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ALMOST PERIODICITY OF ABSTRACT VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

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ABSTRACT. The main purpose of this paper is to investigate almost periodic properties of various classes of (a, k)-regularized C-resolvent families in Banach spaces. We contemplate the work of many other authors working in this field, giving also some original contributions and applications. In general case, (a, k)-regularized C-resolvent families under our considerations are degenerate and their subgenerators are multivalued linear operators or pairs of closed linear operators. We also consider the class of (a, k)-regularized (C_1, C_2)-existence and uniqueness families, where the operators C_1 and C_2 are not necessarily injective, and provide several illustrative examples of abstract Volterra integro-differential equations which do have almost periodic solutions.

1. INTRODUCTION AND PRELIMINARIES

There is an enormous literature devoted to the study of various types of almost periodic properties of abstract integro-differential equations in Banach spaces (for abstract differential equations of first and second order, we refer the reader to [4]-[5], [7], [11]-[13], [16], [23], [27], [36] and [43]; there is a huge number of other research papers that we cannot cite here because of space limitations). Comprehensive survey of results on abstract non-degenerate almost periodic differential equations can be found in the monograph [28] by Y. Hino, T. Naito, N. V. Minh and J. S. Shin.

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The genesis of this paper is motivated by the fact that we have not been able to locate any significant reference which treats the almost periodicity of (a,k)-regularized C-resolvent families in Banach spaces, even in the case that they are non-degenerate in time or that $C = I$. We focus special attention on the analysis of almost periodic properties of abstract degenerate Volterra integro-differential equations, which has not been the usual case with the investigations carried out so far. In the present state of our knowledge, we do know to quote only two research papers concerning almost periodic and asymptotically almost periodic properties of abstract degenerate differential equations: the paper [41] by Q.-P. Vu (devoted to the study of asymptotical almost periodicity) and the paper [34] by N. T. Lan; in both papers, the authors have considered abstract degenerate differential equations of first order.

In [37, Section 11.4], J. Prüss has analyzed the almost periodic solutions, Stepanov almost periodic solutions and asymptotically almost periodic solutions of the following abstract non-degenerate Cauchy problem

$$u'(t) = \int_0^\infty A_0(s)u'(t-s)\,ds + \int_0^\infty dA_1(s)u(t-s) + f(t), \quad t \in \mathbb{R},$$

where $A_0 \in L^1([0,\infty) : L(Y,X))$, $A_1 \in BV([0,\infty) : L(Y,X))$, X and Y are Banach spaces such that Y is densely and continuously embedded into X. Unquestionably, this was the first work where the existence and uniqueness of various types of almost periodic solutions of abstract non-degenerate Volterra integro-differential equations have been considered. Only a year after the appearing the monograph [37], Q.-P. Vu [42] enquired into the almost periodicity of the abstract Cauchy problems like

$$u'(t) = Au(t) + \int_0^\infty dBu(\tau)u(t-\tau) + f(t), \quad t \in \mathbb{R},$$

where A is a closed linear operator acting on a Banach space X, $(B(t))_{t\geq 0}$ is a family of closed linear operators on X and $f : \mathbb{R} \to X$ is continuous. Mention should also be made of paper [2] by R. Agarwal, B. de Andrade and C. Cuevas, where the authors have considered various types of periodicity for solutions of the following fractional differential equation

$$D_t^\alpha u(t) = Au(t) + D^{\alpha-1}f(t,u(t)), \quad t \in \mathbb{R},$$

where $1 < \alpha < 2$, $D_t^\alpha u(t)$ is a Riemann-Liouville fractional type derivative of order α, $A : D(A) \subseteq X \to X$ is a linear, densely defined, sectorial operator on a complex Banach space X, and $f : \mathbb{R} \times X \to X$ is a pseudo-almost periodic function satisfying suitable conditions in the space variable x. Further information on various types of almost periodic solutions of abstract non-degenerate Volterra equations and abstract non-degenerate fractional differential equations can be obtained by consulting the references [3], [15] and [17], written by C. Lizama and his collaborators.

The organization and main ideas of this paper are described as follows. After giving some preliminary results and definitions, in Section 2 we recollect the most important facts about multivalued linear operators in Banach spaces (for further information concerning the theory of multivalued linear operators, we refer the reader to the monographs [14] by R. Cross, [22] by A. Favini and A. Yagi, [32] by the author, and references cited therein). In a separate subsection, we consider multivalued linear operators as subgenerators of various types of (a, k)-regularized (C_1, C_2)-existence and uniqueness families, with the operators C_1 and C_2 being not necessarily injective. After that, we single out the class of (a, k)-regularized C-resolvent families for special considerations. In Definition 2.6 and Proposition 2.7 (Definition 4.10 and Proposition 4.11), we introduce the class of degenerate K-convoluted C-groups (degenerate (a, k)-regularized C-resolvent group families) and prove its composition property. The main aim of Section 3 is to observe that a great number of structural results proved by Q. Zheng, L. Liu [45] and T.-J. Xiao, J. Liang [43, Section 7.1.1] continue to hold in degenerate case. This section is almost completely written in expository manner and the proofs are given only for a few results. Our main contributions are presented in Section 4, where we investigate almost periodic properties of (a, k)-regularized C-resolvent families subgenerated by multivalued linear operators and almost periodic properties of (a, k)-regularized C-resolvent families generated by a pair of closed linear operators A, B with domains and ranges contained in a complex Banach space X. The abstract results obtained in Section 4 seem to be completely new even in non-degenerate case and they can be simply incorporated in the study of existence and uniqueness of almost periodic solutions of the following abstract degenerate Volterra equation

$$Bu(t) = \int_0^t a(t-s)Au(s)\,ds + f(t),\ t \geq 0,$$

and the following abstract Volterra inclusion

$$u(t) \in \mathcal{A}\int_0^t a(t-s)u(s)\,ds + f(t),\ t \geq 0,$$

where $a \in L^1_{loc}([0,\infty))$, $a \neq 0$, $f : [0,\infty) \to X$ is continuous and $\mathcal{A}$ is a closed multivalued linear operator on X.

We use the standard notation throughout the paper. Unless specifed otherwise, we shall always assume henceforth that X is a complex Banach space. If Y is also such a space, then we denote by $L(X, Y)$ the space of all continuous linear mappings from X into Y; $L(X) \equiv L(X, X)$. If A is a closed linear operator acting on X, then the domain, kernel space and range of A will be denoted by $D(A)$, $N(A)$ and $R(A)$, respectively. Since no confusion seems likely, we will identify A with its graph. By $[D(A)]$ we denote the Banach space $D(A)$ equipped with the graph norm $\|x\|_{[D(A)]} := \|x\| + \|Ax\|$, $x \in D(A)$. By X^* we denote the dual space of X.

Given $s \in \mathbb{R}$ in advance, set $\lceil s \rceil := \inf\{l \in \mathbb{Z} : s \leq l\}$. By $C([0,\tau) : X)$, where $0 < \tau \leq \infty$, we denote the space consisting of all X-valued continuous functions on the interval $[0,\tau)$. The Gamma function is denoted by $\Gamma(\cdot)$ and the principal branch is always used to take the powers; the convolution like mapping $*$ is given by $f * g(t) := \int_0^t f(t-s)g(s)\,ds$. Set $g_\zeta(t) := t^{\zeta-1}/\Gamma(\zeta)$, $\zeta > 0$.

Fairly complete information about fractional calculus and fractional differential equations can be obtained by consulting [9], [19], [29]-[32] and references cited therein. The Mittag-Leffler function $E_{\alpha,\beta}(z)$, defined by

$$E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)}, \quad z \in \mathbb{C},$$

is known to play a crucial role in the anaysis of fractional differential equations. For the basic properties of Mittag-Leffler functions, we refer the reader to [9] and [19].

Throughout the paper, we shall always assume that the function $k(t)$ is a scalar-valued continuous kernel on $[0,\infty)$. The following condition on function $k(t)$ will be used occasionally:

(P1): $k(t)$ is Laplace transformable, i.e., it is locally integrable on $[0,\infty)$ and there exists $\beta \in \mathbb{R}$ such that $\tilde{k}(\lambda) := \mathcal{L}(k)(\lambda) := \lim_{b\to\infty} \int_0^b e^{-\lambda t}k(t)\,dt := \int_0^\infty e^{-\lambda t}k(t)\,dt$ exists for all $\lambda \in \mathbb{C}$ with $\Re\lambda > \beta$. Put $abs(k) :=\inf\{\Re\lambda : \tilde{k}(\lambda) \text{ exists}\}$, $\tilde{\delta}(\lambda) := 1$ and denote by $\mathcal{L}^{-1}$ the inverse Laplace transform.

We refer the reader to [4], [43, Chapter 1] and [31, Section 1.2] for further information concerning the vector-valued Laplace transform.

The basic facts about the theory of abstract degenerate differential equations of first and second order can be obtained by consulting the monographs [22] by A. Favini, A. Yagi and [39] by G. A. Sviridyuk, V. E. Fedorov. The theory of abstract degenerate Volterra integro-differential equations is an active field of research. We can recommend for the reader the forthcoming monograph [32].

Let $I = \mathbb{R}$ or $I = [0,\infty)$, and let $f : I \to X$ be continuous. Given $\epsilon > 0$, we call $\tau > 0$ an ϵ-period for $f(\cdot)$ iff

$$\|f(t+\tau) - f(t)\| \le \epsilon, \quad t \in I. \tag{1.1}$$

The set constituted of all ϵ-periods for $f(\cdot)$ is denoted by $\vartheta(f,\epsilon)$. It is said that $f(\cdot)$ is almost periodic, a.p. for short, iff for each $\epsilon > 0$ the set $\vartheta(f,\epsilon)$ is relatively dense in I, which means that there exists $l > 0$ such that any subinterval of I of length l meets $\vartheta(f,\epsilon)$. We call $f(\cdot)$ weakly almost periodic, w.a.p. for short, iff for each $x^* \in X^*$ the function $x^*(f(\cdot))$ is almost periodic. A family of functions $\mathcal{F} \subseteq X^I$ is said to be uniformly almost periodic iff for each $\epsilon > 0$ there exists $l > 0$ such that any subinterval of I of length l contains a number $\tau > 0$ such that (1.1) holds for all $f \in \mathcal{F}$.

By $AP(I : X)$ we denote the vector space consisting of all almost periodic functions from the interval I into X. Equipped with the sup-norm, $AP(I : X)$ becomes a Banach space.

In [32, Theorem 2.10.16], we have recently reconsidered S. El Mourchid's result [20, Theorem 2.1] concerning the connection between the imaginary point spectrum and hypercyclicity of strongly continuous semigroups. The analysis contained in [32, Example 2.10.17] enables one to simpy construct examples of degenerate first order Cauchy problems $(\mathrm{DFP})_R$ and $(\mathrm{DFP})_L$ whose strong solutions exist and are almost periodic for all initial values belonging a non-trivial subspace X_0 of the pivot space $X = BUC(\mathbb{R})$; cf. Section 2 for the notion, and Example 4.15-Example 4.16 for similar applications.

For the sequel, we need the following definition.

Definition 1.1. Let $I = \mathbb{R}$ or $I = [0, \infty)$, and let $(F(t))_{t \in I} \subseteq L(X)$ be a strongly continuous operator family. It is said that $(F(t))_{t \in I}$ is (weakly) almost periodic iff for each $x \in X$ the mapping $t \mapsto F(t)x$, $t \in I$ is (weakly) almost periodic. It is said that $(F(t))_{t \in I}$ is uniformly almost periodic iff the family $\{F(\cdot)x : \|x\| \leq 1\}$ is uniformly almost periodic.

The concept of almost periodicity was first studied by H. Bohr in 1925 and later generalized by V. Stepanov, H. Weyl and A. S. Besicovitch, amongst many others. Almost periodic Banach space valued functions has been investigated in [8], [24], [35] and [44]; concerning almost automorphic and almost periodic functions in Banach spaces, we can also recommend the monographs [24]-[25] by G. M. N'Guérékata and [18] by T. Diagana.

The most intriguing properties of almost periodic vector-valued functions are collected in the following lemma.

Lemma 1.2. *Let $f \in AP(\mathbb{R} : X)$. Then the following holds:*

(i) *$f(t)$ is bounded, i.e., $\sup_{t \in \mathbb{R}} \|f(t)\| < \infty$;*
(ii) *if $g \in AP(\mathbb{R} : X)$, $h \in AP(\mathbb{R} : \mathbb{C})$, then $f + g$ and $hf \in AP(\mathbb{R} : X)$;*
(iii) *$P_r(f) := \lim_{t \to \infty} \frac{1}{t} \int_0^t e^{-irs} f(s)\, ds$ exists for all $r \in \mathbb{R}$ (Bohr's transform of $f(\cdot)$) and $P_r(f) := \lim_{t \to \infty} \frac{1}{t} \int_{\alpha}^{t+\alpha} e^{-irs} f(s)\, ds$ for all $\alpha,\ r \in \mathbb{R}$;*
(iv) *if $P_r(f) = 0$ for all $r \in \mathbb{R}$, then $f(t) = 0$ for all $t \in \mathbb{R}$;*
(v) *$\sigma(f) := \{r \in \mathbb{R} : P_r(f) \neq 0\}$ is at most countable;*
(vi) *if $c_0 \nsubseteq X$, which means that X does not contain an isomorphic copy of c_0, and $g(t) = \int_0^t f(s)\, ds$ $(t \in \mathbb{R})$ is bounded, then $g \in AP(\mathbb{R} : X)$;*
(vii) *if $(g_n)_{n \in \mathbb{N}}$ is a sequence in $AP(\mathbb{R} : X)$ and $(g_n)_{n \in \mathbb{N}}$ converges uniformly to g, then $g \in AP(\mathbb{R} : X)$;*
(viii) *if $f' \in BUC(\mathbb{R} : X)$, then $f' \in AP(\mathbb{R} : X)$.*

Before proceeding any further, we would like to mention that the necessary and sufficient condition for X to contain c_0 is given in [4, Theorem 4.6.14]: $c_0 \subseteq X$ iff there exists a divergent series $\sum_{n=1}^{\infty} x_n$ in X which is unconditionally bounded, i.e., there exists $M \geq 0$ such that

$$\left\| \sum_{j=1}^{m} x_{n_j} \right\| \leq M,$$

whenever $n_j \in \mathbb{N}$ $(j = 1, 2, \cdots, m)$ such that $n_1 < n_2 < \cdots < n_m$.

Let us recall that a non-empty subset Λ of $\mathbb{R}$ is called harmonious iff for each $\epsilon > 0$ the set

$$\bigcap_{\lambda \in \Lambda} \left\{ \tau \in \mathbb{R} : \left| e^{i\lambda\tau} - 1 \right| \leq \epsilon \right\}$$

is relatively dense in $\mathbb{R}$. It is well known that a subset of a harmonious set Λ is harmonious as well as that for any finite set F the set $\Lambda + F$ is also harmonious. Any non-empty finite set and certain lacunary infinite sequences are harmonious, as well.

Almost periodicity of functions with values in a general vector topological space has been introduced and analyzed for the first time by G. M. N'Guérékata in [26]; see also [10], [24] and references cited therein. For the sake of convenience and better exposition, our results will be formulated in the setting of Banach spaces.

2. Multivalued linear operators in Banach spaces

Let X and Y be Banach spaces. A multivalued map (multimap) $\mathcal{A} : X \to P(Y)$ is said to be a multivalued linear operator (MLO) iff the following holds:

(i) $D(\mathcal{A}) := \{x \in X : \mathcal{A}x \neq \emptyset\}$ is a linear subspace of X;
(ii) $\mathcal{A}x + \mathcal{A}y \subseteq \mathcal{A}(x+y)$, $x,\ y \in D(\mathcal{A})$ and $\lambda\mathcal{A}x \subseteq \mathcal{A}(\lambda x)$, $\lambda \in \mathbb{C}$, $x \in D(\mathcal{A})$.

If $X = Y$, then we say that $\mathcal{A}$ is an MLO in X. It is well-known that, for any $x,\ y \in D(\mathcal{A})$ and $\lambda,\ \eta \in \mathbb{C}$ with $|\lambda| + |\eta| \neq 0$, we have $\lambda\mathcal{A}x + \eta\mathcal{A}y = \mathcal{A}(\lambda x + \eta y)$. If $\mathcal{A}$ is an MLO, then $\mathcal{A}0$ is a linear submanifold of Y and $\mathcal{A}x = f + \mathcal{A}0$ for any $x \in D(\mathcal{A})$ and $f \in \mathcal{A}x$. Set $R(\mathcal{A}) := \{\mathcal{A}x : x \in D(\mathcal{A})\}$. Then the set $\mathcal{A}^{-1}0 = \{x \in D(\mathcal{A}) : 0 \in \mathcal{A}x\}$ is called the kernel of $\mathcal{A}$ and it is denoted by either $N(\mathcal{A})$ or $\mathrm{Kern}(\mathcal{A})$. The inverse $\mathcal{A}^{-1}$ of an MLO is defined by $D(\mathcal{A}^{-1}) := R(\mathcal{A})$ and $\mathcal{A}^{-1}y := \{x \in D(\mathcal{A}) : y \in \mathcal{A}x\}$. It can be simply verified that $\mathcal{A}^{-1}$ is an MLO in X, as well as that $N(\mathcal{A}^{-1}) = \mathcal{A}0$ and $(\mathcal{A}^{-1})^{-1} = \mathcal{A}$. If $N(\mathcal{A}) = \{0\}$, i.e., if $\mathcal{A}^{-1}$ is single-valued, then $\mathcal{A}$ is said to be injective. It is worth noting that $\mathcal{A}x = \mathcal{A}y$ for some two elements x and $y \in D(\mathcal{A})$, iff $\mathcal{A}x \cap \mathcal{A}y \neq \emptyset$; furthermore, if $\mathcal{A}$ is injective, then the equality $\mathcal{A}x = \mathcal{A}y$ holds iff $x = y$.

For any mapping $\mathcal{A} : X \to P(Y)$ we define $\check{\mathcal{A}} := \{(x, y) : x \in D(\mathcal{A}),\ y \in \mathcal{A}x\}$. Then $\mathcal{A}$ is an MLO iff $\check{\mathcal{A}}$ is a linear relation in $X \times Y$, i.e., iff $\check{\mathcal{A}}$ is a linear subspace of $X \times Y$.

If $\mathcal{A},\ \mathcal{B} : X \to P(Y)$ are two MLOs, then we define its sum $\mathcal{A} + \mathcal{B}$ by $D(\mathcal{A} + \mathcal{B}) := D(\mathcal{A}) \cap D(\mathcal{B})$ and $(\mathcal{A} + \mathcal{B})x := \mathcal{A}x + \mathcal{B}x$, $x \in D(\mathcal{A} + \mathcal{B})$. It is clear that $\mathcal{A} + \mathcal{B}$ is likewise an MLO.

Let $\mathcal{A} : X \to P(Y)$ and $\mathcal{B} : Y \to P(Z)$ be two MLOs, where Z is an SCLCS. The product of $\mathcal{A}$ and $\mathcal{B}$ is defined by $D(\mathcal{BA}) := \{x \in D(\mathcal{A}) : D(\mathcal{B}) \cap \mathcal{A}x \neq \emptyset\}$ and $\mathcal{BA}x := \mathcal{B}(D(\mathcal{B}) \cap \mathcal{A}x)$. Then $\mathcal{BA} : X \to P(Z)$ is an MLO and $(\mathcal{BA})^{-1} = \mathcal{A}^{-1}\mathcal{B}^{-1}$. The scalar multiplication of an MLO $\mathcal{A} : X \to P(Y)$ with the number $z \in \mathbb{C}$, $z\mathcal{A}$ for short, is defined by $D(z\mathcal{A}) := D(\mathcal{A})$ and $(z\mathcal{A})(x) := z\mathcal{A}x$, $x \in D(\mathcal{A})$. It is clear that $z\mathcal{A} : X \to P(Y)$ is an MLO and $(\omega z)\mathcal{A} = \omega(z\mathcal{A}) = z(\omega\mathcal{A})$, $z,\ \omega \in \mathbb{C}$.

The integer powers of an MLO $\mathcal{A} : X \to P(X)$ are defined recursively as follows: $\mathcal{A}^0 =: I$; if $\mathcal{A}^{n-1}$ is defined, set

$$D(\mathcal{A}^n) := \big\{x \in D(\mathcal{A}^{n-1}) : D(\mathcal{A}) \cap \mathcal{A}^{n-1}x \neq \emptyset\big\},$$

and

$$\mathcal{A}^n x := \big(\mathcal{A}\mathcal{A}^{n-1}\big)x = \bigcup_{y \in D(\mathcal{A}) \cap \mathcal{A}^{n-1}x} \mathcal{A}y, \quad x \in D(\mathcal{A}^n).$$

We can prove inductively that $(\mathcal{A}^n)^{-1} = (\mathcal{A}^{n-1})^{-1}\mathcal{A}^{-1} = (\mathcal{A}^{-1})^n =: \mathcal{A}^{-n}$, $n \in \mathbb{N}$ and $D((\lambda - \mathcal{A})^n) = D(\mathcal{A}^n)$, $n \in \mathbb{N}_0$. Let us recall that, if $\mathcal{A}$ is single-valued, then the above definitions are consistent with the usual definition of powers of $\mathcal{A}$.

If $\mathcal{A} : X \to P(Y)$ and $\mathcal{B} : X \to P(Y)$ are two MLOs, then we write $\mathcal{A} \subseteq \mathcal{B}$ iff $D(\mathcal{A}) \subseteq D(\mathcal{B})$ and $\mathcal{A}x \subseteq \mathcal{B}x$ for all $x \in D(\mathcal{A})$. Assume now that a linear single-valued operator $S : D(S) \subseteq X \to Y$ has domain $D(S) = D(\mathcal{A})$ and $S \subseteq \mathcal{A}$, where $\mathcal{A} : X \to P(Y)$ is an MLO. Then S is called a section of $\mathcal{A}$; in this case, we have $\mathcal{A}x = Sx + \mathcal{A}0$, $x \in D(\mathcal{A})$ and $R(\mathcal{A}) = R(S) + \mathcal{A}0$.

Suppose that $\mathcal{A}$ is an MLO in X. Then we say that a point $\lambda \in \mathbb{C}$ is an eigenvalue of $\mathcal{A}$ iff there exists a vector $x \in X \setminus \{0\}$ such that $\lambda x \in \mathcal{A}x$; we call x an eigenvector of operator $\mathcal{A}$ corresponding to the eigenvalue λ. Observe that, in purely multivalued case, a vector $x \in X \setminus \{0\}$ can be an eigenvector of operator $\mathcal{A}$ corresponding to different values of scalars λ. The point spectrum of $\mathcal{A}$, $\sigma_p(\mathcal{A})$ for short, is defined as the union of all eigenvalues of $\mathcal{A}$.

It is said that an MLO $\mathcal{A} : X \to P(Y)$ is closed if for any two sequences (x_n) in $D(\mathcal{A})$ and (y_n) in Y such that $y_n \in \mathcal{A}x_n$ for all $n \in \mathbb{N}$ we have that the preassumptions $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$ imply $x \in D(\mathcal{A})$ and $y \in \mathcal{A}x$.

If $\mathcal{A} : X \to P(Y)$ is an MLO, then we define the adjoint $\mathcal{A}^* : Y^* \to P(X^*)$ of $\mathcal{A}$ by its graph

$$\mathcal{A}^* := \Big\{ \big(y^*, x^*\big) \in Y^* \times X^* : \big\langle y^*, y\big\rangle = \big\langle x^*, x\big\rangle \text{ for all pairs } (x, y) \in \mathcal{A} \Big\}.$$

It is simpy verified that $\mathcal{A}^*$ is a closed MLO, and that $\langle y^*, y\rangle = 0$ whenever $y^* \in D(\mathcal{A}^*)$ and $y \in \mathcal{A}0$.

We will use the following important lemmae.

Lemma 2.1. *(cf.* [32, Section 1.1]*) Let Ω be a locally compact, separable metric space, and let μ be a locally finite Borel measure defined on Ω. Suppose that $\mathcal{A} : X \to P(Y)$ is a closed* MLO. *Let $f : \Omega \to X$ and $g : \Omega \to Y$ be μ-integrable, and let $g(x) \in \mathcal{A}f(x)$, $x \in \Omega$. Then $\int_\Omega f\,d\mu \in D(\mathcal{A})$ and $\int_\Omega g\,d\mu \in \mathcal{A}\int_\Omega f\,d\mu$.*

Lemma 2.2. *(cf.* [32, Section 1.2]*) Suppose that $\mathcal{A} : X \to P(Y)$ is a closed* MLO. *Assume, further, that $x_0 \in X$, $y_0 \in Y$ and $\langle x^*, x_0\rangle = \langle y^*, y_0\rangle$ for all pairs $(x^*, y^*) \in X^* \times Y^*$ satisfying that $\langle x^*, x\rangle = \langle y^*, y\rangle$ whenever $y \in \mathcal{A}x$. Then $y_0 \in \mathcal{A}x_0$.*

Before we switch to Subsection 2.1, we need to remind ourselves of the basic facts regarding the C-resolvent sets of multivalued linear operators in Banach spaces and the well-posedness of related abstract degenerate Volterra inclusions. Our standing assumptions is that $\mathcal{A}$ is an MLO in X, as well as that $C \in L(X)$ and $C\mathcal{A} \subseteq \mathcal{A}C$ (this is equivalent to say that, for any $(x, y) \in X \times X$, we have the implication $(x, y) \in \mathcal{A} \Rightarrow (Cx, Cy) \in \mathcal{A}$; by induction, we immediately get that $C\mathcal{A}^k \subseteq \mathcal{A}^kC$ for all $k \in \mathbb{N}$). It is worth noting here that we allow C to be possibly non-injective. Then the C-resolvent set of $\mathcal{A}$, $\rho_C(\mathcal{A})$ for short, is defined as the union of those complex numbers $\lambda \in \mathbb{C}$ for which

(i) $R(C) \subseteq R(\lambda - \mathcal{A})$;
(ii) $(\lambda - \mathcal{A})^{-1}C$ is a single-valued bounded operator on X.

The operator $\lambda \mapsto (\lambda - \mathcal{A})^{-1}C$ is called the C-resolvent of $\mathcal{A}$ ($\lambda \in \rho_C(\mathcal{A})$); the resolvent set of $\mathcal{A}$ is defined by $\rho(\mathcal{A}) := \rho_I(\mathcal{A})$, $R(\lambda : \mathcal{A}) \equiv (\lambda - \mathcal{A})^{-1}$ ($\lambda \in \rho(\mathcal{A})$). The basic properties of C-resolvent sets of single-valued linear operators continue to hold in our framework; for example, the Hilbert resolvent formula and the generalized resolvent equation formulae are still valid in our setting ([30]-[31]).

In [32], we have recently considered the C-wellposedness of following abstract degenerate Volterra inclusion:

$$\mathcal{B}u(t) \subseteq \mathcal{A} \int_0^t a(t-s)u(s)\,ds + \mathcal{F}(t),\ t \in [0,\tau), \tag{2.1}$$

where $a \in L^1_{loc}([0,\tau))$, $a \neq 0$, $\mathcal{A} : X \to P(X)$ and $\mathcal{B} : X \to P(X)$ are given multivalued linear operators, and $\mathcal{F} : X \to P(X)$ is a given mutivalued mapping, as well as the following fractional Sobolev inclusions:

$$(\mathrm{DFP})_R : \begin{cases} \mathbf{D}_t^\alpha Bu(t) \in \mathcal{A}u(t) + \mathcal{F}(t), \quad t \geq 0, \\ (Bu)^{(j)}(0) = Bx_j,\ 0 \leq j \leq \lceil \alpha \rceil - 1, \end{cases}$$

where we assume that $B = \mathcal{B}$ is single-valued, and

$$(\mathrm{DFP})_L : \begin{cases} \mathcal{B}\mathbf{D}_t^\alpha u(t) \subseteq \mathcal{A}u(t) + \mathcal{F}(t), \quad t \geq 0, \\ u^{(j)}(0) = x_j,\ 0 \leq j \leq \lceil \alpha \rceil - 1, \end{cases}$$

where $\alpha > 0$ and $\mathbf{D}_t^\alpha$ denotes the Caputo fractional derivative ([9], [31]).

In the following general definition, we introduce various types of solutions to the abstract degenerate inclusion (2.1).

Definition 2.3. ([32]) A function $u \in C([0,\tau) : X)$ is said to be a pre-solution of (2.1) iff $(a*u)(t) \in D(\mathcal{A})$ and $u(t) \in D(\mathcal{B})$ for $t \in [0,\tau)$, as well as (2.1) holds. By a solution of (2.1), we mean any pre-solution $u(\cdot)$ of (2.1) satisfying additionally that there exist functions $u_{\mathcal{B}} \in C([0,\tau) : X)$ and $u_{a,\mathcal{A}} \in C([0,\tau) : X)$ such that $u_{\mathcal{B}}(t) \in \mathcal{B}u(t)$ and $u_{a,\mathcal{A}}(t) \in \mathcal{A}\int_0^t a(t-s)u(s)\,ds$ for $t \in [0,\tau)$, as well as

$$u_{\mathcal{B}}(t) \in u_{a,\mathcal{A}}(t) + \mathcal{F}(t), \quad t \in [0,\tau).$$

Strong solution of (2.1) is any function $u \in C([0,\tau) : X)$ satisfying that there exist two continuous functions $u_{\mathcal{B}} \in C([0,\tau) : X)$ and $u_{\mathcal{A}} \in C([0,\tau) : Y)$ such that $u_{\mathcal{B}}(t) \in \mathcal{B}u(t)$, $u_{\mathcal{A}}(t) \in \mathcal{A}u(t)$ for all $t \in [0,\tau)$, and

$$u_{\mathcal{B}}(t) \in (a * u_{\mathcal{A}})(t) + \mathcal{F}(t), \quad t \in [0,\tau).$$

2.1. **Degenerate (a,k)-regularized C-resolvent family.** Let X and Y be two complex Banach spaces. In [32], we have recently introduced the following definitions:

Definition 2.4. Suppose $0 < \tau \leq \infty$, $k \in C([0,\tau))$, $k \neq 0$, $a \in L^1_{loc}([0,\tau))$, $a \neq 0$, $\mathcal{A} : X \to P(X)$ is an MLO, $C_1 \in L(Y,X)$, and $C_2 \in L(X)$.

(i) Then it is said that $\mathcal{A}$ is a subgenerator of a (local, if $\tau < \infty$) mild (a,k)-regularized (C_1,C_2)-existence and uniqueness family $(R_1(t),R_2(t))_{t\in[0,\tau)} \subseteq L(Y,X) \times L(X)$ iff the mappings $t \mapsto R_1(t)y$, $t \geq 0$ and $t \mapsto R_2(t)x$, $t \in [0,\tau)$ are continuous for every fixed $x \in X$ and $y \in Y$, as well as the following conditions hold:

$$\left(\int_0^t a(t-s)R_1(s)y\,ds, R_1(t)y - k(t)C_1y\right) \in \mathcal{A},\ t \in [0,\tau),\ y \in Y \text{ and} \tag{2.2}$$

$$\int_0^t a(t-s)R_2(s)y\,ds = R_2(t)x - k(t)C_2x, \text{ whenever } t \in [0,\tau) \text{ and } (x,y) \in \mathcal{A}. \tag{2.3}$$

(ii) Let $(R_1(t))_{t\in[0,\tau)} \subseteq L(Y,X)$ be strongly continuous. Then it is said that $\mathcal{A}$ is a subgenerator of a (local, if $\tau < \infty$) mild (a,k)-regularized C_1-existence family $(R_1(t))_{t\in[0,\tau)}$ iff (2.2) holds.

(iii) Let $(R_2(t))_{t\in[0,\tau)} \subseteq L(X)$ be strongly continuous. Then it is said that $\mathcal{A}$ is a subgenerator of a (local, if $\tau < \infty$) mild (a,k)-regularized C_2-uniqueness family $(R_2(t))_{t\in[0,\tau)}$ iff (2.3) holds.

Definition 2.5. Suppose that $0 < \tau \leq \infty$, $k \in C([0,\tau))$, $k \neq 0$, $a \in L^1_{loc}([0,\tau))$, $a \neq 0$, $\mathcal{A} : X \to P(X)$ is an MLO, $C \in L(X)$ and $C\mathcal{A} \subseteq \mathcal{A}C$. Then it is said that a strongly continuous operator family $(R(t))_{t\in[0,\tau)} \subseteq L(X)$ is an (a,k)-regularized C-resolvent family with a subgenerator $\mathcal{A}$ iff $(R(t))_{t\in[0,\tau)}$ is a mild (a,k)-regularized C-uniqueness family having $\mathcal{A}$ as subgenerator, $R(t)C = CR(t)$ and $R(t)\mathcal{A} \subseteq \mathcal{A}R(t)$ $(t \in [0,\tau))$.

Any (a,k)-regularized C-resolvent family under our consideration will be also a mild (a,k)-regularized C-existence family and the condition $0 \in \operatorname{supp}(a)$ will be assumed.

We say that an (a,k)-regularized C-resolvent family $(R(t))_{t\geq 0}$ is exponentially bounded (bounded) iff there exists $\omega \in \mathbb{R}$ $(\omega = 0)$ such that the family $\{e^{-\omega t}R(t) : t \geq 0\} \subseteq L(X)$ is bounded. If $k(t) = g_{\alpha+1}(t)$, where $\alpha \geq 0$, then it is also said that $(R(t))_{t\in[0,\tau)}$ is an α-times integrated (a,C)-resolvent family; 0-times integrated (a,C)-resolvent family is further abbreviated to (a,C)-resolvent family. We pay special attention to the case $a(t) \equiv 1$, resp. $a(t) \equiv t$, when we say that $(R(t))_{t\geq 0}$ is an α-times integrated C-semigroup (C-semigroup, if $\alpha = 0$), resp. an α-times integrated C-cosine function (C-cosine function, if $\alpha = 0$). Similar terminological agreement is accepted for the class of mild (a,k)-regularized (C_1,C_2)-existence and uniqueness families.

By $\chi(R)$ we denote the set consisting of all subgenerators of $(R(t))_{t\in[0,\tau)}$. It is clear that for each subgenerator $\mathcal{A} \in \chi(R)$ we have $\overline{\mathcal{A}} \in \chi(R)$. The set $\chi(R)$ can have infinitely many elements; furthermore, if $\mathcal{A} \in \chi(R)$, then $\mathcal{A} \subseteq \mathcal{A}_{int}$, where the integral generator of $(R(t))_{t\in[0,\tau)}$ is defined by

$$\mathcal{A}_{int} := \left\{(x,y) \in X\times X : R(t)x - k(t)Cx = \int_0^t a(t-s)R(s)y\,ds \text{ for all } t \in [0,\tau)\right\}.$$

The integral generator $\mathcal{A}_{int}$ of $(R(t))_{t\in[0,\tau)}$ is always a closed subgenerator of $(R(t))_{t\in[0,\tau)}$, provided that $\tau = \infty$. If $\mathcal{A}$ and $\mathcal{B}$ are two subgenerators of $(R(t))_{t\in[0,\tau)}$ and $\alpha,\ \beta \in \mathbb{C}$ with $\alpha + \beta = 1$, then $C(D(\mathcal{A})) \subseteq D(\mathcal{B})$, $\mathcal{A}_{int} \subseteq C^{-1}\mathcal{A}C$ and $\alpha\mathcal{A} + \beta\mathcal{B}$ is also a subgenerator of $(R(t))_{t\in[0,\tau)}$; furthermore, if C is injective, then $\mathcal{A}_{int} = C^{-1}\mathcal{A}C$. We similarly define the notion of integral generator of a mild (a,k)-regularized C_2-uniqueness family $(R_2(t))_{t\in[0,\tau)}$.

In this paper, we will use the following definition of a (local) K-convoluted C-group; cf. [30, Section 2.6] for more details about non-degenerate case.

Definition 2.6. Let $C \in L(X)$ and $K \in L^1_{loc}([0,\tau))$, $K \neq 0$. Suppose that $\tau \in (0,\infty]$ and $\pm\mathcal{A}$ are the integral generators of K-convoluted C-semigroups $(S_{K,\pm}(t))_{t\in[0,\tau)}$. Put $S_K(t) := S_{K,+}(t)$, $t \in [0,\tau)$ and $S_K(t) := S_{K,-}(-t)$, $t \in (-\tau,0)$. Then we say that $(S_K(t))_{t\in(-\tau,\tau)}$ is a K-convoluted C-group with the integral generator $\mathcal{A}$.

Any (local, degenerate or non-degenerate in time) K-convoluted C-semigroup $(S_K(t))_{t\in[0,\tau)}$, resp. K-convoluted C-cosine function $(C_K(t))_{t\in[0,\tau)}$, where C is not necessarily injective, satisfies the well known composition properties stated in [30, Proposition 2.1.5, resp. Theorem 2.1.13]. Similar composition properties hold for (local) C-semigroups and (local) C-cosine functions. Although we do not intend to analyze the class of degenerate K-convoluted C-groups in more detail, we will use hereafter some special cases of the general composition property of degenerate K-convoluted C-groups. This composition property is stated in the following proposition, which can be viewed of some independent interest (the continuity of mapping $t \mapsto S_K(t)x$, $t \in (-\tau,\tau)$ for $x \in \overline{D(\mathcal{A})}$ is irrelevant here; cf. also the short discussion after Definition 4.10).

Proposition 2.7. *Suppose that $(S_K(t))_{t\in(-\tau,\tau)}$ is a K-convoluted C-group with the integral generator $\mathcal{A}$. Then, for every t, $s \in (-\tau,\tau)$ with $t < 0 < s$ and $x \in X$, one has:*

$$S_K(t)S_K(s)x = S_K(s)S_K(t)x$$
$$= \begin{cases} \int_{t+s}^{s} K(r-t-s)S_K(r)Cx\,dr + \int_{t}^{0} K(t+s-r)S_K(r)Cx\,dr, & t+s \geq 0, \\ \int_{t}^{t+s} K(t+s-r)S_K(r)Cx\,dr + \int_{0}^{s} K(r-t-s)S_K(r)Cx\,dr, & t+s < 0. \end{cases}$$

Proof. Let $-\tau < t < 0 < s < \tau$ and $t+s \geq 0$. Proceeding as in the proof of [30, Theorem 2.6.7], we obtain similarly as in non-degenerate case that, for every $x \in X$,

$$\begin{aligned} S_K(t)\int_0^s S_K(\sigma)x\,d\sigma &= \int_{t+s}^{s} \Theta(r)S_K(t+s-r)Cx\,dr + \int_{t+s}^{s} K(r-s-t)C\int_0^r S_K(\sigma)x\,d\sigma\,dr \\ &= \int_{t}^{0} \Theta(t+s-r)S_K(r)Cx\,dr + \int_{t+s}^{s} K(r-s-t)C\int_0^r S_K(\sigma)x\,d\sigma\,dr, \end{aligned}$$

where $\Theta(t) = \int_0^t K(s)\,ds$, $t \in [0,\tau)$. Applying the partial integration on the second addend in the above equality and using a straightforward computation, we get that, for every $x \in X$,

$$
\begin{aligned}
&S_K(t)S_K(s)x \\
&\quad = \frac{d}{ds}\left[S_K(t)\int_0^s S_K(\sigma)x\,d\sigma\right] \\
&\quad = \int_t^0 K(t+s-r)S_K(r)Cx\,dr \\
&\quad + \frac{d}{ds}\left[\Theta(-t)\int_0^s S_K(r)Cx\,dr - \int_{t+s}^s \Theta(r-s-t)S_K(r)Cx\,dr\right] \\
&\quad = \int_t^0 K(t+s-r)S_K(r)Cx\,dr + \int_s^{t+s} K(r-s-t)S_K(r)Cx\,dr.
\end{aligned}
$$

We can analogously prove that, for $t+s<0$ and $x \in X$, we have

$$
S_K(t)S_K(s)x = \int_t^{t+s} K(t+s-r)S_K(r)Cx\,dr + \int_0^s K(r-t-s)S_K(r)Cx\,dr.
$$

Since $(S_K(-t))_{t\in(-\tau,\tau)}$ is a K-convoluted C-group with the integral generator $-\mathcal{A}$, the obtained composition properties imply that $S_K(t)S_K(s) = S_K(s)S_K(t)$ for all $t,\ s \in (-\tau,\tau)$. The proof of the theorem is thereby completed. $\square$

In the case that $\pm\mathcal{A}$ are the integral generators of C-semigroups $(T_\pm(t))_{t\in[0,\tau)}$, then the C-group $(T(t))_{t\in(-\tau,\tau)}$, defined similarly as above, satisfies the much simpler group property $T(t+s)C = T(t)T(s)$ for all $t,\ s \in (-\tau,\tau)$ with $t+s \in (-\tau,\tau)$. We will use this functional equality for transferring [45, Theorem 2.1] to degenerate C-groups (see Theorem 3.1 below).

Henceforward, we will investigate only the global case $\tau = \infty$.

3. Almost periodicity of abstract degenerate first and second order Cauchy problems

We start this section by reconsidering the structural results proved by Q. Zheng and L. Liu in [45]. The assumption on denseness of $R(C)$ is crucial in this paper, but the careful inspections of proofs show that the injectivity of regularizing operator C is superfluous in almost all places (if $D(\mathcal{A})$ is dense in X, $\mathcal{A}$ generates a C-semigroup $(T(t))_{t\geq 0}$ (C-cosine function $(C(t))_{t\geq 0}$) and C is injective, then $(T(t))_{t\geq 0}$ is non-degenerate since $T(0) = C$ ($C(0) = C$), which automatically implies that $\mathcal{A}$ is single-valued).

Unless specified otherwise, we assume that $(T(t))_{t\in\mathbb{R}}$ is a global C-group with the integral generator $\mathcal{A}$; as explained above, this means that $\mathcal{A}$ generates a C-semigroup $(T(t))_{t\geq 0}$ and $-\mathcal{A}$ generates a C-semigroup $(T(-t))_{t\geq 0}$ (a class of very simple counterexamples shows that the equality $T_+(t)T_-(t) = C^2$ stated in the proof of implication (b) $\Rightarrow$ (c) of [45, Theorem 3.1] does not hold for C-degenerate groups, even in the case that $C = I$). Then it is very simple to prove that the assumption $irx \in \mathcal{A}x$ for some $r \in \mathbb{R}$ implies that $T(t)x = e^{irt}Cx$, $t \in \mathbb{R}$.

Keeping in mind this fact, as well as Lemma 1.2, the parts (i)-(ii), (iv) and (vii), it is straightforward to extend the assertion of [45, Theorem 2.1] to degenerate C-groups; cf. also Theorem 3.3, Theorem 3.4, Proposition 4.1 and Theorem 4.5 below.

Theorem 3.1. *Suppose that $\overline{R(C)} = \overline{D(\mathcal{A})} = X$. Then $(T(t))_{t\in\mathbb{R}}$ is almost periodic iff $(T(t))_{t\in\mathbb{R}}$ is bounded and the set D consisting of all eigenvectors of operator $\mathcal{A}$ which corresponds to purely imaginary eigenvalues of operator $\mathcal{A}$ is total in X (i.e., the linear span of D is dense in X).*

Before proceeding further, we would like to point out that Theorem 3.1 does not hold without assuming the denseness of $\mathcal{A}$. For example, the a.p. degenerate group $(T(t) \equiv 0)_{t\in\mathbb{R}}$ has the integral generator $\{0\} \times X$ but the set D is empty. Furthermore, almost periodic C-groups $(T(t))_{t\in\mathbb{R}}$ for which $\overline{R(C)} = \overline{D(\mathcal{A})} = X$ and $\mathcal{A}$ is not single-valued really exist. To see this, observe that for each $C \in L(X)$ we have that $(T(t) \equiv C)_{t\in\mathbb{R}}$ is a global C-group whose integral generator $\mathcal{A}$ is given by $\mathcal{A} = X \times N(C)$; this simply implies that $\rho_C(\mathcal{A}) = \emptyset$, provided that the operator C is not injective.

Arguing as in [45], we can prove the following:

1. If $c_0 \not\subseteq X$, $\pm\mathcal{A}$ are the integral generators of global bounded C-uniqueness families $(T(\pm t))_{t\geq 0}$ and the mapping $t \mapsto T(t)y$, $t \in \mathbb{R}$ is almost periodic for all $y \in R(\mathcal{A})$, then the mapping $t \mapsto T(t)x$, $t \in \mathbb{R}$ is almost periodic for all $x \in \overline{D(\mathcal{A})}$ (cf. [45, Proposition 2.4]). Here we use the parts (vi) and (vii) of Lemma 1.2.
2. Let us recall that X is weakly sequentially complete iff every weak Cauchy sequence in X converges weakly. If this is the case, then the denseness of $R(C)$ and $D(\mathcal{A})$ in X implies that the concepts weak almost periodicity and almost periodicity of a global C-group $(T(t))_{t\in\mathbb{R}}$ are mutually equivalent; furthermore, the almost periodicity of $(T(t))_{t\in\mathbb{R}}$ implies its uniform periodicity provided, in addition to the above, that $\{e^{\lambda t} : \lambda \in \sigma_p(\mathcal{A})\}$ is uniformly almost periodic. Keeping in mind Lemma 1.2, the proofs of these statements are almost the same as those of parts (a) and (b) in [45, Theorem 2.5]; cf. also the proof of [7, Theorem 3] and observe that for each $r \in \mathbb{R}$ the graph of $ir - \mathcal{A}$ is a closed convex subset of $X \times X$, as well as that the partial integration and Lemma 2.1 together imply that for each $r \in \mathbb{R}$ we have
$$\frac{1}{t}\big[Cx - e^{-irt}T(t)x\big] \in \lim_{t\to\infty}(ir - \mathcal{A})\frac{1}{t}\int_0^t e^{-irs}T(s)x\,ds, \quad x \in X.$$

For the sequel, we need some preliminaries from the pioneering paper [7] by H. Bart and S. Goldberg. By $AP(\Lambda : X)$, where Λ is a non-empty subset of $\mathbb{R}$, we denote the vector subspace of $AP(\mathbb{R} : X)$ consisting of all functions $f \in AP(\mathbb{R} : X)$ for which the inclusion $\sigma(f) \subseteq \Lambda$ holds good. It can be easily seen that $AP(\Lambda : X)$ is a closed subspace of $AP(\mathbb{R} : X)$ and therefore Banach space itself.

The translation semigroup $(W(t))_{t\geq 0}$ on $AP([0,\infty):X)$, given by $[W(t)f](s) := f(t+s)$, $t\geq 0$, $s\geq 0$, $f\in AP([0,\infty):X)$ is consisted solely of surjective isometries $W(t)$ $(t\geq 0)$ and can be extended to a C_0-group $(W(t))_{t\in\mathbb{R}}$ of isometries on $AP([0,\infty):X)$, where $W(-t):=W(t)^{-1}$ for $t>0$. Furthermore, the mapping $E: AP([0,\infty):X)\to AP(\mathbb{R}:X)$, defined by

$$[Ef](t) := [W(t)f](0), \quad t\in\mathbb{R},\ f\in AP([0,\infty):X),$$

is a linear surjective isometry and Ef is the unique continuous almost periodic extension of a function f from $AP([0,\infty):X)$ to the whole real line. We have that $[E(Bf)] = B(Ef)$ for all $B\in L(X)$ and $f\in AP([0,\infty):X)$.

The following is an extension of [45, Theorem 3.3] to degenerate C-semigroups.

Theorem 3.2. *Suppose that $(T(t))_{t\geq 0}$ is an almost periodic C-semigroup with the integral generator $\mathcal{A}$. Then there exists a bounded, strongly continuous, almost periodic operator family $(S(t))_{t\in\mathbb{R}}\subseteq L(X)$ commuting with C and satisfying:*

$$S(t)=T(t),\ t\geq 0 \text{ and } S(t)S(s)=S(t+s)C,\ t,\ s\in\mathbb{R}. \tag{3.1}$$

Furthermore, if $(T(t))_{t\geq 0}$ is uniformly almost periodic, then $(S(t))_{t\in\mathbb{R}}$ is likewise uniformly almost periodic. In the case that C is injective, we have that $(S(t))_{t\in\mathbb{R}}$ is a global C-group with the integral generator $\mathcal{A}$ (i.e., $(S(-t))_{t\geq 0}$ is a global C-semigroup with the integral generator $-\mathcal{A}$).

Proof. Since $(T(t))_{t\geq 0}$ is almost periodic, it must be uniformly bounded. Let $M:=\sup_{t\geq 0}\|T(t)\|$. Define $S(t)x := [E(T_x(\cdot))](t)$, $t\in\mathbb{R}$, $x\in X$, where $T_x(t) := T(t)x$, $x\in X$, $t\geq 0$. Since E is a linear surjective isometry between the spaces $AP([0,\infty):X)$ and $AP(\mathbb{R}:X)$, we have that

$$\|S(t)x\| \leq \sup_{s\in\mathbb{R}}\|S(s)x\| = \sup_{s\geq 0}\|S(s)x\| = \sup_{s\geq 0}\|T(s)x\| \leq M\|x\|, \quad x\in X,\ t<0,$$

so that $S(t)\in L(X)$ for all $t\in\mathbb{R}$, and $\sup_{t\in\mathbb{R}}\|S(t)\| = M<\infty$. It can be easily seen that $S(\cdot)$ commutes with C. On the other hand, we have that

$$\begin{aligned}
&S(t)S(s)x\\
&\quad = S(t)\big[E(T_x(\cdot))\big](s) = \big[E(S(t)T_x(\cdot))\big](s) = \big[E(CW(t)T_x(\cdot))\big](s)\\
&\quad = C\big[E(W(t)T_x(\cdot))\big](s) = C\big[W(s)W(t)T_x(\cdot)\big](0) = C\big[W(t+s)T_x(\cdot)\big](0)\\
&\quad = C\big[E(T_x(\cdot))\big](t+s) = CS(t+s)x, \quad t\geq 0,\ s\leq 0.
\end{aligned}$$

Since $S(t)$ and $S(s)$ commute, the proof of (3.1) is completed, which simply implies that $(S(t))_{t\in\mathbb{R}}$ is uniformly almost periodic provided that $(T(t))_{t\geq 0}$ is. If the operator C is injective, a simple computation shows that the integral generator $\mathcal{B}$ of a global C-semigroup $(S(-t))_{t\geq 0}$ equals $-\mathcal{A}$ (in general case, we can only prove that $\mathcal{B}\subseteq C^{-1}[-\mathcal{A}]C$ and $-\mathcal{A}\subseteq C^{-1}\mathcal{B}C$). $\square$

If C is injective, then $R(C)$ endowed with the norm $\|\cdot\|_{R(C)} := \|C^{-1}\cdot\|$ becomes a Banach space; we will denote this space simply by $[R(C)]$.

Now we are ready to prove the following extension of [45, Theorem 3.4]:

Theorem 3.3. *Suppose that $(T(t))_{t\geq 0}$ is a global C-semigroup with the integral generator $\mathcal{A}$, as well as $\overline{R(C)} = \overline{D(\mathcal{A})} = X$. Then $(T(t))_{t\geq 0}$ is almost periodic* *iff* $(T(t))_{t\geq 0}$ *is bounded and the set* D *consisting of all eigenvectors of operator* $\mathcal{A}$ *which correspond to purely imaginary eigenvalues of operator* $\mathcal{A}$ *is total in* X. *Furthermore, in the case that the operator* C *is injective, then we have that* $\mathcal{A} = A$ *is single-valued as well as that:*

(i) $\mathbb{C} \setminus i\mathbb{R} \subseteq \rho_C(A)$; *if the number* ir *is a pole of* $\rho_C(A)$, *then* ir *is its simple pole (in particular,* $ir \in \sigma_p(A)$*), and its residue is* P_r *defined by*

$$P_r x = \lim_{t\to+\infty} \frac{1}{t} \int_0^t e^{-irs} T(s)x\,ds, \quad x \in X.$$

(ii) *If* $\sigma_p(A)$ *is bounded, then* $A \in L([R(C)], X)$.

Proof. Suppose that $(T(t))_{t\geq 0}$ is almost periodic. Then it is clear that $(T(t))_{t\geq 0}$ is bounded. Let $(S(t))_{t\in\mathbb{R}}$ be given by Theorem 3.2. Repeating literally the proof of Necessity in [45, Theorem 3.1], with $T(\cdot)$ replaced by $S(\cdot)$ therein, we obtain that the set D is total in X. For the converse, we can use the arguments contained in the proof of Sufficiency in the above-mentioned theorem since Lemma 1.2(vii) holds for the functions defined on the semi-axis $[0,\infty)$ (recall that the mapping E defined above is a linear surjective isometry) and the assumption $ir \in \mathcal{A}x$ for some $r \in \mathbb{R}$ and $x \in X$ implies $T(t)x = e^{irt}Cx$, $t \geq 0$. The remnant is a part of [45, Theorem 3.3]. □

3. The periodicity of abstract (degenerate) Volterra integro-differential equations is not our focus here (cf. the paper [6] by V. Barbu and A. Favini for some interesting applications given in this direction). We only want to observe the following: Suppose that $(T(t))_{t\in\mathbb{R}}$ is a global C-group with the integral generator $\mathcal{A}$. Then we say that a number $p > 0$ is a period of $(T(t))_{t\in\mathbb{R}}$ iff $T(t+p) = T(t)$ for all $t \in \mathbb{R}$. If this is the case, then it can be easily seen that
$$\left(Cx, \frac{\int_0^p e^{-\lambda s} T(s)x\,ds}{1-e^{-\lambda p}}\right) \in \lambda - \mathcal{A}, \quad x \in X,\ \lambda \in \mathbb{C} \setminus 2\pi i p^{-1}\mathbb{Z};$$
furthermore, if the set D_p consisting of all eigenvectors of operator $\mathcal{A}$ which corresponds to eigenvalues $\lambda \in \mathbb{C} \setminus 2\pi i p^{-1}\mathbb{Z}$ of operator $\mathcal{A}$ is total in X, then p is a period of $(T(t))_{t\in\mathbb{R}}$ (see [45, Theorem 5.1]). Similar statements can be proved for global C-cosine functions (see [45, Theorem 5.2-Theorem 5.3]).

In the remaining part of this section, we analyze almost periodicity of abstract degenerate second order Cauchy problems. Let $\mathcal{A}$ be the integral generator of a C-cosine function $(C(t))_{t\geq 0}$. Set $C(-t) := C(t)$, $S(t) := \int_0^t C(s)\,ds$ and $S(-t) := -S(t)$ $(t \geq 0)$. As it is well known, $S(\cdot)$ is said to be a sine function associated with $C(\cdot)$. It can be easily seen that d'Alambert functional equation $2C(t)C(s) = C(t+s)C + C(t-s)C$ holds for all $t,\ s \in \mathbb{R}$.

The assertion of [45, Theorem 4.1] can be reformulated for degenerate cosine functions in the following way; the proof is standard and omitted therefore (the only thing worth noting is that the assumption $-r^2 x \in \mathcal{A}x$ for some $r \in \mathbb{R}$ and $x \in X$ implies $C(t)x = \cos(rt)Cx$, $t \in \mathbb{R}$):

Theorem 3.4. *Let $\overline{R(C)} = \overline{D(\mathcal{A})} = X$. Then $(C(t))_{t\in\mathbb{R}}$ ($(S(t))_{t\in\mathbb{R}}$) is almost periodic iff $(C(t))_{t\in\mathbb{R}}$ ($(S(t))_{t\in\mathbb{R}}$) is bounded and the set D consisted of all eigenvectors of $\mathcal{A}$ which corresponds to the real non-positive eigenvalues of $\mathcal{A}$ is total in X.*

Remark 3.5. Observe that the injectivity of C (we do not need the condition $\overline{R(C)} = \overline{D(\mathcal{A})} = X$ here) and the almost periodicity of $(S(t))_{t\in\mathbb{R}}$ imply that there exists $a > 0$ such that $\sigma_p(\mathcal{A}) \subseteq (-\infty, -a^2]$; cf. the final part of proof of [45, Theorem 4.1] and the proof of Sufficiency in [43, Theorem 1.2, pp. 242-243] for non-degenerate case.

It is well known that the almost periodicity of $(C(t))_{t\in\mathbb{R}}$ does not imply the almost periodicity of $(S(t))_{t\in\mathbb{R}}$, even in the case that $C = I$ ([21]); the most simplest counterexample is: $C(t) = I$, $t \in \mathbb{R}$ and $S(t) = tI$, $t \in \mathbb{R}$. Using the argumentation already employed in non-degenerate case [45], we can deduce the following:

4. If $c_0 \nsubseteq X$, then the almost periodicity of $(C(t))_{t\in\mathbb{R}}$ taken together with the boundedness of $(S(t))_{t\in\mathbb{R}}$ imply the almost periodicity of $(S(t))_{t\in\mathbb{R}}$; see [45, Theorem 4.3(a)-(b)].
5. If $c_0 \nsubseteq X$, the integral generator of a global bounded C-cosine function $(C(t))_{t\in\mathbb{R}}$ is densely defined and the mapping $t \mapsto S(t)y$, $t \in \mathbb{R}$ is almost periodic for all $y \in R(\mathcal{A})$, then $(C(t))_{t\in\mathbb{R}}$ is almost periodic (cf. the proof of [45, Proposition 2.4]).
6. In the case that the state space X is weakly sequentially complete, as well as that $\overline{R(C)} = \overline{D(\mathcal{A})} = X$, then any w.a.p. C-sine (or C-cosine) function is automatically a.p.
7. The almost periodicity of $(C(t))_{t\in\mathbb{R}}$ ($(S(t))_{t\in\mathbb{R}}$) implies its uniform periodicity provided, in addition to the weak sequential completeness of X, that the family $\{e^{\lambda t} : \lambda^2 \in \sigma_p(\mathcal{A})\}$ is uniformly almost periodic. The proofs of [6./7.] follows similarly as in that of [45, Theorem 4.4].
8. Any degenerate semigroup $(T(t))_{t\geq 0}$ is exponentially bounded. By the proof of [4, Lemma 3.14.3], the above is also true for degenerate cosine functions. Keeping in mind this fact, we can repeat almost literally the proof of [43, Theorem 1.6, pp. 247-249] in order to see that the (weak) almost periodicity of a sine function $(S(t))_{t\in\mathbb{R}}$ implies that for each $u \in D(\mathcal{A})$ the mapping $t \mapsto C(t)u$, $t \in \mathbb{R}$ is (weakly) almost periodic. Strictly speaking, a more general result holds true: Suppose that $\mathcal{A}$ is the integral generator of an exponentially bounded C-cosine function $(C(t))_{t\geq 0}$ (we do not need the denseness of $\mathcal{A}$ nor the range of C in X, C can be non-injective). Then the (weak) almost periodicity of a C-sine function $(S(t))_{t\in\mathbb{R}}$ implies that for each $u \in D(\mathcal{A})$ the mapping $t \mapsto C(t)u$, $t \in \mathbb{R}$ is (weakly) almost periodic. This can be seen by considering the function $g(t,x) := S(t)y - \omega_0^2 S(t)x$, $t \geq 0$, where $y \in \mathcal{A}x$ is arbitrarily chosen and ω_0 is strictly greater than the exponential type of $(C(t))_{t\geq 0}$.

Then the computation given on pp. 247-248 of [43] shows that

$$-\frac{1}{2\omega_0}\int_{-\infty}^{\infty} e^{-\omega_0|t-s|}g(s,x)\,ds = S(t)x, \quad t \in \mathbb{R},$$

and the final conclusion follows as in non-degenerate case.

Suppose now that $k \in \mathbb{N}$ and A is a closed, single-valued linear operator with non-empty resolvent set, say $\lambda_0 \in \rho(A)$. Then it is well known that:

(i) A generates a global exponentially bounded k-times integrated semigroup iff A generates a global exponentially bounded $(\lambda_0 - A)^{-k}$-semigroup.

(ii) A generates a global exponentially bounded $(2k)$-times $((2k+1)$-times$)$ integrated cosine function iff A generates a global exponentially bounded $(\lambda_0 - A)^{-k}$-cosine function $((\lambda_0 - A)^{-k-1}$-cosine function$)$.

Slight extensions of the above statements are clarified in [30, Proposition 2.3.12, Proposition 2.3.13]. The fomulae obtained in these propositions show that it is very difficult to expect the boundedness of induced $(\lambda_0 - A)^{-k}$-semigroups, resp. $(\lambda_0 - A)^{-k}$-cosine functions $((\lambda_0 - A)^{-k-1}$-cosine functions$)$, in the case that A does not generate a strongly continuous semigroup (cosine operator function). Nevertheless, Q. Zheng and L. Liu have investigated in [45, Section 6] various questions about almost periodicity of induced regularized semigroups and cosine functions, giving also a necessary and sufficient condition for the generation of almost periodic tempered distribution semigroups.

9. Let us recall that X. Gu, M. Li and F. Huang have investigated the almost periodicity of C-semigroups, integrated semigroups and C-cosine groups in [23], by assuming that the range of C is not necessarily dense in X. Their results lean heavily on the use of Hille-Yosida's spaces for closed single-valued linear operators which do not have eigenvalues in $(0,\infty)$, and we would like to point out that these results cannot be so easily reformulated in degenerate case.

It is also worth observing the following:

10. Let $\mathcal{A}$ be an MLO. Denote by D (E) the set consisting of all eigenvectors of operator $\mathcal{A}$ which correspond to purely imaginary eigenvalues of operator $\mathcal{A}$ (to non-positive real eigenvalues of operator $\mathcal{A}$). By D_0 (E_0) we denote the set consisting of all eigenvectors of operator $\mathcal{A}$ which correspond to purely imaginary non-zero eigenvalues of operator $\mathcal{A}$ (to negative real eigenvalues of operator $\mathcal{A}$).

    Suppose that $n \in \mathbb{N}_0$, $\mathcal{A}$ is a subgenerator of an n-times integrated C_2-uniqueness family $(S_n(t))_{t\geq 0}$, $r \in \mathbb{R}$ and $irx \in \mathcal{A}x$. Then $S_n(t)x - g_{n+1}(t)C_2x = ir\int_0^t S_n(s)x\,ds$, $t \geq 0$, which simply implies that the mapping $t \mapsto S_n(t)x$, $t \geq 0$ is infinitely differentiable with all derivatives at zero of order less than or equal to $n-1$ being zeroes. Hence, the mapping $t \mapsto (d^n/dt^n)S_n(t)x = e^{irt}C_2x$, $t \geq 0$ is almost periodic for all $y \in span(D)$ and the mapping $t \mapsto (d^{n-1}/dt^{n-1})S_n(t)x$, $t \geq 0$ is almost periodic for all $y \in span(D_0)$. Similarly, if $\mathcal{A}$ is a subgenerator of an n-times integrated C_2-cosine uniqueness family $(C_n(t))_{t\geq 0}$, $r \in \mathbb{R}$ and $-r^2x \in \mathcal{A}x$, then $C_n(t)x - g_{n+1}(t)C_2x = ir\int_0^t (t-s)C_n(s)x\,ds$, $t \geq 0$, which simply implies that the mapping $t \mapsto C_n(t)x$, $t \geq 0$ is infinitely differentiable with all derivatives at zero of order less than or equal to $n-1$ being zeroes. Hence, the mapping $t \mapsto (d^n/dt^n)C_n(t)x = \cos(rt)C_2x$, $t \geq 0$ is almost periodic for all $y \in span(E)$, as well as the mappings $t \mapsto (d^{n-1}/dt^{n-1})C_n(t)x$, $t \geq 0$ and $t \mapsto (d^{n-2}/dt^{n-2})C_n(t)x$, $t \geq 0$ are almost periodic for all $y \in span(E_0)$. Here, $d^l/dt^l \cdot = g_{-l} * \cdot$ for $l \in -\mathbb{N}$.

In [12], I. Cioranescu has investigated the conditions under which the abstract non-degenerate inhomogeneous Cauchy problem $(\mathrm{DFP})_L$ of first and second order ($\alpha \in \{1, 2\}$) has a unique strong almost periodic solution for all initial values $x \in D_\infty(A)$. The notions of (bounded, almost periodic) distribution group and (bounded, almost periodic) cosine distribution have been introduced for the first time in this paper (cf. [30] for further information concerning these subjects), and the spectral characterizations of generators of such distribution groups and cosine distributions have been given. We will further reconsider the structural results proved by I. Cioranescu in our forthcoming paper [33].

4. Almost periodic solutions of abstract Volterra integro-differential equations

We start our work in this section by stating the following simple but important result.

Proposition 4.1. *Suppose that $abs(|a|) < \infty$, $abs(k) < \infty$ and $\mathcal{A}$ is a subgenerator of a mild, strongly Laplace transformable, (a, k)-regularized C_2-uniqueness family $(R_2(t))_{t\geq 0}$. Denote by D the set consisting of all eigenvectors x of operator $\mathcal{A}$ which corresponds to eigenvalues $\lambda \in \mathbb{C}$ of operator $\mathcal{A}$ for which the mapping*

$$f_{\lambda,x}(t) := \mathcal{L}^{-1}\left(\frac{\tilde{k}(z)}{1 - \lambda\tilde{a}(z)}\right)(t)C_2x, \quad t \geq 0$$

is almost periodic. Then the mapping $t \mapsto R_2(t)x$, $t \geq 0$ is almost periodic for all $x \in span(D)$; furthermore, the mapping $t \mapsto R_2(t)x$, $t \geq 0$ is almost periodic for all $x \in \overline{span(D)}$ provided additionally that $(R_2(t))_{t\geq 0}$ is bounded.

Proof. Let $x \in D$ be an eigenvector of operator $\mathcal{A}$ which corresponds to an eigenvalue $\lambda \in \mathbb{C}$ of operator $\mathcal{A}$. Then

$$\lambda \int_0^t a(t-s)R_2(s)x\,ds = R_2(t)x - k(t)C_2x, \quad t \geq 0.$$

Performing the Laplace transform, we get that $R_2(t)x = f_{\lambda,x}(t)$, $t \geq 0$. This immediately implies the final conclusions since the parts (ii) and (vii) of Lemma 1.2 hold for the functions defined on the semi-axis $[0, \infty)$. □

Remark 4.2. Suppose that $R_2(t)\mathcal{A} \subseteq \mathcal{A}R_2(t)$, $t \geq 0$ and $x \in span(D)$. Then the mapping $t \mapsto u(t) := R_2(t)x$, $t \geq 0$ is a strong solution of the abstract Cauchy inclusion (2.1) with $\mathcal{B} = I$ and $\mathcal{F}(t) = k(t)C_2x$, $t \geq 0$.

Remark 4.3. Suppose that $x \in D$, $\lambda x \in \mathcal{A}x$ and $C_2 x \neq 0$. Then the scalar-valued function

$$\vartheta(t) := t \mapsto \mathcal{L}^{-1}\left(\frac{\tilde{k}(z)}{1-\lambda\tilde{a}(z)}\right)(t), \quad t \geq 0 \tag{4.1}$$

is almost periodic. Since it is well known that almost periodic functions are uniform limits of trigonometric polynomials in $BUC(\mathbb{R})$, the most important case in which the above holds is that there exist integer $n \in \mathbb{N}$, real numbers $r_1(\lambda), \cdots, r_n(\lambda)$, positive real number $\omega(\lambda)$, and complex numbers $\alpha_1(\lambda), \cdots, \alpha_n(\lambda)$, such that

$$\frac{\tilde{k}(z)}{1-\lambda\tilde{a}(z)} = \frac{\alpha_1(\lambda)}{z - ir_1(\lambda)} + \cdots + \frac{\alpha_n(\lambda)}{z - ir_n(\lambda)}, \quad \Re z > \omega(\lambda). \tag{4.2}$$

It is worth noting that (4.2) holds for substantially large classes of kernels $a(t)$ and regularizing functions $k(t)$; for example, (4.2) holds in the case that $a(t) = k(t) = \sin t$, $\lambda \in (-\infty, 1) \subseteq \sigma_p(\mathcal{A})$, $n = 2$, $ir_{1,2}(\lambda) = \pm\sqrt{\lambda - 1}$, $\alpha_{1,2}(\lambda) = \pm 2^{-1}\sqrt{\lambda-1}^{-1}$. If $a(t) = g_n(t) + \sum_{j=0}^{n-1} a_j g_j(t)$, $k(t) = g_n(t) + \sum_{j=0}^{n-1} b_j g_j(t)$, where $a_j,\ b_j \in \mathbb{C}$ for $1 \leq j \leq n-1$ and there exists a non-empty subset Ω of $\sigma_p(\mathcal{A})$ such that for each $\lambda \in \Omega$ the polynomial

$$P_\lambda(z) = z^n - \lambda \sum_{j=1}^{n-1} a_{n-j} z^j - \lambda, \quad z \in \mathbb{C}$$

has purely imaginary, pairwise disjoint, roots $ir_1(\lambda), \cdots,\ ir_n(\lambda)$, then (4.2) holds with appropriately chosen complex numbers $\alpha_1(\lambda), \cdots, \alpha_n(\lambda)$.

The situation is much more complicated in purely fractional case. For example, if $a(t) = g_\alpha(t)$, where $\alpha \in (0, \infty) \setminus \mathbb{N}$, and (4.2) holds for pairwise disjoint numbers $r_j(\lambda)$ and non-zero complex numbers $\alpha_j(\lambda)$ $(1 \leq j \leq n)$, then it is necessary to have $\lambda \in (i\mathbb{R})^\alpha$, $\lambda = (ir_j(\lambda))^\alpha$ $(1 \leq j \leq n)$, and the independence of function

$$k(t) = \sum_{j=1}^{n} \alpha_j(\lambda)\left[e^{ir_j(\lambda)t} - \lambda t^\alpha E_{1,1+\alpha}\big(ir_j(\lambda)t\big)\right], \quad t \geq 0$$

on parameter λ; cf. the Laplace transform identity [9, (1.26)]. From an application point of view, this is very difficult to be satisfied.

The following useful lemma is probably known in the existing literature; we will include the proof for the sake of clarity (cf. also [38, Theorem 1.3(iii)] for a similar result).

Lemma 4.4. *Let* $(R(t))_{t\geq 0} \subseteq L(X)$ *be a bounded strongly continuous operator family, and let* $x \in X$. *If* $a \in L^1_{loc}([0,\tau))$, $a \neq 0$, $\tilde{a}(ir)$ *exists for some* $r \in \mathbb{R}$ *and*

$$P_r^R x := \lim_{t\to\infty} \frac{1}{t}\int_0^t e^{-irs} R(s)x\, ds$$

exists, then $P_r^{a*R}x = \lim_{t\to\infty} \frac{1}{t}\int_0^t e^{-irs}(a*R)(s)x\,ds$ *exists as well, and the following holds:*

$$P_r^{a*R}x = \tilde{a}(ir)P_r^R x. \tag{4.3}$$

Proof. Let $\epsilon > 0$ be given, and let $\|R(t)\| \le M'$, $t \ge 0$. Then there exists $M > 0$ such that

$$\left|\int_M^\infty e^{-irs}a(s)\,ds\right| < \epsilon. \tag{4.4}$$

Furthermore,

$$\begin{aligned}
&\frac{1}{t}\int_0^t e^{-irs}(a*R)(s)x\,ds \\
&\qquad = \frac{1}{t}\int_0^t \int_0^s e^{-ir(s-v)}e^{-irv}R(v)x\,dv\,ds \\
&\qquad = \frac{1}{t}\int_0^t \int_t^v e^{-ir(s-v)}e^{-irv}R(v)x\,ds\,dv \\
&\qquad = \frac{1}{t}\int_0^t \left[\int_0^{t-v} e^{-irs}a(s)\,ds\right] e^{-irv}R(v)x\,dv.
\end{aligned}$$

Hence, for any $t > M$, we have

$$\begin{aligned}
&\left\|\frac{1}{t}\int_0^t e^{-irs}(a*R)(s)x\,ds - \tilde{a}(ir)P_r^R x\right\| \\
&\quad = \left\|\frac{1}{t}\int_0^t \left[\int_{t-v}^\infty e^{-irs}a(s)\,ds\right] e^{-irv}R(v)x\,dv\right\| \\
&\quad = \left\|\frac{1}{t}\left[\int_0^{t-M} + \int_{t-M}^t\right]\left[\int_{t-v}^\infty e^{-irs}a(s)\,ds\right] e^{-irv}R(v)x\,dv\right\|.
\end{aligned}$$

Now the equality (4.3) follows from (4.4) and the following estimates:

$$\begin{aligned}
&\left\|\frac{1}{t}\int_0^{t-M} \left[\int_{t-v}^\infty e^{-irs}a(s)\,ds\right] e^{-irv}R(v)x\,dv\right\| \\
&\qquad \le \frac{1}{t}\int_0^{t-M} \epsilon M'\,dv = \epsilon M'\frac{t-M}{t},
\end{aligned}$$

$$\begin{aligned}
&\left\|\frac{1}{t}\int_{t-M}^t \left[\int_{t-v}^M e^{-irs}a(s)\,ds\right] e^{-irv}R(v)x\,dv\right\| \\
&\qquad \le \frac{1}{t}\int_t^{t-M} \left[\int_0^M |a(s)|\,ds\right] M'\,dv = \left[\int_0^M |a(s)|\,ds\right] M'\frac{M}{t},
\end{aligned}$$

$$\left\| \frac{1}{t} \int_{t-M}^{t} \left[\int_{M}^{\infty} e^{-irs} a(s)\, ds \right] e^{-irv} R(v) x\, dv \right\| \\ \leq \frac{1}{t} \int_{t}^{t-M} \epsilon M'\, dv = \epsilon M' \frac{M}{t}.$$

□

Now we are ready to state the following necessary conditions for an (a,k)-regularized C-resolvent family $(R(t))_{t\geq 0}$ to be almost periodic. In some sense, this is a converse to Proposition 4.1.

Theorem 4.5. *Let $\mathcal{A}$ be the integral generator of an almost periodic (a,k)-regularized C-resolvent family $(R(t))_{t\geq 0}$, let $\overline{R(C)} = \overline{D(\mathcal{A})} = X$, and let $k(0) \neq 0$. Denote*

$$\mathcal{R} := \{ r \in \mathbb{R} : \tilde{a}(ir) \text{ exists} \}. \tag{4.5}$$

Suppose that $k(t)$ and $|a|(t)$ satisfy (P1), $\lim_{\Re z \to \infty} \tilde{a}(z) = 0$ *as well as that*

$$P_r^k = \lim_{t\to\infty} \frac{1}{t} \int_0^t e^{-irs} k(s)\, ds = 0, \quad r \in \mathcal{R}. \tag{4.6}$$

Then $(R(t))_{t\geq 0}$ is bounded and (Q) *holds, where*

(Q): $P_r^R x \in \mathcal{A}[\tilde{a}(ir) P_r^R x]$, $r \in \mathcal{R}$, $x \in X$ *and the mapping*

$$R(t) P_r^R x = \mathcal{L}^{-1}\left(\frac{\tilde{k}(z)\tilde{a}(ir)}{\tilde{a}(ir) - \tilde{a}(z)} \right)(t) C P_r^R x, \quad t \geq 0,\ x \in X,$$

is almost periodic for all $r \in \mathcal{R}$ and $x \in X$.

Suppose, in addition, that

$$R(t) P_r^R x = k(t) C P_r^R x, \quad t \geq 0,\ r \in \mathbb{R} \setminus \mathcal{R},\ x \in X. \tag{4.7}$$

Then the set D consisting of all eigenvectors of operator $\mathcal{A}$ which corresponds to eigenvalues $\lambda \in \{0\} \cup \{\tilde{a}(ir)^{-1} : r \in \mathcal{R},\ \tilde{a}(ir) \neq 0\}$ of operator $\mathcal{A}$ is total in X.

Proof. The boundedness of $(R(t))_{t\geq 0}$ follows from Lemma 1.2(i) and the uniform boundedness principle. Let $x \in X$. Since $\overline{R(C)} = \overline{D(\mathcal{A})} = X$, we have that $R(0) = k(0)C$. By [32, (271)], $(R(t))_{t\geq 0}$ satisfies the following functional equality:

$$R(t)(a * R)(s)x - k(t)C(a * R)(s)x = (a * R)(t)R(s)x - k(s)(a * R)(t)Cx, \tag{4.8}$$

for all $t,\ s \geq 0$. On the other hand, Lemma 4.4 implies that, for every $r \in \mathcal{R}$, we have that $P_r^{a*R}x$ exists and (4.3) holds. Using this equation as well as (4.6) and

(4.8) we get that, for every $r \in \mathcal{R}$,

$$
\begin{aligned}
&R(t)\tilde{a}(ir)P_r^R x - k(t)C\tilde{a}(ir)P_r^R x\\
&\quad = R(t)P_r^{a*R}x - k(t)CP_r^{a*R}x\\
&\quad = \lim_{\sigma\to\infty}\frac{1}{\sigma}\int_0^\sigma e^{-irs}\big[R(t)(a*R)(s)x - k(t)C(a*R)(s)x\big]\,ds\\
&\quad = \lim_{\sigma\to\infty}\frac{1}{\sigma}\int_0^\sigma e^{-irs}\big[(a*R)(t)R(s)x - k(s)C(a*R)(t)x\big]\,ds\\
&\quad = (a*R)(t)\lim_{\sigma\to\infty}\frac{1}{\sigma}\int_0^\sigma e^{-irs}\big[R(s)x - k(s)Cx\big]\,ds\\
&\quad = (a*R)(t)P_r^R x, \quad t \geq 0.
\end{aligned}
$$

Hence, $P_r^R x \in \mathcal{A}[\tilde{a}(ir)P_r^R x]$, $r \in \mathcal{R}$ and performing the Laplace transform we get that the condition (Q) holds; notice only that the condition $\lim_{\Re z\to\infty}\tilde{a}(z) = 0$ implies that for each $r \in \mathcal{R}$ with $\tilde{a}(ir) \neq 0$ we have that $\tilde{a}(ir) \neq \tilde{a}(z)$ on some right half plane. Assume now that $\tilde{a}(ir) = 0$ for some $r \in \mathcal{R}$. Then the previous computation gives $(a*R)(t)P_r^R x = 0$, $t \geq 0$ so that $R(t)P_r^R x = 0$, $t \geq 0$ as well as $R(0)P_r^R x = k(0)CP_r^R x$ and $CP_r^R x = 0$ due to condition $k(0) \neq 0$. This simply implies $0 = R(t)P_r^R x = k(t)CP_r^R x$, $t \geq 0$ and therefore $0 \in \mathcal{A}P_r^R x$. The validity of (4.7) implies that $0 \in \mathcal{A}P_r^R x$ for all $r \in \mathbb{R} \setminus \mathcal{R}$, as well. Therefore, if $P_r^R x \neq 0$ for some $r \in \mathbb{R}$, then $P_r^R x \in D$. Suppose that $x^* \in X^*$ and $\langle x^*, y\rangle = 0$ for all $y \in D$. Then

$$
\lim_{\sigma\to\infty}\frac{1}{\sigma}\int_0^\sigma e^{-irs}\big\langle x^*, R(s)x\big\rangle\,ds = \Big\langle x^*, P_r^R x\Big\rangle = 0, \quad r \in \mathbb{R},
$$

so that the almost periodicity of mapping $t \mapsto R(t)x$, $t \geq 0$ yields by Lemma 1.2(iv) that $\langle x^*, R(t)x\rangle = 0$ for all $t \geq 0$. Especially, $\langle x^*, k(0)Cx\rangle = \langle x^*, Cx\rangle = 0$, whence we may conclude by our assumption $\overline{R(C)} = X$ that $x^* = 0$. This completes the proof of theorem. □

Remark 4.6. (i) Suppose that $\mathcal{A} = A$ is single-valued and generates an almost periodic (a,k)-regularized C-resolvent family $(R(t))_{t\geq 0}$. Since the equation [31, (22), Proposition 2.1.3] holds in our framework, we have that $R(0) = k(0)C$. Therefore, if we disregard the condition $\overline{R(C)} = \overline{D(\mathcal{A})} = X$ and accept all other conditions from the first part of formulation of Theorem 4.5, then $(R(t))_{t\geq 0}$ is still bounded and (Q) still holds. Let it be the case, and let (4.7) be fulfilled. Then the final part of proof of Theorem 4.5 shows that $span(D)^\circ \subseteq R(C)^\circ$, which simply implies by the bipolar theorem that $\overline{R(C)} \subseteq \overline{span(D)}$.

(ii) We feel duty bound to say that the condition (4.7) from the formulation of Theorem 4.5 seems to be slightly redundant. This condition is satisfied in the usual considerations of almost periodicity of various types of semigroups and cosine operator functions.

In the following two propositions, we will reconsider our conclusions from the points [2,6,7] for (a,k)-regularized C-resolvent families.

Proposition 4.7. *Let $\mathcal{A}$ be the integral generator of a weak almost periodic (a,k)-regularized C-resolvent family $(R(t))_{t\geq 0}$, let $\overline{R(C)} = \overline{D(\mathcal{A})} = X$, and let $k(0) \neq 0$. Suppose that $k(t)$ is almost periodic, $|a|(t)$ satisfies* (P1), $\lim_{\Re z \to \infty} \tilde{a}(z) = 0$, (4.6) *holds (see* (4.5)*), and*

$$\lim_{\sigma\to\infty} \frac{1}{\sigma} \int_0^\sigma e^{-irs} \bigl\langle x^*, R(s)x \bigr\rangle \, ds = 0, \quad r \in \mathbb{R} \setminus \mathcal{R},\ x^* \in R\bigl(\mathcal{A}^*\bigr),\ x \in X. \tag{4.9}$$

Let X be weakly sequentially complete. Then $(R(t))_{t\geq 0}$ is almost periodic.

Proof. Since $(R(t))_{t\geq 0}$ is weakly almost periodic, the uniform boundedness theorem and Mackey's theorem together imply that $(R(t))_{t\geq 0}$ is bounded. Fix an element $x \in X$. Then the weak sequential completeness of X in combination with the weak almost periodicity of $(R(t))_{t\geq 0}$ yields that for each number $r \in \mathbb{R}$ there exists an element $M_r x \in X$ such that

$$\bigl\langle x^*, M_r x \bigr\rangle = \lim_{\sigma\to\infty} \frac{1}{\sigma} \int_0^\sigma e^{-irs} \bigl\langle x^*, R(s)x \bigr\rangle \, ds, \quad x^* \in X^*. \tag{4.10}$$

Let a pair $(y^*, x^*) \in \mathcal{A}^*$ and a number $r \in \mathcal{R}$ be fixed. Then

$$\bigl\langle y^*, R(t)x - k(t)Cx \bigr\rangle = \bigl\langle x^*, (a * R)(t)x \bigr\rangle, \quad t \geq 0.$$

Using this equality, as well as Lemma 4.4 and (4.6), it readily follows that

$$\bigl\langle y^*, M_r x \bigr\rangle = \bigl\langle x^*, \tilde{a}(ir) M_r x \bigr\rangle, \quad r \in \mathcal{R}.$$

Owing to Lemma 2.2, the above implies

$$M_r x \in \mathcal{A}\bigl[\tilde{a}(ir) M_r x\bigr], \quad r \in \mathcal{R}. \tag{4.11}$$

The proof of Theorem 4.5 shows that $0 \in \mathcal{A} M_r x$ for all $r \in \mathcal{R}$ with $\tilde{a}(ir) = 0$, as well as that $R(t) M_r x = \mathcal{L}^{-1}\bigl(\frac{\tilde{k}(z)\tilde{a}(ir)}{\tilde{a}(ir) - \tilde{a}(z)}\bigr)(t) C P_r^R x$, $t \geq 0$, $r \in \mathcal{R}$. Using Lemma 2.2 and (4.9), we get that

$$0 \in \mathcal{A} M_r x \text{ and } R(t) M_r x = k(t) C M_r x, \quad r \in \mathbb{R} \setminus \mathcal{R},\ t \geq 0. \tag{4.12}$$

Repeating literally the final part of proof of Theorem 4.5, we get that the set D consisting of all eigenvectors of operator $\mathcal{A}$ which corresponds to eigenvalues $\lambda \in \{0\} \cup \{\tilde{a}(ir)^{-1} : r \in \mathcal{R},\ \tilde{a}(ir) \neq 0\}$ of operator $\mathcal{A}$ is total in X. Now the final conclusion follows by applying Proposition 4.1. □

Proposition 4.8. *Let $\mathcal{A}$ be the integral generator of an almost periodic (a,k)-regularized C-resolvent family $(R(t))_{t\geq 0}$. Suppose that $k(t)$ and $|a|(t)$ satisfy* (P1), (4.6) *holds, and the set*

$$\Lambda := \bigl\{ r \in \mathcal{R} : \tilde{a}(ir) \neq 0,\ \tilde{a}(ir)^{-1} \in \sigma_p(\mathcal{A}) \bigr\} \cup \bigl\{ r \in \mathcal{R} : \tilde{a}(ir) = 0 \bigr\} \cup (\mathbb{R} \setminus \mathcal{R})$$

is harmonious. Then $(R(t))_{t\geq 0}$ is uniformly almost periodic.

Proof. Without loss of generality, we may assume that $R(\cdot)$ is defined on the whole real line $\mathbb{R}$ and strongly almost periodic there. Suppose that $P_r^R x \neq 0$ for some $r \in \mathbb{R}$ and $x \in X$ with $\|x\| \leq 1$. Then the proof of Theorem 4.5 shows that $r \in \Lambda$. Since $(R(t))_{t\in\mathbb{R}}$ is almost periodic, this inclusion in combination with the uniform boundedness theorem yields that $\{R(\cdot)x : x \in X,\ \|x\| \leq 1\}$ is a bounded subset of $AP(\Lambda : X)$. Since Λ is harmonious, the claimed assertion follows from [7, Theorem 13]. $\square$

Remark 4.9. If $\mathcal{A} = A$ is single-valued, then it is sufficient to assume that the set

$$\Lambda' := \{r \in \mathcal{R} : \tilde{a}(ir) \neq 0,\ \tilde{a}(ir)^{-1} \in \sigma_p(\mathcal{A})\} \cup (\mathbb{R} \setminus \mathcal{R})$$

is harmonious. Speaking-matter-of-factly, due to the proof of Theorem 4.5 we have that the assumption $P_r^R x \neq 0$ for some $r \in \mathcal{R}$ with $\tilde{a}(ir) = 0$ and some $x \in X$ with $\|x\| \leq 1$ implies $P_r^R x \in A[\tilde{a}(ir)P_r^R x] = A0 = \{0\}$, which is a contradiction.

Now we would like to point out a few important facts concerning the possibilities of transferring the assertion of Theorem 3.2 to (a, k)-regularized C-resolvent families. First or all, we need to extend the notion introduced in Definition 2.6.

Definition 4.10. Suppose $0 < \tau \leq \infty$, $k \in C([0, \tau))$, $k \neq 0$, $a \in L^1_{loc}([0, \tau))$, $a \neq 0$, $C \in L(X)$, $C\mathcal{A} \subseteq \mathcal{A}C$ and $\pm\mathcal{A}$ are the integral generators of (a, k)-regularized C-resolvent families $(R_\pm(t))_{t\in[0,\tau)}$. Put $R(t) := R_+(t)$, $t \in [0, \tau)$ and $R(t) := R(-t)$, $t \in (-\tau, 0)$. Then we say that $(R(t))_{t\in(-\tau,\tau)}$ is an (a, k)-regularized C-resolvent group family with the integral generator $\mathcal{A}$.

Observe that the mapping $t \mapsto R(t)x$, $t \in (-\tau, \tau)$ is continuous for all $x \in \overline{D(\mathcal{A})}$ and that the strong continuity of $(R(t))_{t\in(-\tau,\tau)}$ in degenerate case (at zero) is not automatically guaranteed by Definition 4.10. Nevertheless, a composition property of $(R(t))_{t\in(-\tau,\tau)}$ can be deduced even in the case that the mapping $t \mapsto R(t)x$, $t \in (-\tau, \tau)$ is not continuous for some $x \in X \setminus \overline{D(\mathcal{A})}$ (we can similarly introduce the class of (a, k)-regularized (C_1, C_2)-existence and uniqueness group families and prove an analogous composition property; cf. [31, Section 2.8] for more details):

Proposition 4.11. *Let $(R(t))_{t\in(-\tau,\tau)}$ be an (a, k)-regularized C-resolvent group family with the integral generator $\mathcal{A}$. Set*

$$k_g(t) := k(t) \text{ for } t \in [0, \tau) \text{ and } k_g(t) := k(-t) \text{ for } t \in (-\tau, 0]$$

and, for every $x \in X$,

$$\big(a *_g R\big)(t)x := \int_0^t a(t - s)R(s)x\, ds \text{ for } t \in [0, \tau),$$

$$\big(a *_g R\big)(t)x := \int_0^t a(s - t)R(s)x\, ds \text{ for } t \in (-\tau, 0].$$

Then we have, for $-\tau < t,\ s < \tau$ and $x \in X$,

$$\begin{aligned}\big(a *_g R\big)(s)R(t)x - &R(s)\big(a *_g R\big)(t)x \\ &= k_g(t)\big(a *_g R\big)(s)Cx - k_g(s)C\big(a *_g R\big)(t)x. \qquad (4.13)\end{aligned}$$

Proof. Let $x \in X$ be fixed. We will prove the composition property only in the case that $-\tau < s \leq 0$ and $0 \leq t < \tau$; the proof in all other cases is similar. Carrying out a straigtforward computation, we obtain that

$$\big(a *_g R\big)(s)y = R(s)x - k(-s)Cx, \text{ whenever } (x, y) \in \mathcal{A}.$$

Using this equality and elementary definitions, we get that:

$$\begin{aligned}\big(a*_g R\big)(s)R(t)x &\ni \big(a*_g R\big)(s)\big[\mathcal{A}\big(a*_g R\big)(t)x+k(t)Cx\big]\\ &= k(t)\big(a*_g R\big)(s)Cx+\big(a*_g R\big)(s)\mathcal{A}\big(a*R\big)(t)x\\ &= k(t)\big(a*_g R\big)(s)Cx+R(s)\big(a*_g R\big)(t)x-k(-s)C\big(a*_g R\big)(t)x.\end{aligned}$$

This immediately implies (4.13). □

Remark 4.12. The composition property already established for K-convoluted C-groups and the corresponding composition property presented in (4.13) are not the same. Further comparisons of these composition properties are without scope of this paper.

Suppose now that $\mathcal{A}$ is the integral generator of an almost periodic (a,k)-regularized C-resolvent family $(R(t))_{t\geq 0}$ satisfying that $R(t)R(s)=R(s)R(t)$ for all $t,\ s\geq 0$. Set $S(t)x:=[E(R_x(\cdot))](t)$, $t\in\mathbb{R}$, $x\in X$, where $R_x(t):=R(t)x$, $x\in X$, $t\geq 0$. Arguing as in the proof of Theorem 3.2, we get that $(S(t))_{t\in\mathbb{R}}\subseteq L(X)$ is a bounded, strongly continuous, almost periodic operator family satisfying that $S(t)=R(t)$ for all $t\geq 0$ and $S(t)S(s)=S(s)S(t)$ for all $t,\ s\in\mathbb{R}$. Furthermore, the uniform almost periodicity of $(R(t))_{t\geq 0}$ is equivalent with that of $(S(t))_{t\in\mathbb{R}}$. But, the equation (4.13) cannot be expected with $(R(t))_{t\in\mathbb{R}}$ replaced by $(S(t))_{t\in\mathbb{R}}$. To explain this in more detail, suppose that the functions $t\mapsto k(t)$, $t\geq 0$ and $t\mapsto (a*R)(t)x$, $t\geq 0$ are almost periodic, as well. Let $t\geq 0$ and $s\leq 0$. Then (4.13) and the properties of extension mapping $E:AP([0,\infty):X)\rightarrow AP(\mathbb{R}:X)$ show that

$$\begin{aligned}\big(a*_g S\big)(t)S(s)x &= \Big[W(s)\big(a*_g S\big)(t)R_x(\cdot)\Big](0)\\ &= \Big[W(s)\Big\{S(t)\big(a*_g S\big)(\cdot)x\\ &\qquad +k(\cdot)\big(a*_g S\big)(t)Cx-k(t)C\big(a*_g S\big)(\cdot)x\Big\}\Big](0)\\ &= S(t)\Big[E\big(\big(a*_g S\big)(\cdot)x\big)\Big](s)\\ &\qquad +[E(k)](s)\big(a*_g S\big)(t)x-k(t)C\Big[E\big(\big(a*_g S\big)(\cdot)x\big)\Big](s).\end{aligned}$$

In general case, $[E(k)](s)\neq k(-s)$ for $s<0$ and the obtained composition property is clearly different from (4.13). Additional problem is how to compute $[E((a*_g S)(\cdot)x)](s)$.

For the sequel, we need to remind ourselves of the notion of an exponentially bounded (a,k)-regularized C-resolvent families generated by a pair of closed linear operators $A,\ B$ acting on X; cf. [32, Subsection 2.3.3] for more details.

Definition 4.13. ([32]) Suppose that the functions $a(t)$ and $k(t)$ satisfy (P1), as well as that $R(t)\in L(X,[D(B)])$ for all $t\geq 0$. Let $C\in L(X)$ be injective, and let $CA\subseteq AC$ and $CB\subseteq BC$. Then the operator family $(R(t))_{t\geq 0}$ is said to be an exponentially bounded (a,k)-regularized C-resolvent family generated by $A,\ B$ iff there exists $\omega\geq\max(0,\operatorname{abs}(a),\operatorname{abs}(k))$ such that the following holds:

(i) The mappings $t \mapsto R(t)x$, $t \geq 0$ and $t \mapsto BR(t)x$, $t \geq 0$ are continuous for every fixed element $x \in X$.
(ii) The family $\{e^{-\omega t}R(t) : t \geq 0\} \subseteq L(X, [D(B)])$ is bounded.
(iii) For every $\lambda \in \mathbb{C}$ with $\Re\lambda > \omega$ and $\tilde{k}(\lambda) \neq 0$, the operator $B - \tilde{a}(\lambda)A$ is injective, $R(C) \subseteq R(B - \tilde{a}(\lambda)A)$ and

$$\tilde{k}(\lambda)\big(B - \tilde{a}(\lambda)A\big)^{-1}Cx = \int_0^\infty e^{-\lambda t}R(t)x\,dt, \quad x \in X.$$

Before proceeding further, we want to mention that the class of exponentially bounded (a,k)-regularized C-resolvent family generated by A, B, where the operator $C \in L(X)$ is not injective, has not deserved the attention of authors so far.

The most important properties of exponentially bounded (a,k)-regularized C-resolvent families generated by A, B are clarified in [32, Theorem 2.3.15]. For our purposes, it will be sufficient to recall the following facts:

(T1): Let $x \in X$. Then the function $t \mapsto u(t)$, $t \geq 0$, defined by $u(t) := R(t)x$, $t \geq 0$ is a solution of problem (2.1) with $\mathcal{A} = A$, $\mathcal{B} = B$ and $\mathcal{F}(t) = k(t)Cx$, $t \geq 0$.
(T2): Let $x \in D(A) \cap D(B)$. Then the function $t \mapsto u(t)$, $t \geq 0$, defined by $u(t) := R(t)Bx$, $t \geq 0$ is a strong solution of problem (2.1) with $\mathcal{A} = A$, $\mathcal{B} = B$ and $\mathcal{F}(t) = k(t)CBx$, $t \geq 0$. Furthermore,

$$u(t) = k(t)Cx + \int_0^t a(t-s)R(s)Ax\,ds, \quad t \geq 0. \tag{4.14}$$

Taking the Laplace transform of both sides of (4.14), it is straightforward to prove the following analogue of Proposition 4.1:

Proposition 4.14. *Suppose that $abs(|a|) < \infty$, $abs(k) < \infty$ and $(R(t))_{t\geq 0}$ is an exponentially bounded (a,k)-regularized C-resolvent family generated by A, B. Denote by D the set consisting of all non-zero vectors $x \in D(A) \cap D(B)$ such that there exists $\lambda \in \mathbb{C}$ satisfying that $\lambda Bx = Ax$ and the mapping $\vartheta(t)$, defined through (4.1), is almost periodic. Then the mapping $t \mapsto R(t)Bx$, $t \geq 0$ is almost periodic for all $x \in span(D)$; furthermore, the mapping $t \mapsto R(t)y$, $t \geq 0$ is almost periodic for all $y \in \overline{B(span(D))}$ provided additionally that $(R(t))_{t\geq 0}$ is bounded.*

Concerning the statement of Theorem 4.5, it is very difficult to say what will be the consequences of almost periodicity of an exponentially bounded (a,k)-regularized C-resolvent family $(R(t))_{t\geq 0}$ generated by A, B (with the exception of its boundedness, which is a trivial thing that must be satisfied). But, as a simple consequence of Theorem 4.5, we can clarify some necessary conditions for the operator family $(BR(t))_{t\geq 0} \subseteq L(X)$ to be almost periodic. More precisely, we always have that $(BR(t))_{t\geq 0}$ is an exponentially bounded (a,k)-regularized C-resolvent family generated by the multivalued linear operator $\overline{AB^{-1}}$ (recall that AB^{-1} is closed provided that $C = I$, as well as that any multivalued linear operator is closable); cf. [32, Remark 3.2.6(iv)]. Since $BR(0) = k(0)C$, in the corresponding reformulation we do not need to employ the condition on the denseness of domain of the multivalued linear operator AB^{-1}. The interested reader may try to rephrase the assertions of Proposition 4.7-Proposition 4.8 for the operator family $(BR(t))_{t\geq 0}$.

We close the paper by giving two illustrative examples. In the first one, we continue the analysis of the existence and uniqueness of entire solutions to the abstract Barenblatt-Zheltov-Kochina equation [32, Example 2.3.49].

Example 4.15. Suppose that $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$. By $\{\lambda_k\}$ $[= \sigma(\Delta)]$ we denote the eigenvalues of the Dirichlet Laplacian Δ in $X := L^2(\Omega)$ (recall that $0 < -\lambda_1 \leq -\lambda_2 \cdots \leq -\lambda_k \leq \cdots \to +\infty$ as $k \to \infty$; see [32] for further information), numbered in nonascending order with regard to multiplicities. By $\{\phi_k\} \subseteq C^\infty(\Omega)$ we denote the corresponding set of mutually orthogonal [in the sense of $L^2(\Omega)$] eigenfunctions.

In the afore-mentioned example, we have analyzed the entire solutions of the Barenblatt-Zheltov-Kochina equation

$$(\lambda - \Delta)u_t(t,x) = \zeta\Delta u(t,x), \quad t \in \mathbb{R},\ x \in \Omega,$$

equipped with the following initial conditions

$$u(0,x) = u_0(x), \quad x \in \Omega;\ u(t,x) = 0,\ (t,x) \in \mathbb{R} \times \partial\Omega, \tag{4.15}$$

where $\zeta \in \mathbb{R} \setminus \{0\}$ and $\lambda = \lambda_{k_0} \in \sigma(\Delta)$. We have constructed an entire, exponentially bounded, $(1,t)$-regularized I-resolvent family $(W^1(t))_{t\geq 0}$ generated by $A := \zeta\Delta,\ B := \lambda - \Delta$, which additionally satisfies that there exists a finite constant $\omega > 0$ such that the operator families $\{e^{-\omega z}W^1(z) : z \in \mathbb{C}\} \subseteq L(X)$ and $\{e^{-\omega z}BW^1(z) : z \in \mathbb{C}\} \subseteq L(X)$ are bounded (here, by entireness we mean that, for every $f \in X$, the mappings $z \mapsto W^1(z)f$, $z \in \mathbb{C}$ and $z \mapsto BW^1(z)f$, $z \in \mathbb{C}$ are holomorphic).

(i) Let $\zeta \in \mathbb{R} \setminus \{0\}$ and $\lambda = \lambda_{k_0} \in \sigma(\Delta)$. Consider the equation

$$(\lambda - \Delta)u_t(t,x) = i\zeta\Delta u(t,x), \quad t \in \mathbb{R},\ x \in \Omega, \tag{4.16}$$

equipped with the initial conditions of type (4.15). It can be easily seen that $(i^{-1}W^1(ti))_{t\geq 0}$ is an entire, exponentially bounded, $(1,t)$-regularized I-resolvent family generated by $iA,\ B$. Furthermore, if $r = -\zeta^{-1}\lambda_k^{-1}(\lambda - \lambda_k)$ for some $k \in \mathbb{N}$ with $k \neq k_0$, then $(ir)^{-1}B\phi_k = (iA)\phi_k$. Hence, Proposition 4.14 implies that the mapping $t \mapsto i^{-1}W^1(ti)Bf$, $t \geq 0$, appearing in (T2), is almost periodic for any $f \in \mathit{span}(\{\phi_k : k \in \mathbb{N},\ k \neq k_0\})$. By the representation formula from [1, Theorem 2.3], it readily follows that there exists a sufficiently large real number $\lambda_0 > 0$ such that for each $f \in H^2(\Omega) \cap H_0^1(\Omega)$, the expression

$$u(t) = \bigl(\lambda_0 B - iA\bigr)^{-1}Bf + \Bigl(\lambda_0\bigl(\lambda_0 B - iA\bigr)^{-1}B - I\Bigr)i^{-1}W^1(ti)Bf, \quad t \geq 0$$

defines a unique strong, almost periodic, solution of problem (4.16), with the initial value $u_0 = \bigl(\lambda_0 B - iA\bigr)^{-1}Bf$ in (4.15). In conclusion, we obtain that for all initial values of $u_0 \in \mathit{span}(\{\phi_k : k \neq k_0\})$, the unique strong solution of (4.16) is almost periodic. It is clear that we cannot expect the almost perodicity of mappings like $t \mapsto W^1(t)f$, $t \geq 0$ for all $f \in X$ because $W^1(\cdot)$ is entire.

(ii) In the following example of general type, we would like to exhibit an idea concerning the use of regularizing functions $k(t)$ of the form

$$k(t) = \mathcal{L}^{-1}\left(\frac{1}{(z - ia_1)(z - ia_2)\cdots(z - ia_n)}\right)(t), \quad t \geq 0, \tag{4.17}$$

where $a_1,\ a_2, \cdots, a_n$ are real numbers. Suppose $P(z),\ P_1(z),\ Q(z),\ Q_1(z)$ are non-zero complex polynomials, $dg(P_1) < dg(Q_1)$, $A := P(\Delta)$, $B := Q(\Delta)$, there exist a non-empty set $\Omega \subseteq \mathbb{C}$ and a finite constant $M > 0$ such that the operator $\lambda B - A$ is not injective, as well as all complex roots of the polynomial $z \mapsto Q_1(z) - \lambda P_1(z)$, $z \in \mathbb{C}$ belong to the interval $[-iM, iM]$ and are pairwise disjoint $(\lambda \in \Omega)$. Let $a(t) := \mathcal{L}^{-1}(P_1(z)/Q_1(z))(t)$, $t \geq 0$ (for example, $a(t) = \sin t$ or $a(t) = \cos t$). Suppose, further, that there exist $N \in \mathbb{N}$ and $\omega > 0$ such that $\|(B - \tilde{a}(z)A)^{-1}\| + \|B(B - \tilde{a}(z)A)^{-1}\| = O(1 + |z|^N)$, $\Re z > \omega$. Let $n \geq N + 2$ and let $|a_i| > M$ $(1 \leq i \leq n)$ in (4.17). Then the complex characterization theorem for the Laplace transform shows that there exists an exponentially bounded (a, k)-regularized resolvent family generated by A, B; furthermore, the function $\vartheta(\cdot)$ defined above is almost periodic for all $\lambda \in \Omega$. Hence, Proposition 4.14 is susceptible to applications.

The existence and uniqueness of almost periodic solutions to abstract degenerate higher order Cauchy problems with integer order derivatives have not attracted the attention of authors so far. In the second example, we will point out a few relevant facts concerning almost periodic solutions of the abstract linearized Boussinesq-Love equation; cf. the paper [40] by G. A. Sviridyuk and A. A. Zamyshlyaeva for more details.

Example 4.16. Suppose that $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$, and $X = L^2(\Omega)$. Of concern is the following Cauchy-Dirichlet problem for linearized Boussinesq-Love equation:

$$(\lambda - \Delta)u_{tt}(t, x) - \alpha(\Delta - \lambda')u_t(t, x) = \beta(\Delta - \lambda'')u(t, x) + f(t, x),\ t \in \mathbb{R},\ x \in \Omega, \tag{4.18}$$

$$u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x),\ (t, x) \in \mathbb{R} \times \Omega;\ u(t, x) = 0,\ (t, x) \in \mathbb{R} \times \partial\Omega, \tag{4.19}$$

where $\lambda,\ \lambda',\ \lambda'' \in \mathbb{R}$, $\alpha,\ \beta \in \mathbb{R}$ and $\alpha,\ \beta \neq 0$. We will use the same terminology as in Example 4.15.

In [40, Theorem 5.1], the authors have proved the well-posedness results for degenerate Cauchy problem (4.18)-(4.19) under the following conditions:

(i) $\lambda \in \rho(\Delta)$, or

(ii) $\lambda \in \sigma(\Delta)\ \wedge\ \lambda = \lambda'\ \wedge\ \lambda \neq \lambda''$.

As observed in [32, Example 2.3.48], [40, Theorem 5.1] is inapplicable in the following case:

(iii) $\lambda \in \sigma(\Delta)\ \wedge\ \lambda \neq \lambda'\ \wedge\ (\alpha = 0 \Rightarrow \lambda \neq \lambda'')$.

If so, the existence and uniqueness of entire solutions of problem (4.18)-(4.19) (cf. [32, Definition 2.3.45] for the notion) for all initial values of $u_0,\ u_1 \in H^2(\Omega) \cap H_0^1(\Omega)$ follows from an application of [32, Theorem 2.3.46].

In what follows, we will consider the well-posedness of homogeneous problem (4.18)-(4.19) in case (i), with $u_0,\ u_1 \in H^2(\Omega) \cap H_0^1(\Omega)$. By [40, Theorem 5.1(i)], there exists a unique solution of this problem and has the following form:

$$u(t) = \sum_{k=1}^{\infty} \left[\frac{\mu_k^1}{\mu_k^1 - \mu_k^2} e^{\mu_k^1 t} - \frac{\mu_k^2}{\mu_k^1 - \mu_k^2} e^{\mu_k^2 t} \right] \langle \phi_k, u_0 \rangle \phi_k$$
$$+ \sum_{k=1}^{\infty} \frac{e^{\mu_k^1 t} - e^{\mu_k^2 t}}{(\lambda - \lambda_k)(\mu_k^1 - \mu_k^2)} \langle \phi_k, u_1 \rangle \phi_k, \quad t \in \mathbb{R},$$

where

$$\mu_k^{1,2} := \frac{-\alpha(\lambda' - \lambda_k) \pm \sqrt{\alpha^2(\lambda' - \lambda_k)^2 - 4\beta(\lambda - \lambda_k)(\lambda'' - \lambda_k)}}{2(\lambda - \lambda_k)}, \quad k \in \mathbb{N}.$$

Suppose that the following condition holds:

$$\alpha^2(\lambda' - \lambda_k)^2 \geq 4\beta(\lambda - \lambda_k)(\lambda'' - \lambda_k), \quad k \in \mathbb{N}.$$

Then $\mu_k^{1,2} \in \mathbb{R}$, $k \in \mathbb{R}$ and the function $t \mapsto u(it)$, $t \in \mathbb{R}$ is almost periodic for all $u_0,\ u_1 \in span(\{\phi_k : k \in \mathbb{N}\})$, which is clearly dense in X. This fact cannot be established in case (ii) since then there exists a strong solution of (4.18)-(4.19) only for initial values of $u_0,\ u_1 \in H^2(\Omega) \cap H_0^1(\Omega)$ that are orthogonal to the functions ϕ_k for $\lambda = \lambda_k$ (cf. [40, Theorem 5.1(ii)]).

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