U}_2 = \overline{U}$ and $\widetilde{V}_2 = \overline{V}$. Then $\widetilde{U}_1^\dagger \widetilde{V}_1 = B^\dagger(B - A) = B^\dagger B - B^\dagger A = A^\dagger A - (I - H)A^\dagger A = HA^\dagger A = H \geq 0$. The condition $U_k^\dagger \geq \overline{U}^\dagger$ implies $\widetilde{U}_1^\dagger \geq \widetilde{U_2}^\dagger$, $\widetilde{V}_2 \geq 0$. Hence, $\rho(H) \leq \rho(\overline{U}^\dagger \overline{V})$ by [14], Theorem 3.4 (iii).

(ii) Define $\widetilde{U}_1 = \underline{U}$, $\widetilde{V}_1 = \underline{V}$ and $\widetilde{U}_2 = B$, and on applying [14], Theorem 3.4 (ii), we obtain $\rho(\underline{U}^\dagger \underline{V}) \leq \rho(H)$. □

For a square nonsingular matrix A, the above result reduces to the following corollary.

Corollary 3.22. *Let $(U_k, V_k, E_k)_{k=1}^p$ be a weak regular multisplitting of a monotone matrix $A \in \mathbb{R}^{n\times n}$ and $\underline{U}, \overline{U} \in \mathbb{R}^{n\times n}$ such that*

$$\overline{U}^{-1} \leq U_k^{-1} \leq \underline{U}^{-1}, \text{ for each } k = 1, 2, \ldots, p.$$

(i) If $A = \overline{U} - \overline{V}$ is a regular splitting, then

$$\rho(H) \leq \rho(\overline{U}^{-1}\overline{V}).$$

(ii) If $A = \underline{U} - \underline{V}$ is a regular splitting, then

$$\rho(\underline{U}^{-1}\underline{V}) \leq \rho(H).$$

The spectral radii of iteration matrices of two proper weak regular multisplittings of the same coefficient matrix A is compared below.

Theorem 3.23. *Let $(U_k^{(i)}, V_k^{(i)}, E_k)_{k=1}^p$, $i = 1, 2$, be two proper weak regular multisplittings of a non-negative semi-monotone matrix $A \in \mathbb{R}^{m\times n}$ such that $R(E_k) \subseteq R(A^T)$, for each $k = 1, 2, \ldots, p$. If $V_k^{(2)} \geq V_k^{(1)}$, for each $k = 1, 2, \ldots, p$, then*

$$\rho(H_1) \leq \rho(H_2) < 1,$$

where $H_i = \sum_{k=1}^{p} E_k [U_k^{(i)}]^\dagger V_k^{(i)}$, *for each* $i = 1, 2$.

Proof. From $V_k^{(2)} \geq V_k^{(1)}$, for each $k = 1, 2, \ldots, p$, we obtain

$$U_k^{(2)} \geq U_k^{(1)}, \text{ for each } k = 1, 2, \ldots, p.$$

Since $R(U_k^{(1)}) = R(U_k^{(2)})$ and $N(U_k^{(1)}) = N(U_k^{(2)})$ by [14], Lemma 3.16, it follows that

$$[U_k^{(1)}]^\dagger \geq [U_k^{(2)}]^\dagger, \text{ for each } k = 1, 2, \ldots, p.$$

Consequently,

$$\sum_{k=1}^{p} E_k [U_k^{(1)}]^\dagger \geq \sum_{k=1}^{p} E_k [U_k^{(2)}]^\dagger,$$

i.e.,

$$B_1^\dagger \geq B_2^\dagger.$$

By Theorem 3.19, the splittings $A = B_1 - C_1 = B_2 - C_2$ induced by H_1 and H_2 are proper regular splittings. Hence, by Theorem 2.2, we obtain $\rho(H_1) \leq \rho(H_2) < 1$. □

We have the following corollary.

Corollary 3.24. *Let* $(U_k^{(i)}, V_k^{(i)}, E_k)_{k=1}^p,\ i = 1, 2,$ *be two weak regular multisplittings of a non-negative monotone matrix* $A \in \mathbb{R}^{n\times n}$. *If* $V_k^{(2)} \geq V_k^{(1)}$, *for each* $k = 1, 2, \ldots, p$, *then*

$$\rho(H_1) \leq \rho(H_2) < 1,$$

where $H_i = \sum_{k=1}^{p} E_k[U_k^{(i)}]^{-1}V_k^{(i)}$, *for each* $i = 1, 2$.

Remark 3.25. Theorem 3.19 and Theorem 3.23 are also true if we assume $G^\dagger \geq 0$ instead of $A \geq 0$.

Next result compares the spectral radii of iteration matrices of two proper weak regular multisplittings of the same coefficient matrix A.

Theorem 3.26. *Let* $(U_k^{(i)}, V_k^{(i)}, E_k)_{k=1}^p,\ i = 1, 2,$ *be two proper weak regular multisplittings of a non-negative semi-monotone matrix* $A \in \mathbb{R}^{m\times n}$ *such that* $R(E_k) \subseteq R(A^T)$, *for each* $k = 1, 2, \ldots, p$. *If* $[U_k^{(1)}]^\dagger \geq [U_k^{(2)}]^\dagger$, *for each* $k = 1, 2, \ldots, p$, *then*

$$\rho(H_1) \leq \rho(H_2) < 1.$$

Proof. By Theorem 3.19, the splittings $A = B_1 - C_1 = B_2 - C_2$ induced by $H_1 = \sum_{k=1}^{p} E_k[U_k^{(1)}]^\dagger V_k^{(1)}$ and $H_2 = \sum_{k=1}^{p} E_k[U_k^{(2)}]^\dagger V_k^{(2)}$ are proper regular splittings. From

$$[U_k^{(1)}]^\dagger \geq [U_k^{(2)}]^\dagger, \text{ for each } k = 1, 2, \ldots, p,$$

we have

$$\sum_{k=1}^{p} E_k[U_k^{(1)}]^\dagger \geq \sum_{k=1}^{p} E_k[U_k^{(2)}]^\dagger, \text{ for each } k = 1, 2, \ldots p,$$

i.e.,

$$B_1^\dagger \geq B_2^\dagger.$$

Hence, by Theorem 2.2, we obtain $\rho(H_1) \leq \rho(H_2) < 1$. □

The following example illustrates Theorem 3.23 and Theorem 3.26.

Example 3.27. Let $A = \begin{pmatrix} 3 & 0 \\ 0 & 2 \\ 0 & 3 \end{pmatrix}$. Set $U_1^{(1)} = \begin{pmatrix} 6 & 0 \\ 0 & 2 \\ 0 & 3 \end{pmatrix}, U_1^{(2)} = \begin{pmatrix} 9 & 0 \\ 0 & 2 \\ 0 & 3 \end{pmatrix}$, $U_2^{(1)} = \begin{pmatrix} 12 & 0 \\ 0 & 4 \\ 0 & 6 \end{pmatrix}, U_2^{(2)} = \begin{pmatrix} 16 & 0 \\ 0 & 4 \\ 0 & 6 \end{pmatrix}, E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $E_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, respectively. Then $(U_k^{(1)}, V_k^{(1)}, E_k)_{k=1}^2$ and $(U_k^{(2)}, V_k^{(2)}, E_k)_{k=1}^2$ are two proper weak regular multisplittings of a non-negative semi-monotone matrix A with $R(E_k) \subseteq R(A^T)$. Also, $V_1^{(2)} = \begin{pmatrix} 9 & 0 \\ 0 & 2 \\ 0 & 3 \end{pmatrix} \geq \begin{pmatrix} 3 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = V_1^{(1)}, V_2^{(2)} = \begin{pmatrix} 13 & 0 \\ 0 & 2 \\ 0 & 3 \end{pmatrix} \geq$

$\begin{pmatrix} 6 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = V_2^{(1)}$, and $[U_1^{(1)}]^\dagger = \begin{pmatrix} 0.1667 & 0 & 0 \\ 0 & 0.1538 & 0.2308 \end{pmatrix} \geq$
$\begin{pmatrix} 0.1111 & 0 & 0 \\ 0 & 0.1538 & 0.2308 \end{pmatrix} = [U_1^{(2)}]^\dagger,\ [U_2^{(1)}]^\dagger = \begin{pmatrix} 0.0833 & 0 & 0 \\ 0 & 0.0769 & 0.1154 \end{pmatrix} \geq$
$\begin{pmatrix} 0.0625 & 0 & 0 \\ 0 & 0.0769 & 0.1154 \end{pmatrix} = [U_2^{(2)}]^\dagger$.

Thus, all the conditions of Theorem 3.23 and Theorem 3.26 are satisfied. Note that $0.5 = \rho(H_1) \leq \rho(H_2) = 0.75 < 1$.

The following corollary follows immediately from the above result when a square nonsingular system of linear equations is considered.

Corollary 3.28. *Let $(U_k^{(i)}, V_k^{(i)}, E_k)_{k=1}^p,\ i = 1, 2,$ be two weak regular multisplittings of a non-negative monotone matrix $A \in \mathbb{R}^{n\times n}$. If $[U_k^{(1)}]^{-1} \geq [U_k^{(2)}]^{-1}$, for each $k = 1, 2, \ldots, p$, then*

$$\rho(H_1) \leq \rho(H_2) < 1.$$

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