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TWO-WEIGHT NORM INEQUALITIES FOR THE HIGHER-ORDER COMMUTATORS OF FRACTIONAL INTEGRAL OPERATORS

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Abstract. In this paper, we obtain several sufficient conditions such that the higher-order commutators $I^m_{\alpha,b}$ generated by I_α and $b \in \mathrm{BMO}(\mathbb{R}^n)$ is bounded from $L^p(v)$ to $L^q(u)$, where $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ and $0 < \alpha < n$.

1. Introduction and main results

For $0 < \alpha < n$ and $f \in C_0^\infty$, the fractional integral operator is defined by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} \mathrm{d}y.$$

It is a classical operator and has close connections with the partial differential equations. Many mathematicians have investigated its boundedness on various settings, which can be found in [1, 2, 3, 4, 5, 6, 7] and their references and so on.

Let μ be a nonnegative locally integrable function on $\mathbb{R}^n$. A function $b \in \mathrm{BMO}(\mu)$, if there is a constant $C > 0$ such that

$$\int_B |b(x) - b_B| \mathrm{d}x \leqslant C \int_{\mathrm{B}} \mu(x) \mathrm{d}x,$$

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where $b_B = \frac{1}{|B|}\int_B b(x)\mathrm{d}x$ and B is any ball in $\mathbb{R}^n$. Obviously, when $\mu = 1$, $\mathrm{BMO}(\mu)$ is the classical BMO spaces with

$$\|b\|_* := \|b\|_{\mathrm{BMO}} = \sup_B \frac{1}{|B|}\int_B |b(x) - b_B|\mathrm{d}x < \infty.$$

Let m be a nonnegative integer. The m-order commutator $I^m_{\alpha,b}$, generated by I_α and $b \in \mathrm{BMO}(\mu)$, is defined by

$$I^m_{\alpha,b}f(x) = \int_{\mathbb{R}^n} \frac{(b(x) - b(y))^m}{|x-y|^{n-\alpha}} f(y)\mathrm{d}y.$$

When $b \in \mathrm{BMO}$, the boundedness of $I^m_{\alpha,b}$ from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ have been established. We refer readers to [2] and its references.

To state the following results, let us recall the definition of $A_{p,q}$ weights. A nonnegative locally integrable function u on $\mathbb{R}^n$ is said to belong to $A_{p,q}$ $(1 < p,\, q < \infty)$ if there exists a constant $C > 0$ such that

$$\sup_Q \left(\frac{1}{|Q|}\int_Q u(x)^q \mathrm{d}x\right)^{\frac{1}{q}} \left(\frac{1}{|Q|}\int_Q u(x)^{-p'}\mathrm{d}x\right)^{\frac{1}{p'}} \leqslant C,$$

where $\frac{1}{p} + \frac{1}{p'} = 1$ and Q is a cube in $\mathbb{R}^n$ with its sides parallel to the coordinate axes.

The $(L^p(w^p), L^q(w^q))$ boundedness of $I^m_{\alpha,b}$ can be seen in [2] and its references and so on.

In 1999, Y. Ding and S. Z. Lu [8] gave the weighted boundedness of higher-order commutators for a class of rough operators. As a special case of the results in [8], the follow theorem offered a sufficient condition which can ensure the boundedness of $I^m_{\alpha,b}$ from $L^p(u^p)$ to $L^q(v^q)$.

Theorem A [8] *Suppose $0 < \alpha < n$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. If $b \in \mathrm{BMO}(\mu)$, $u(x)^s, v(x)^s \in A_{\frac{p}{s},\frac{q}{s}}$ for some $s \in [1,\, p)$ and $u(x)v(x)^{-1} = \mu(x)^m$, then there exists constant $C > 0$ independent of f such that $I^m_{\alpha,b}$ satisfies*

$$\left(\int_{\mathbb{R}^n} \left(|I^m_{\alpha,b}f(x)|v(x)\right)^q \mathrm{d}x\right)^{\frac{1}{q}} \leqslant C\left(\int_{\mathbb{R}^n} \left(|f(x)|u(x)\right)^p \mathrm{d}x\right)^{\frac{1}{p}}.$$

A function $\mathbf{B} : [0, +\infty) \to [0, \infty)$ is said to be a Young function if it is convex, increasing, $\mathbf{B}(0) = 0$ and $\mathbf{B}(t) \to \infty$ as $t \to \infty$. Given a Young function $\mathbf{B}$, the mean Luxembourg norm of f on a cube Q is defined by

$$\|f\|_{\mathbf{B},Q} = \inf\left\{\lambda > 0 : \frac{1}{|Q|}\int_Q \mathbf{B}\left(\frac{|f(y)|}{\lambda}\right)\mathrm{d}y \leqslant 1\right\}.$$

For $0 < \alpha < n$, $1 < p < \infty$ and $b \in \mathrm{BMO}$, Z. G. Liu and S. Z. Lu [9] proved that for some $r > 1$ and all cubes Q, if

$$|Q|^{\frac{\alpha}{n}} \left(\frac{1}{|Q|}\int_Q u(x)^r \mathrm{d}x\right)^{\frac{1}{rp}} \|v^{-\frac{1}{p}}\|_{\Psi,Q} < C,$$

then $I^1_{\alpha,b}$ is bounded from $L^p(v)$ to $WL^p(u)$, where $\Psi(t) = t^{p'}(1 + log^+ t)^{p'}$.

W. M. Li [10] gave a sufficient condition for the two-weight (p,q) inequalities for the commutators of potential type integral operators. As an application of this result, he obtained another sufficient condition for the boundedness of $I^m_{\alpha,b}$ from $L^p(u^p)$ to $L^q(v^q)$, and his conditions are expressed by the mean Luxembourg norm of u,v on any cube Q. More detail can be found in [10].

Rakotonratsimba [11] studied the two-weight inequality for the commutators of singular integral operators. Inspired by [11], we also consider the $(L^p(v),L^q(u))$ boundedness for $I^m_{\alpha,b}$ and obtain several different sufficient conditions in this paper.

Throughout this paper, u, v, w are nonnegative locally integral functions. Denote $B=B(x,\,R)=\{z\in\mathbb{R}^n:\,|x-z|<R\}$ with $R>0$ and $x\in\mathbb{R}^n$. For any $k\in\mathbb{Z}$, define

$$B_k:=\{x\in\mathbb{R}^n:|x|<2^k\},\quad E_k:=\{x\in\mathbb{R}^n:\,2^k<|x|\leqslant 2^{k+1}\},$$

$$b_k:=b_{B(0,\,2^k)}=\frac{1}{|B(0,\,2^k)|}\int_{B(0,\,2^k)}b(x)\mathrm{d}x$$

and $\chi_k(x):=\chi_{E_k}(x)$ is the characteristic function of E_k.

Definition 1.1. Given $0<\alpha<n$, $1<p<\frac{n}{\alpha}$, $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$ and $b\in\mathrm{BMO}$.

(1) Let

$$\begin{aligned}A(j,k):=&2^{-k(n-\alpha)}\Big[\Big(\int_{E_k}|b(x)-b_j|^{mq}u(x)\mathrm{d}x\Big)^{\frac{1}{q}}\Big(\int_{E_j}v(z)^{1-p'}\mathrm{d}z\Big)^{\frac{1}{p'}}\\&+\Big(\int_{E_k}u(x)\mathrm{d}x\Big)^{\frac{1}{q}}\Big(\int_{E_j}|b(z)-b_j|^{mp'}v(z)^{1-p'}\mathrm{d}z\Big)^{\frac{1}{p'}}\Big].\end{aligned}$$

(u,v,b) is said to belong to $\mathcal{A}(j,k)$ if there exist some constants ρ, η, $A>0$ such that

$$A(j,k)\leqslant A(k-j)^{\eta}2^{(j-k)n\rho}\|b\|_*^m.$$

(2) Let

$$\begin{aligned}A^*(j,k):=&2^{-k(n-\alpha)}\Big[\Big(\int_{E_j}|b(x)-b_j|^{mq}u(x)\mathrm{d}x\Big)^{\frac{1}{q}}\Big(\int_{E_k}v(z)^{1-p'}\mathrm{d}z\Big)^{\frac{1}{p'}}\\&+\Big(\int_{E_j}u(x)\mathrm{d}x\Big)^{\frac{1}{q}}\Big(\int_{E_k}|b(z)-b_j|^{mp'}v(z)^{1-p'}\mathrm{d}z\Big)^{\frac{1}{p'}}\Big].\end{aligned}$$

(u,v,b) is said to belong to $\mathcal{A}^*(j,k)$ if there exist some constants τ, η, $A>0$ such that

$$A^*(j,k)\leqslant A(k-j)^{\eta}2^{(j-k)n\tau}\|b\|_*^m.$$

$\mathcal{A}^*(j,k)$ can be seen as a dual of $\mathcal{A}(j,k)$. Our main results are as follows:

Theorem 1.2. *Given $0<\alpha<n$, $1<p<\frac{n}{\alpha}$, $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$ and $b\in\mathrm{BMO}$. If there exist some constants $\rho,\tau,\,\eta,\,A>0$ such that $(u,v,b)\in\mathcal{A}(j,k)\cap\mathcal{A}^*(j,k)$ for $j\leqslant k-2$, and*

(a) $u(x)\big(\sup_{\{y:\ 4^{-1}|x|<|y|<4|x|\}}\frac{1}{v(y)}\big)\leqslant A^q$, *a.e.* $x\in\mathbb{R}^n$,

(b) $\sup_{z\in E_k}u(z)\leqslant C\big(\sup_{z\in E_k}u(z)\big)^{\frac{q}{p}}$, $k\in\mathbb{Z}$.

Then $I^m_{\alpha,b}$ *is bounded from* $L^p(udx)$ *to* $L^q(udx)$*. Condition* (a) *can be replaced by*

(a′) $\big(\sup_{\{y:\ 4^{-1}|x|<|y|<4|x|\}} u(y)\big)\frac{1}{v(x)} \le A^q, \quad a.e. \quad x \in \mathbb{R}^n.$

And the above results remain true when $(u, v, b) \in \mathcal{A}(j, k) \cap \mathcal{A}^*(j, k)$ *for* $j \leqslant k - n_0$*, where integer* $n_0 \geqslant 2$*.*

Theorem 1.3. *Given* $0 < \alpha < n$*,* $1 < p < \frac{n}{\alpha}$*,* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ *and* $b \in \mathrm{BMO}$*. For* $j \in \mathbb{Z}$*, define*

$$U_j = \sup_{x\in E_j} u(x), \qquad V_j = \sup_{x\in E_j} v(x)^{1-p'}.$$

Then $I^m_{\alpha,b}$ *is bounded from* $L^p(u\mathrm{d}x)$ *to* $L^q(u\mathrm{d}x)$ *whenever for some constant* $A > 0$*, the conditions* (a) (b) *or* (a′) (b) *in Theorem 1.2 are satisfied and*

(c) $U_k^{\frac{1}{q}} V_j^{\frac{1}{p'}} \leqslant C2^{(j-k)na}A, \quad U_j^{\frac{1}{q}} V_k^{\frac{1}{p'}} \leqslant C2^{(j-k)na^*}A$

for all integers k *and* j *with* $j \leqslant k - 2$*. Here* a, a^* *are constants with* $-\frac{1}{p'} < a$ *and* $-\frac{1}{q} < a^*$*.*

Let M be the classical Hardy-Littlewood maximal operator. We say $w(\cdot) \in \mathcal{H}$ if there exists constant $C > 0$ such that

$$\sup_{\{z:\ \frac{1}{4}R<|z|<4R\}} w(z) \leqslant C\frac{1}{R^n}\int_{\frac{1}{2^N}R<|y|<2^N R} w(y)\mathrm{d}y$$

for any $R > 0$ and some integer $N \geqslant 3$.

It is easy to see $w \in \mathcal{H}$ whenever w is radical and monotone. However $w \in \mathcal{H}$ is not necessarily a monotone weight (see [12]).

We say $w \in RD_\gamma$ (the reverse doubling condition) if there exist nonnegative constant C and γ such that

$$\int_{B(0,\,tR)} w(y)\mathrm{d}y \leqslant Ct^{n\gamma}\int_{B(0,\,R)} w(y)\mathrm{d}y$$

for all $R > 0$ and $0 < t < 1$.

It has been proved that each doubling weight (in particular each A_∞-weight) satisfies necessarily the reverse doubling condition RD (see [13][14]), but there exist reverse doubling weights which are not doubling ones (see [13][15]).

By Theorem 1.3 we can get another two sufficient conditions such that $I^m_{\alpha,b}$ is bounded from $L^p(udx)$ to $L^q(udx)$.

Corollary 1.4. *Let* $b \in \mathrm{BMO}$*. If* u *satisfies condition* (b) *in Theorem 1.2 and* $u \in \mathcal{H} \cap RD_\gamma$ *for some* $\gamma > 0$*, then* $I^m_{\alpha,b}$ *is bounded from* $L^p(Mu)$ *to* $L^q(u)$*. Here,* $0 < \alpha < \min\{n, n\gamma\}$*,* $1 < p < \min\{\frac{n\gamma}{\alpha}, \frac{n}{\alpha}\}$*, and* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$*.*

Corollary 1.5. *Let* $b \in \mathrm{BMO}$*,* $u \in \mathcal{H} \cap RD_\gamma$ *and* $v^{1-p'} \in \mathcal{H} \cap RD_\gamma$ *for some* $\gamma > 0$*. If* u *satisfies condition* (b) *in Theorem 1.2 and there exists constant* $A > 0$ *such that for all* $R > 0$

$$\Big(\frac{1}{R^n}\int_{|x|<R} u(x)\mathrm{d}x\Big)^{\frac{1}{p}}\Big(\frac{1}{R^n}\int_{|x|<R} v(x)^{1-p'}\mathrm{d}x\Big)^{\frac{1}{p'}} \leqslant A, \tag{1.1}$$

then $I^m_{\alpha,b}$ *is bounded from* $L^p(v\mathrm{d}x)$ *to* $L^q(u\mathrm{d}x)$*. Here* $0<\alpha<\min\{n,n\gamma\}$, $1<p<\min\{\frac{n\gamma}{\alpha},\frac{n}{\alpha}\}$, $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$.

Remark 1.6. The set of (u,v,b) satisfying the conditions in Theorem 1.2, Theorem 1.3 or Corollary 1.4 is not empty, and the conditions in Theorem 1.2 are different from that in Theorem A. For example:

(i) Let $u(x)\equiv 1$, $v(x)=\begin{cases}1, & |x|\leqslant 1\\ |x|, & |x|>1\end{cases}$, $b(x)=\begin{cases}x, & |x|\leqslant 1\\ 1, & |x|>1\end{cases}$ for $x\in\mathbb{R}$, then $(u,\,v,\,b)$ satisfies the conditions in Theorem 1.2. Furthermore, $(u,\,v,\,b)$ does not satisfy the condition in Theorem A for $b\notin BMO\big((uv^{-1})^{\frac{1}{m}}\big)$ for any $m\geqslant 1$.

(ii) Let $u(x)=\begin{cases}|x|, & |x|\leqslant 1\\ 1, & |x|>1\end{cases}$, $v(x)=\begin{cases}1, & |x|\leqslant 1\\ |x|, & |x|>1\end{cases}$, then $(u,\,v)$ satisfies the conditions in Theorem 1.3 and $(u,\,v,\,b)$ satisfies the conditions in Theorem 1.2 for any $b\in\mathrm{BMO}$.

(iii) Let $u(x)=\begin{cases}|x|, & |x|\leqslant 1\\ 1, & |x|>1\end{cases}$, then u satisfies the conditions in Corollary 1.4 and $(Mu,\,u)$ satisfies the conditions in Theorem 1.3.

Remark 1.7. It would be noted that all conditions in above results are not expressed in term of arbitrary cubes as in the classical A_p condition, and they are also not expressed in term of the Luxembourg norm. These conditions involving annuli and balls centered at the origin are easy to be checked in many situations.

2. Proof of the Main Results

Proof of Theorem 1.2. For $f\in C_0^\infty(\mathbb{R}^n)$, by the definition of $I^m_{\alpha,b}$ and the Minkowski inequality, we have

$$
\begin{aligned}
&\left(\int_{\mathbb{R}^n}\left|I^m_{\alpha,b}f(x)\right|^q u(x)\mathrm{d}x\right)^{\frac{1}{q}}\\
&\quad=\Bigg(\sum_{k=-\infty}^{\infty}\int_{E_k}\Big|I^m_{\alpha,b}(f\chi_{|\cdot|<2^{k-1}})(x)+I^m_{\alpha,b}(f\chi_{2^{k-1}\leqslant|\cdot|\leqslant 2^{k+2}})(x)\\
&\qquad+I^m_{\alpha,b}(f\chi_{|\cdot|>2^{k+2}})(x)\Big|^q u(x)\,\mathrm{d}x\Bigg)^{\frac{1}{q}}\\
&\quad\leqslant C(S_1+S_2+S_3)^{\frac{1}{q}},
\end{aligned}
$$

where

$$S_1 := \sum_{k=-\infty}^{\infty} \int_{E_k} \Big| I_{\alpha,b}^m (f\chi_{|\cdot|<2^{k-1}})(x)\Big|^q u(x)\,\mathrm{d}x,$$

$$S_2 := \sum_{k=-\infty}^{\infty} \int_{E_k} \Big| I_{\alpha,b}^m (f\chi_{2^{k-1}\leqslant|\cdot|\leqslant 2^{k+2}})(x)\Big|^q u(x)\,\mathrm{d}x,$$

$$S_3 := \sum_{k=-\infty}^{\infty} \int_{E_k} \Big| I_{\alpha,b}^m (f\chi_{|\cdot|>2^{k+2}})(x)\Big|^q u(x)\,\mathrm{d}x.$$

First, we give the estimate for S_1 . It is easy to see that

$$\Big| I_{\alpha,b}^m (f\chi_{|\cdot|<2^{k-1}})(x)\Big| \leqslant \sum_{j=-\infty}^{k-2} \int_{E_j} |b(x)-b(y)|^m |x-y|^{-(n-\alpha)} |f(y)|\,\mathrm{d}y.$$

For $j \leqslant k-2$, if $x \in E_k, y \in E_j$, then $|x-y| \geqslant |x|-|y| > |x|/2$ and $|x-y|^{-(n-\alpha)} \leqslant C|x|^{-(n-\alpha)}$. Applying $m \geqslant 1$, $q \geqslant 1$ we obtain

$$\begin{aligned}
&S_1\\
&\leqslant C \sum_{k=-\infty}^{\infty} \int_{E_k} \Big(\sum_{j=-\infty}^{k-2} \int_{E_j} |b(x)-b(y)|^m |f(y)|\,\mathrm{d}y\Big)^q |x|^{-(n-\alpha)q} u(x)\,\mathrm{d}x\\
&\leqslant C \sum_{k=-\infty}^{\infty} \int_{E_k} \Big(\sum_{j=-\infty}^{k-2} \int_{E_j} (|b(x)-b_j|^m + |b(y)-b_j|^m) |f(y)|\,\mathrm{d}y\Big)^q |x|^{-(n-\alpha)q} u(x)\,\mathrm{d}x\\
&\leqslant C\Big(\sum_{k=-\infty}^{\infty} \int_{E_k} \Big(\sum_{j=-\infty}^{k-2} |b(x)-b_j|^m \int_{E_j} |f(y)|\,\mathrm{d}y\Big)^q |x|^{-(n-\alpha)q} u(x)\,\mathrm{d}x\\
&\quad + \sum_{k=-\infty}^{\infty} \Big(\sum_{j=-\infty}^{k-2} \int_{E_j} |b(y)-b_j|^m |f(y)|\,\mathrm{d}y\Big)^q \int_{E_k} |x|^{-(n-\alpha)q} u(x)\,\mathrm{d}x\Big)\\
&=: C(S_{11}+S_{12}).
\end{aligned}$$

Obviously,

$$S_{12} \leqslant C \sum_{k=-\infty}^{\infty} 2^{-(n-\alpha)kq} \Big(\int_{E_k} u(x)\,\mathrm{d}x\Big) \Big(\sum_{j=-\infty}^{k-2} \int_{E_j} |b(y)-b_j|^m |f(y)|\,\mathrm{d}y\Big)^q.$$

By the Hölder inequality, we have

$$\begin{aligned}
&\sum_{j=-\infty}^{k-2} \int_{E_j} |b(y)-b_j|^m |f(y)|\,\mathrm{d}y\\
&\leqslant C \sum_{j=-\infty}^{k-2} \Big(\int_{E_j} |f(y)|^p v(y)\,\mathrm{d}y\Big)^{\frac{1}{p}} \Big(\int_{E_j} |b(z)-b_j|^{mp'} v(z)^{1-p'}\,\mathrm{d}z\Big)^{\frac{1}{p'}}. \qquad (2.1)
\end{aligned}$$

Therefore by (2.1) and $(u,v,b)\in\mathcal{A}(j,k)$ for $j\leqslant k-2$, we can get

$$S_{12}\leqslant CA^q\|b\|_*^{mq}\sum_{k=-\infty}^{\infty}\Big(\sum_{j=-\infty}^{k-2}(k-j)^\eta 2^{(j-k)n\rho}\Big(\int_{E_j}|f(y)|^p v(y)\,\mathrm{d}y\Big)^{\frac{1}{p}}\Big)^q. \tag{2.2}$$

Applying the Hölder inequality,

$$\begin{aligned}&\Big(\sum_{j=-\infty}^{k-2}(k-j)^\eta 2^{(j-k)n\rho}\Big(\int_{E_j}|f(y)|^p v(y)\,\mathrm{d}y\Big)^{\frac{1}{p}}\Big)^q\\&\leqslant C\Big(\sum_{j=-\infty}^{k-2}(k-j)^\eta 2^{(j-k)n\rho}\Big)^{q-1}\sum_{j=-\infty}^{k-2}(k-j)^\eta 2^{(j-k)n\rho}\Big(\int_{E_j}|f(y)|^p v(y)\,\mathrm{d}y\Big)^{\frac{q}{p}}.\end{aligned} \tag{2.3}$$

Put (2.3) into (2.2) and exchange the order of summation, we can obtain

$$\begin{aligned}S_{12}&\leqslant CA^q\|b\|_*^{mq}\sum_{j=-\infty}^{\infty}\Big(\int_{E_j}|f(y)|^p v(y)\,\mathrm{d}y\Big)^{\frac{q}{p}}\Big(\sum_{k=j+2}^{\infty}(k-j)^\eta 2^{(j-k)n\rho}\Big)^q\\&\leqslant CA^q\|b\|_*^{mq}\Big(\int_{\mathbb{R}^n}|f(x)|^p v(x)\,\mathrm{d}x\Big)^{\frac{q}{p}}.\end{aligned}$$

For

$$S_{11}=\sum_{k=-\infty}^{\infty}\int_{E_k}\Big(\sum_{j=-\infty}^{k-2}|b(x)-b_j|^m\int_{E_j}|f(y)|\,\mathrm{d}y\Big)^q|x|^{-(n-\alpha)q}u(x)\,\mathrm{d}x,$$

if we consider $w(x):=|x|^{-(n-\alpha)q}u(x)$ as a weight function, then

$$\begin{aligned}&\int_{E_k}\Big(\sum_{j=-\infty}^{k-2}|b(x)-b_j|^m\int_{E_j}|f(y)|\,\mathrm{d}y\Big)^q|x|^{-(n-\alpha)q}u(x)\,\mathrm{d}x\\&=\Big\|\sum_{j=-\infty}^{k-2}|b(x)-b_j|^m\int_{E_j}|f(y)|\,\mathrm{d}y\Big\|_{L^q(E_k,w\mathrm{d}x)}^q.\end{aligned}$$

Therefore, using the Minkowski inequality, the Hölder inequality and $(u, v, b) \in \mathcal{A}(j, k)$ for $j \leqslant k-2$, we have

$$\begin{aligned}
S_{11} &\leqslant C \sum_{k=-\infty}^{\infty} \bigg(\sum_{j=-\infty}^{k-2} \int_{E_j} |f(y)| \,\mathrm{d}y \Big(\int_{E_k} |b(x)-b_j|^{mq} |x|^{-(n-\alpha)q} u(x) \,\mathrm{d}x \Big)^{\frac{1}{q}} \bigg)^q \\
&\leqslant C \sum_{k=-\infty}^{\infty} \bigg(\sum_{j=-\infty}^{k-2} \Big(\int_{E_j} |f(y)|^p v(y) \,\mathrm{d}y \Big)^{\frac{1}{p}} \Big(\int_{E_j} v(z)^{1-p'} \,\mathrm{d}z \Big)^{\frac{1}{p'}} \\
&\quad \times 2^{-k(n-\alpha)} \Big(\int_{E_k} |b(x)-b_j|^{mq} u(x) \,\mathrm{d}x \Big)^{\frac{1}{q}} \bigg)^q \\
&\leqslant C A^q \|b\|_*^{mq} \sum_{k=-\infty}^{\infty} \bigg(\sum_{j=-\infty}^{k-2} (k-j)^{\eta} 2^{(j-k)n\rho} \Big(\int_{E_j} |f(y)|^p v(y) \,\mathrm{d}y \Big)^{\frac{1}{p}} \bigg)^q \\
&\leqslant C A^q \|b\|_*^{mq} \Big(\int_{\mathbb{R}^n} |f(x)|^p v(x) \,\mathrm{d}x \Big)^{\frac{q}{p}},
\end{aligned}$$

where the last inequality follows from the similar estimate as that for (2.2). Next we consider the estimate for S_3. It is easy to see that

$$|I_{\alpha,b}^m (f\chi_{|\cdot|>2^{k+2}})(x)| \leqslant \sum_{j=k+2}^{+\infty} \int_{E_j} |b(x)-b(y)|^m |x-y|^{-(n-\alpha)} |f(y)| \,\mathrm{d}y.$$

When $j \geqslant k+2$, if $x \in E_k, y \in E_j$, then $|x-y| \geqslant |y|-|x| > \frac{|y|}{2}$ and $|x-y|^{-(n-\alpha)} < C|y|^{-(n-\alpha)}$. Therefore by $m \geqslant 1, q \geqslant 1$, we get

$$\begin{aligned}
&S_3 \\
&\leqslant \sum_{k=-\infty}^{\infty} \int_{E_k} \bigg(\sum_{j=k+2}^{\infty} \int_{E_j} (b(x)-b(y))^m |y|^{-(n-\alpha)} |f(y)| \,\mathrm{d}y \bigg)^q u(x) \,\mathrm{d}x \\
&\leqslant C \sum_{k=-\infty}^{\infty} \int_{E_k} \Big(\sum_{j=k+2}^{\infty} \int_{E_j} \big(|b(x)-b_k|^m + |b(y)-b_k|^m\big) |y|^{-(n-\alpha)} |f(y)| \,\mathrm{d}y \Big)^q u(x) \,\mathrm{d}x \\
&\leqslant C \sum_{k=-\infty}^{\infty} \int_{E_k} \Big(\sum_{j=k+2}^{\infty} |b(x)-b_k|^m \int_{E_j} |y|^{-(n-\alpha)} |f(y)| \,\mathrm{d}y \Big)^q u(x) \,\mathrm{d}x \\
&\quad + C \sum_{k=-\infty}^{\infty} \Big(\sum_{j=k+2}^{\infty} \int_{E_j} |b(y)-b_k|^m |y|^{-(n-\alpha)} |f(y)| \,\mathrm{d}y \Big)^q \int_{E_k} u(x) \,\mathrm{d}x \\
&=: C(S_{31}+S_{32}).
\end{aligned}$$

By the Hölder inequality and $(u, v, b) \in \mathcal{A}^*(k, j)$ for $k \leqslant j-2$, we get

$$\begin{aligned}
S_{32} &\leqslant C \sum_{k=-\infty}^{\infty} \Big(\sum_{j=k+2}^{\infty} 2^{-j(n-\alpha)} \int_{E_j} |b(y)-b_k|^m |f(y)| \,\mathrm{d}y \Big(\int_{E_k} u(x)\,\mathrm{d}x \Big)^{\frac{1}{q}} \Big)^q \\
&\leqslant C \sum_{k=-\infty}^{\infty} \Big(\sum_{j=k+2}^{\infty} \Big(\int_{E_j} |f(y)|^p v(y)\,\mathrm{d}y \Big)^{\frac{1}{p}} \\
&\quad \times 2^{-j(n-\alpha)} \Big(\int_{E_j} |b(z)-b_k|^{mp'} v(z)^{1-p'}\,\mathrm{d}z \Big)^{\frac{1}{p'}} \Big(\int_{E_k} u(x)\,\mathrm{d}x \Big)^{\frac{1}{q}} \Big)^q \\
&\leqslant CA^q \|b\|_*^{mq} \sum_{k=-\infty}^{\infty} \Big(\sum_{j=k+2}^{\infty} (j-k)^{\eta} 2^{(k-j)n\tau} \int_{E_j} |f(y)|^p v(y)\,\mathrm{d}y \Big)^q \\
&\leqslant CA^q \|b\|_*^{mq} \sum_{j=-\infty}^{\infty} \Big(\int_{E_j} |f(y)|^p v(y)\,\mathrm{d}y \Big)^{\frac{q}{p}} \\
&= CA^q \|b\|_*^{mq} \Big(\int_{\mathbb{R}^n} |f(x)|^p v(x)\,\mathrm{d}x \Big)^{\frac{q}{p}},
\end{aligned}$$

where the last inequality follows from the similar estimate as that for (2.2).
For

$$S_{31} := \sum_{k=-\infty}^{\infty} \int_{E_k} \Big(\sum_{j=k+2}^{\infty} |b(x)-b_k|^m \int_{E_j} |f(y)||y|^{-(n-\alpha)}\,\mathrm{d}y \Big)^q u(x)\,\mathrm{d}x,$$

we can consider $\int_{E_k} \big(\sum_{j=k+2}^{\infty} |b(x)-b_k|^m \int_{E_j} |f(y)||y|^{-(n-\alpha)}\,\mathrm{d}y \big)^q u(x)\,\mathrm{d}x$ as:

$$\Big\| \sum_{j=k+2}^{\infty} |b(x)-b_k|^m \int_{E_j} |f(y)||y|^{-(n-\alpha)}\,\mathrm{d}y \Big\|_{L^q(E_k, u\mathrm{d}x)}^q.$$

Thus, applying the Minkowski inequality, the Hölder inequality and $(u, v, b) \in \mathcal{A}^*(k, j)$ for $k \leqslant j-2$, we have

$$\begin{aligned}
S_{31} &\leqslant C \sum_{k=-\infty}^{\infty} \Big(\sum_{j=k+2}^{\infty} \Big(\int_{E_k} |b(x)-b_k|^{mq} u(x)\,\mathrm{d}x \Big)^{\frac{1}{q}} \int_{E_j} |f(y)||y|^{-(n-\alpha)}\,\mathrm{d}y \Big)^q \\
&\leqslant C \sum_{k=-\infty}^{\infty} \Big(\sum_{j=k+2}^{\infty} 2^{-j(n-\alpha)} \Big(\int_{E_k} |b(x)-b_k|^{mq} u(x)\,\mathrm{d}x \Big)^{\frac{1}{q}} \\
&\quad \Big(\int_{E_j} v(z)^{1-p'}\,dz \Big) \Big(\int_{E_j} |f(y)|^p v(y)\,\mathrm{d}y \Big)^{\frac{1}{p}} \Big)^q \\
&\leqslant CA^q \|b\|_*^{mq} \sum_{k=-\infty}^{\infty} \Big(\sum_{j=k+2}^{\infty} (j-k)^{\eta} 2^{(k-j)n\tau} \Big(\int_{E_j} |f(y)|^p v(y)\,\mathrm{d}y \Big)^{\frac{1}{p}} \Big)^q \\
&\leqslant CA^q \|b\|_*^{mq} \Big(\int_{\mathbb{R}^n} |f(y)|^p v(y)\,\mathrm{d}y \Big)^{\frac{q}{p}}.
\end{aligned}$$

At last, we estimate S_2. Denote $M_k = \{x \in \mathbb{R}^n, 2^{k-1} \leqslant |x| \leqslant 2^{k+2}\}$.

Suppose the conditions (a′) and (b) are satisfied. When $x \in M_k$, $z \in E_k$, we have $\frac{1}{4}|x| \leqslant 2^k < |z| \leqslant 2^{k+1} \leqslant 4|x|$. Then by the $L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)(\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n})$ boundedness of $I^m_{\alpha,b}$ (see [2], Remark 3.6.1) and conditions (a′), (b), we obtain

$$\begin{aligned}
S_2 &\leqslant \sum_{k=-\infty}^{\infty} \sup_{z\in E_k} u(z) \int_{E_k} \left|I^m_{\alpha,b}(f\chi_{M_k})(x)\right|^q \mathrm{d}x \\
&\leqslant C\|b\|_*^{mq} \sum_{k=-\infty}^{\infty} \sup_{z\in E_k} u(z) \Big(\int_{M_k} \left|f(x)\right|^p \mathrm{d}x\Big)^{\frac{q}{p}} \\
&\leqslant C\|b\|_*^{mq} \sum_{k=-\infty}^{\infty} \Big(\int_{M_k} \left|f(x)\right|^p \big(\sup_{z\in E_k} u(z)\big) \mathrm{d}x\Big)^{\frac{q}{p}} \\
&\leq C\|b\|_*^{mq} \sum_{k=-\infty}^{\infty} \Big(\int_{M_k} \left|f(x)\right|^p \Big(\sup_{\{z:\, \frac{1}{4}|x|<|z|<4|x|\}} u(z)\Big) \mathrm{d}x\Big)^{\frac{q}{p}} \\
&\leq CA^q\|b\|_*^{mq} \sum_{k=-\infty}^{\infty} \Big(\int_{M_k} \left|f(x)\right|^p v(x) \mathrm{d}x\Big)^{\frac{q}{p}} \\
&= CA^q\|b\|_*^{mq} \Big(\int_{\mathbb{R}^n} \left|f(x)\right|^p v(x) \mathrm{d}x\Big)^{\frac{q}{p}}.
\end{aligned}$$

Now suppose condition (a) and (b) are satisfied. We have $\frac{1}{4}|z| \leqslant 2^{k-1} \leqslant |y| \leqslant 2^{k+2} \leqslant 4|z|$ for $y \in M_k, z \in E_k$. Therefore $1 = \frac{1}{v(x)} \times v(x) \leqslant \sup_{\{y:\, 4^{-1}|z|<|y|<4|z|\}} \frac{1}{v(y)} \times v(x)$ for $x \in M_k$. By condition (a) and the estimate above, we get

$$\begin{aligned}
S_2 &\leqslant C \sum_{k=-\infty}^{\infty} \Big(\int_{M_k} |f(x)|^p \Big(\sup_{z\in E_k} u(z)\Big) \mathrm{d}x\Big)^{\frac{q}{p}} \\
&\leqslant CA^q\|b\|_*^{mq} \sum_{k=-\infty}^{\infty} \Big(\int_{M_k} |f(x)|^p \Big(\sup_{z\in E_k} \big(u(z) \sup_{\{y:\, 4^{-1}|z|<|y|<4|z|\}} \frac{1}{v(y)}\big)\Big) v(x) \mathrm{d}x\Big)^{\frac{q}{p}} \\
&\leqslant CA^q\|b\|_*^{mq} \sum_{k=-\infty}^{\infty} \Big(\int_{M_k} |f(x)|^p v(x) \mathrm{d}x\Big)^{\frac{q}{p}} \\
&= CA^q\|b\|_*^{mq} \Big(\int_{\mathbb{R}^n} |f(x)|^p v(x) \mathrm{d}x\Big)^{\frac{q}{p}}.
\end{aligned}$$

Applying all the estimates for S_1, S_2, S_3 implies the result. □

The proof of Theorem 1.3. By Theorem 1.2, we only need to prove that if u, v satisfy (c), then $(u, v, b) \in \mathcal{A}(j, k) \cap \mathcal{A}^*(j, k)$ for $j \leqslant k-2$ and any $b \in \mathrm{BMO}$.

Let $b \in \mathrm{BMO}$ and $1 \leqslant r < \infty$. Then there exist constants C_1, $C_2 > 0$ such that for $j \in \mathbb{Z}$,

$$\int_{E_j} |b(x) - b_j|^r \mathrm{d}x \leqslant 2^{(j+1)n} \frac{1}{|B_{j+1}|} \int_{B_{j+1}} |b(x) - b_j|^r \mathrm{d}x \leqslant C_1 \|b\|_*^r \times 2^{jn}, \tag{2.4}$$

$$|b_j - b_{j-1}| \leqslant \frac{1}{|B_{j-1}|} \int_{B_{j-1}} |b(x) - b_j| \mathrm{d}x \leqslant C \frac{1}{|B_j|} \int_{B_j} |b(x) - b_j| \mathrm{d}x \leqslant C\|b\|_*. \tag{2.5}$$

Let $j,\ k$ be integers with $j \leqslant k-2$. By (2.4), the Hölder inequality and (2.5),

$$\begin{aligned}\int_{E_k} |b(x)-b_j|^r \mathrm{d}x \leqslant & C\Big(\int_{E_k} |b(x)-b_k|^r \mathrm{d}x + |b_k-b_j|^r\Big)\\ \leqslant & C\|b\|_*^r \times 2^{kn} + C(k-j+1)^{r-1}\sum_{i=j+1}^{k} |b_i-b_{i-1}|^r\\ \leqslant & C\|b\|_*^r \times 2^{kn} + C(k-j)^r\|b\|_*^r\\ \leqslant & C\|b\|_*^r (k-j)^r 2^{kn}. \end{aligned} \tag{2.6}$$

By the condition (c), (2.4) and (2.6),

$$\begin{aligned} A(j,k) \leqslant & C2^{-k(n-\alpha)}\Big[U_k^{\frac{1}{q}}\big(2^{jn}V_j\big)^{\frac{1}{p'}}\Big(\int_{E_k}|b(x)-b_j|^{mq}\,\mathrm{d}x\Big)^{\frac{1}{q}}\\ & + V_j^{\frac{1}{p'}}\big(2^{kn}U_k\big)^{\frac{1}{q}}\Big(\int_{E_j}|b(z)-b_j|^{mp'}\,\mathrm{d}z\Big)^{\frac{1}{p'}}\Big]\\ \leqslant & CA\|b\|_*^m 2^{-k(n-\alpha)}\Big(\big(k-j\big)^m 2^{\frac{kn}{q}+\frac{jn}{p'}+(j-k)na} + 2^{\frac{jn}{p'}+\frac{kn}{q}+(j-k)na}\Big)\\ \leq & CA\|b\|_*^m (k-j)^m 2^{(j-k)n(a+\frac{1}{p'})}. \end{aligned}$$

Therefore, $(u,v,b)\in\mathcal{A}(j,k)$ with $\eta=m$ and $\rho=\frac{1}{p'}+a$.

Similarly, by condition (c), (2.4) and (2.6), we have

$$\begin{aligned} A^*(j,k) \leqslant & 2^{-k(n-\alpha)}\Big[\Big(2^{jn}U_j\Big)^{\frac{1}{q}}V_k^{\frac{1}{p'}}\Big(\int_{E_k}|b(z)-b_j|^{mp'}\,\mathrm{d}z\Big)^{\frac{1}{p'}}\\ & + \Big(2^{kn}V_k\Big)^{\frac{1}{p'}}\Big(U_j\Big)^{\frac{1}{q}}\Big(\int_{E_j}|b(x)-b_j|^{mq}\,\mathrm{d}x\Big)^{\frac{1}{q}}\Big]\\ \leqslant & CA\|b\|_*^m 2^{-k(n-\alpha)}\Big(\big(k-j\big)^m 2^{\frac{kn}{p'}+\frac{jn}{q}+(j-k)na^*} + 2^{\frac{kn}{p'}+\frac{jn}{q}+(j-k)na^*}\Big)\\ = & CA\|b\|_*^m (k-j)^m 2^{(j-k)n(\frac{1}{q}+a^*)}. \end{aligned}$$

Thus, $(u,v,b)\in\mathcal{A}^*(j,k)$ with $\eta=m$ and $\tau=\frac{1}{q}+a^*$. □

The proof of Corollary 1.4. By Theorem 1.3, it is sufficient to check conditions (a′) and (c) with $v(x)=Mu(x)$.

Since for each $x\in\mathbb{R}^n$ and some integer $N\geqslant 3$,

$$\begin{aligned}\frac{1}{|x|^n}\int_{\frac{1}{2^N}|x|<|z|<2^N|x|} u(z)\mathrm{d}z \leqslant & C\frac{1}{(2^{N+1}|x|)^n}\int_{|x-z|<2^{N+1}|x|} u(z)\mathrm{d}z\\ \leqslant & C\,Mu(x) = C\,v(x),\end{aligned}$$

then by $u\in\mathcal{H}$,

$$\begin{aligned}&\Big(\sup_{\frac{1}{4}|x|<|y|<4|x|} u(y)\Big)\times\frac{1}{v(x)}\\ &\leqslant C\Big(\frac{1}{|x|^n}\int_{\frac{1}{2^N}|x|<|z|<2^N|x|} u(z)\mathrm{d}z\Big)\Big(\frac{1}{|x|^n}\int_{\frac{1}{2^N}|x|<|z|<2^N|x|} u(z)\mathrm{d}z\Big)^{-1} = C.\end{aligned}$$

Therefore, the condition (a′) is satisfied.

Let j, k be integers with $j \leqslant k-2$. For any $y \in E_j$, by (b), $u \in \mathcal{H}$ and $N \geqslant 3$,

$$\begin{aligned}
U_k \leqslant & C U_k^{\frac{q}{p}} \leqslant C\Big(\sup_{\{x:\frac{1}{4}\cdot 2^k<|x|<4\cdot 2^k\}} u(x)\Big)^{\frac{q}{p}} \\
\leqslant & C\Big(\frac{1}{2^{kn}}\int_{2^{-N+k}<|x|<2^{N+k}} u(x)\mathrm{d}x\Big)^{\frac{q}{p}} \\
\leqslant & C\Big(\frac{1}{(2^{N+1}2^k)^n}\int_{|y-x|<2^{N+1}2^k} u(x)\mathrm{d}x\Big)^{\frac{q}{p}} \leqslant C\big(Mu(y)\big)^{\frac{q}{p}} = Cv(y)^{\frac{q}{p}}.
\end{aligned}$$

Then

$$V_j = \sup_{y\in E_j} v(y)^{1-p'} \leqslant C\, U_k^{\frac{p}{q}(1-p')} = C\, U_k^{-\frac{p'}{q}}.$$

Therefore

$$U_k^{\frac{1}{q}} V_j^{\frac{1}{p'}} \leqslant C\, U_k^{\frac{1}{q}} U_k^{-\frac{1}{q}} \leqslant C. \tag{2.7}$$

For any $y \in E_k$, by (b), $u \in \mathcal{H} \cap RD_\gamma$ and $N \geqslant 3$,

$$\begin{aligned}
U_j \leqslant & C\Big(\frac{1}{2^{jn}}\int_{2^{-N+j}<|x|<2^{N+j}} u(x)\mathrm{d}x\Big)^{\frac{q}{p}} \\
\leqslant & C\Big(\frac{1}{2^{jn}}\int_{B(0,\,2^{j-k}2^{N+k})} u(x)\mathrm{d}x\Big)^{\frac{q}{p}} \\
\leqslant & C\Big(2^{(j-k)n\gamma}2^{-jn}\int_{|x|<2^{N+k}} u(x)\mathrm{d}x\Big)^{\frac{q}{p}} \\
\leqslant & C\Big(2^{(j-k)n(\gamma-1)}\frac{1}{(2^{N+1}2^k)^n}\int_{|y-x|<2^{N+1}2^k} u(x)\mathrm{d}x\Big)^{\frac{q}{p}} \\
\leqslant & C\Big(2^{(j-k)n(\gamma-1)}Mu(x)\Big)^{\frac{q}{p}} = C\Big(2^{(j-k)n(\gamma-1)}v(x)\Big)^{\frac{q}{p}}.
\end{aligned}$$

Then

$$V_k = \sup_{y\in E_k} v(y)^{1-p'} \leqslant C\, 2^{(j-k)n(\gamma-1)(p'-1)} U_k^{\frac{p}{q}(1-p')} = C\, 2^{(j-k)n(\gamma-1)(p'-1)} U_k^{-\frac{p'}{q}}.$$

Thus

$$U_j^{\frac{1}{q}} V_k^{\frac{1}{p'}} \leqslant C\, 2^{(j-k)n(\gamma-1)\frac{1}{p}} U_k^{\frac{1}{q}} U_k^{-\frac{1}{q}} \leqslant C 2^{(j-k)n(\gamma-1)\frac{1}{p}}. \tag{2.8}$$

By (2.7), (2.8), condition (c) is satisfied with $a = 0$ and $a^* = \frac{\gamma-1}{p}$. □

The proof of Corollary 1.5. By Theorem 1.3, it is sufficient to check conditions (a′) and (c).

Let $x \in \mathbb{R}^n$. Since $v^{1-p'} \in \mathcal{H}$, for some integer $N_1 \geqslant 3$ we have

$$\begin{aligned}
v(x)^{-1} = & \big(v(x)^{1-p'}\big)^{\frac{1}{p'-1}} \leqslant C\Big(\sup_{\frac{1}{4}|x|<|y|<4|x|} v(y)^{1-p'}\Big)^{\frac{1}{p'-1}} \\
\leqslant & C\Big(\frac{1}{|x|^n}\int_{\frac{1}{2^{N_1}}|x|<|z|<2^{N_1}|x|} v(y)^{1-p'}\mathrm{d}y\Big)^{\frac{1}{p'-1}}.
\end{aligned}$$

Then applying $u \in \mathcal{H}$ and (1.1), we obtain that for some integer $N_2 \geqslant 3$,

$$\begin{aligned}
&\Big(\sup_{\frac{1}{4}|x|<|y|<4|x|} u(y)\Big)^{\frac{1}{p}} \times \big(\frac{1}{v(x)}\big)^{\frac{1}{p}}\\
\leqslant & C\Big(\frac{1}{|x|^n}\int_{\frac{1}{2^{N_2}}|x|<|z|<2^{N_2}|x|} u(y)\mathrm{d}y\Big)^{\frac{1}{p}}\Big(\frac{1}{|x|^n}\int_{\frac{1}{2^{N_1}}|x|<|z|<2^{N_1}|x|} v(y)^{1-p'}\mathrm{d}y\Big)^{\frac{1}{p'}}\\
\leqslant & C\Big(\frac{1}{(2^N|x|)^n}\int_{|z|<2^N|x|} u(y)\mathrm{d}y\Big)^{\frac{1}{p}}\Big(\frac{1}{(2^N|x|)^n}\int_{|z|<2^N|x|} v(y)^{1-p'}\mathrm{d}y\Big)^{\frac{1}{p'}}\\
\leqslant & C,
\end{aligned}$$

where $N = \max\{N_1,\ N_2\}$ and C depends on $N_1,\ N_2$. Therefore, the condition (a$'$) is satisfied.

Using condition (b), $u, v^{1-p'} \in \mathcal{H} \cap RD_\gamma$ and (1.1)

$$\begin{aligned}
&U_k^{\frac{1}{q}} V_j^{\frac{1}{p'}} \leqslant U_k^{\frac{1}{p}} V_j^{\frac{1}{p'}}\\
\leqslant & C\Big(\frac{1}{2^{kn}}\int_{2^{-N+k}<|x|<2^{N+k}} u(x)\mathrm{d}x\Big)^{\frac{1}{p}}\Big(\frac{1}{2^{jn}}\int_{2^{-N+j}<|x|<2^{N+j}} v(x)^{1-p'}\mathrm{d}x\Big)^{\frac{1}{p'}}\\
\leqslant & C\Big(\frac{1}{2^{kn}}\int_{|x|<2^{N+k}} u(x)\mathrm{d}x\Big)^{\frac{1}{p}}\Big(2^{(j-k)n\gamma}2^{-jn}\int_{|x|<2^{N+k}} v(x)^{1-p'}\mathrm{d}x\Big)^{\frac{1}{p'}}\\
\leqslant & C2^{(j-k)n(\gamma-1)\frac{1}{p'}}\Big(\frac{1}{2^{(N+k)n}}\int_{|x|<2^{N+k}} u(x)\mathrm{d}x\Big)^{\frac{1}{p}}\Big(\frac{1}{2^{(N+k)n}}\int_{|x|<2^{N+k}} v(x)^{1-p'}\mathrm{d}x\Big)^{\frac{1}{p'}}\\
= & C2^{(j-k)n(\gamma-1)\frac{1}{p'}}A.
\end{aligned}$$

Similarly, by condition (b), $u, v^{1-p'} \in \mathcal{H} \cap RD_\gamma$ and (1.1), we have

$$U_j^{\frac{1}{q}} V_k^{\frac{1}{p'}} \leqslant C2^{(j-k)n(\gamma-1)\frac{1}{p}}A.$$

Thus, condition (c) is satisfied with $a = (\gamma-1)\frac{1}{p'}$ and $a* = (\gamma-1)\frac{1}{p}$. □

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