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A FORMULATION OF THE JACOBI COEFFICIENTS $c_j^l(\alpha, \beta)$ VIA BELL POLYNOMIALS

STUART DAY and ALI TAHERI*

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Abstract. The Jacobi polynomials ($\mathscr{P}_k^{(\alpha,\beta)} : k \geq 0, \alpha, \beta > -1$) are deeply intertwined with the Laplacian on compact rank one symmetric spaces. They represent the *spherical* or zonal functions and as such constitute the main ingredients in describing the spectral measures and spectral projections associated with the Laplacian on these spaces. In this note we strengthen this connection by showing that a set of spectral and geometric quantities associated with Jacobi operator fully describe the Maclaurin coefficients associated with the heat and other related Schwartzian kernels and present an explicit formulation of these quantities using the Bell polynomials.

1. Maclaurin coefficients of Schwartzian kernels

Let $\mathscr{X}$ be a compact rank one symmetric space and let $-\Delta_{\mathscr{X}}$ denote the (positive) Laplace-Beltrami operator on $\mathscr{X}$. By basic spectral theory the heat kernel on $\mathscr{X}$ can be expressed by the spectral sum

$$H_t(x, y) = \sum_{k=0}^{\infty} \frac{M_k(\mathscr{X})}{\mathrm{Vol}(\mathscr{X})} \exp\left(-t\lambda_k^{\mathscr{X}}\right) \mathscr{P}_k^{(\alpha,\beta)}(\cos\theta), \qquad t > 0. \tag{1.1}$$

Here $\mathscr{P}_k^{(\alpha,\beta)}$ (with $\alpha, \beta > -1$ and $k \geq 0$) are the normalised Jacobi polynomials, $\lambda_k^{\mathscr{X}} = k(k + \alpha + \beta + 1)$ are the numerically distinct eigenvalues of $-\Delta_{\mathscr{X}}$, $M_k(\mathscr{X})$

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is multiplicity of $\lambda_k^{\mathscr{X}}$, $\mathrm{Vol}(\mathscr{X})$ is the volume of $\mathscr{X}$ and θ is the geodesic distance between x, y in $\mathscr{X}$ (see Appendix A at the end for more on Jacobi polynomials and their main properties).

On the other hand the Jacobi polynomials can be shown to satisfy a differential-spectral identity (with $k \geq 0$, $l \geq 1$) in the form

$$\frac{d^{2l}}{d\theta^{2l}} \mathscr{P}_k^{(\alpha,\beta)}(\cos\theta)\bigg|_{\theta=0} = \sum_{j=1}^{l} c_j^l(\alpha,\beta)[k(k+\alpha+\beta+1)]^j = \sum_{j=1}^{l} c_j^l(\alpha,\beta)\left[\lambda_k^{\mathscr{X}}\right]^j, \tag{1.2}$$

for suitable choice of scalars $(c_j^l(\alpha,\beta) : 1 \leq j \leq l)$ referred to hereafter as the Jacobi coefficients (see Theorem 2.2 below).

To illustrate the significance of this identity we return to the expression of the heat kernel on the rank one symmetric space $\mathscr{X}$ as given by (1.1). Now since the kernel H_t is an even function of the geodesic distance θ its Maclaurin expansion about $\theta = 0$ takes the form

$$H_t = \sum_{l=0}^{\infty} \frac{\theta^{2l}}{(2l)!} \frac{\partial^{2l}}{\partial\theta^{2l}} H_t\bigg|_{\theta=0} = \sum_{l=0}^{\infty} b_{2l}^n \frac{\theta^{2l}}{(2l)!}, \tag{1.3}$$

where $b_{2l}^n = b_{2l}^n(t)$ $(l \geq 0)$ denote the associated Maclaurin coefficients. Upon invoking the Jacobi coefficients $c_j^l(\alpha,\beta)$ the Maclaurin coefficients b_{2l}^n can now be given the trace formulation

$$b_{2l}^n(t) = \sum_{k=0}^{\infty} \frac{M_k(\mathscr{X}) e^{-t\lambda_k^{\mathscr{X}}}}{\mathrm{Vol}(\mathscr{X})} \sum_{j=1}^{l} c_j^l [\lambda_k^{\mathscr{X}}]^j = \frac{1}{\mathrm{Vol}(\mathscr{X})} \mathrm{tr}\left\{\mathscr{R}_l(-\Delta_{\mathscr{X}}) e^{t\Delta_{\mathscr{X}}}\right\}, \tag{1.4}$$

where $\mathscr{R}_l$ denotes the degree l polynomial in X built out of the Jacobi coefficients by the prescription

$$\mathscr{R}_l(X) = \sum_{j=1}^{l} c_j^l(\alpha,\beta) X^j. \tag{1.5}$$

We remark that this formulation does not restrict to the heat kernel only and one can go beyond, e.g., by taking any suitable function $\Phi = \Phi(X)$ within the functional calculus of $-\Delta_{\mathscr{X}}$; then the Schwartzian kernel of $\Phi(-\Delta_{\mathscr{X}})$ has the Maclaurin expansion

$$K_\Phi(x,y) = \sum_{k=0}^{\infty} \frac{M_k(\mathscr{X})}{\mathrm{Vol}(\mathscr{X})} \Phi(\lambda_k^{\mathscr{X}}) \mathscr{P}_k^{(\alpha,\beta)}(\cos\theta) = \sum_{l=0}^{\infty} b_{2l}^n \frac{\theta^{2l}}{(2l)!}, \tag{1.6}$$

where the associated Maclaurin coefficients in this case are given for $l \geq 0$ (upon agreeing to set $\mathscr{R}_0(X) = 1$) by

$$b_{2l}^n(\Phi) = \frac{1}{\mathrm{Vol}(\mathscr{X})} \mathrm{tr}\,[\mathscr{R}_l \Phi](-\Delta_{\mathscr{X}}). \tag{1.7}$$

A particular class of such Schwartzian kernels K_Φ that directly connect to the heat kernel H_t are those associated with a function $\Phi = \Phi(X)$ of the Laplace transform type, that is,

$$\Phi(X) = \int_0^\infty f(s) e^{-sX}\, ds, \qquad X \geq 0, \tag{1.8}$$

for some suitable integrable function f on $(0, \infty)$. In this case it is not hard to see that the Maclaurin coefficients of K_Φ can be expressed via the trace formula

$$b_{2l}^n = \operatorname{tr}\left[F_l(-\Delta)\right] \tag{1.9}$$

where F_l is in turn the function defined by the integral

$$F_l(X) := \int_0^\infty f(s) \mathscr{R}_l\left(-\frac{d}{ds}\right) e^{-sX}\, ds. \tag{1.10}$$

In this note we give an explicit description of the Jacobi coefficients $c_j^l(\alpha, \beta)$ by utilising the Bell polynomials and the Faà di Bruno's formula. The formulation is stated and proved in Theorem 2.2. We also explicitly give the first few coefficients and the associated polynomials $\mathscr{R}_l$ in the sequence (*see* Section 2 and Appendix B). Before ending this introduction let us note that the compact rank one symmetric spaces of interest are the sphere, the real, complex and quaternionic projective spaces and the Cayley projective plane, specifically, as listed:

- $\mathbb{S}^n = \mathbf{SO}(n+1)/\mathbf{SO}(n)$,
- $\mathbf{P}^n(\mathbb{R}) = \mathbb{S}^n/\{\pm\} = \mathbf{SO}(n+1)/\mathbf{O}(n)$,
- $\mathbf{P}^n(\mathbb{C}) = \mathbf{SU}(n+1)/\mathbf{S}(\mathbf{U}(n) \times \mathbf{U}(1))$ (of real dimension $2n$),
- $\mathbf{P}^n(\mathbb{H}) = \mathbf{Sp}(n+1)/(\mathbf{Sp}(n) \times \mathbf{Sp}(1))$ (of real dimension $4n$),
- $\mathbf{P}^2(\mathrm{Cay}) = \mathbf{F}_4/\mathbf{Spin}(9)$ (of real dimension 16).

For the sake of future reference we next present some of the necessary spectral geometric quantities associated with these symmetric spaces (*see* Table 1 below for the parameter values). The formulation of these in the simply connected case are given, in turn, by the radial part of the Laplacian

$$\frac{\partial^2}{\partial \theta^2} + \left(a \cot\theta + (1/2) b \cot(\theta/2)\right) \frac{\partial}{\partial \theta}; \tag{1.11}$$

the numerically distinct eigenvalues of $-\Delta_{\mathscr{X}}$ by $\lambda_k^{\mathscr{X}} = (\varrho + k)^2 - \varrho^2$ (with $k \geq 0$) where $\varrho = (a + b/2)/2$; the multiplicity of the eigenvalue $\lambda_k^{\mathscr{X}}$ (with $k \geq 0$ and $N = a + b + 1$) by the function

$$M_k(\mathscr{X}) = \frac{2(k+\varrho)\Gamma(k+2\varrho)\Gamma\left((a+1)/2\right)\Gamma\left(k+N/2\right)}{k!\Gamma(2\varrho+1)\Gamma\left(N/2\right)\Gamma\left(k+(a+1)/2\right)}; \tag{1.12}$$

and the volume,

$$\mathrm{Vol}(\mathscr{X}) = 2^N \frac{\pi^{N/2}\Gamma\left((a+1)/2\right)}{\Gamma\left((N+a+1)/2\right)}. \tag{1.13}$$

In the non simply connected case $\mathbf{P}^n(\mathbb{R})$ the counterparts of these are obtained using standard arguments from those of its double cover $\mathbb{S}^n$. In Table 1 below we

All these spaces are simply connected except for the circle $\mathbb{S}^1$ and the real projective spaces $\mathbf{P}^n(\mathbb{R})$ (with $n \geq 1$). Indeed $\pi_1(\mathbb{S}^1) \cong \pi_1(\mathbf{P}^1(\mathbb{R})) \cong \mathbb{Z}$ while $\pi_1(\mathbf{P}^n(\mathbb{R})) \cong \mathbb{Z}_2$ for $n \geq 2$. See [1, 2, 4, 6, 12, 13] for related and further discussion as well as [9, 10] for spectral zetas and determinants of Laplacians.

TABLE 1. The Parameters a, b, N, α, β and $\lambda_k^{\mathscr{X}}$

$\mathscr{X}$	a	b	N	α	β	$\lambda_k^{\mathscr{X}}$
$\mathbb{S}^n$	$n-1$	0	n	$(n-2)/2$	$(n-2)/2$	$k(n+k-1)$
$\mathbf{P}^n(\mathbb{R})$	$n-1$	0	n	$(n-2)/2$	$(n-2)/2$	$2k(n+2k-1)$
$\mathbf{P}^n(\mathbb{C})$	1	$2(n-1)$	$2n$	$n-1$	0	$k(n+k)$
$\mathbf{P}^n(\mathbb{H})$	3	$4(n-1)$	$4n$	$2n-1$	1	$k(2n+k+1)$
$\mathbf{P}^2(\mathrm{Cay})$	7	8	16	7	3	$k(k+11)$

gather together the values of the parameters a, b, N, α and β for the symmetric spaces described above. Note that here N is the real dimension of $\mathscr{X}$ while $\alpha = (N-2)/2$ and $\beta = (a-1)/2$. See, e.g., [1, 4, 6, 13] and the references therein for background and more.

2. A DESCRIPTION OF THE JACOBI COEFFICIENTS $c_j^l(\alpha,\beta)$ FOR $\alpha,\beta > -1$ AND $1 \le j \le l$

Here we give an explicit formulation of the Jacobi coefficients as appearing in (1.2). In order to do so we will make use of the Bell polynomials (*cf.* [3]). Recall that for positive integers m, k the Bell polynomials $B_{m,k}$ is defined as

$$B_{m,k}(\xi) = \sum \frac{m!}{j_1! \dots j_{(m-k+1)}!} \left(\frac{\xi_1}{1!}\right)^{j_1} \cdots \left(\frac{\xi_{m-k+1}}{(m-k+1)!}\right)^{j_{m-k+1}} \tag{2.1}$$

with $\xi = (\xi_1, \dots, \xi_{m-k+1})$ where the summation on the right is taken over all $j_1, \dots, j_{m-k+1}$ such that

$$\sum_{p=1}^{m-k+1} j_p = k, \qquad \sum_{p=1}^{m-k+1} p j_p = m. \tag{2.2}$$

The coefficients of the Bell polynomials relate to the number of ways a given set can be partitioned and thus have many applications in combinatorics (*cf.* [3] for more). We will make use of the Bell polynomials via Faà di Bruno's formula, which is a higher order version of the chain rule and asserts that for two smooth real-valued functions f, g on the line we have

$$\frac{d^m}{dX^m} f(g(X)) = \sum_{k=1}^{m} f^{(k)}(g(X)) B_{m,k}(g^1, \dots, g^{(m-k+1)})(X). \tag{2.3}$$

The following observation simplifies the application of Faà di Bruno's formula.

Lemma 2.1. *Let $l \ge 1$, then $B_{2l,k}(0, \xi_2, \xi_3, \dots, \xi_{2l-k+1}) = 0$ when $k \ge l+1$ for all $\xi_i \in \mathbb{R}$.*

Proof. It suffices to show that all terms in the polynomial $B_{2l,k}$ depend on the first variable. This amounts to showing that if $k \ge l+1$, $(j_p : 1 \le p \le 2l-k+1)$

satisfy (2.2) with $m = 2l$ then $j_1 \neq 0$. Indeed let $(j_p : 1 \leq p \leq 2l - k + 1)$ be non-negative integers such that (2.2) are satisfied but $j_1 = 0$. Then

$$\sum_{p=2}^{2l-k+1} j_p = k \geq l + 1. \tag{2.4}$$

On the other hand because of the second equation in (2.2) being true we have

$$\sum_{p=2}^{2l-k+1} p j_p = \sum_{p=2}^{2l-k+1} (p-2) j_p + 2 \sum_{p=2}^{2l-k+1} j_p \geq 2(l+1) > 2l. \tag{2.5}$$

which is a contradiction. □

We are now in a position to state the main result in this section that gives a computable and explicit expression for the Jacobi coefficients $c_j^l(\alpha, \beta)$.

Theorem 2.2. *Consider the Jacobi polynomial* $\mathscr{P}_k^{(\alpha,\beta)}$ *with integer* $k \geq 0$ *and real* $\alpha, \beta > -1$. *Then we have*

$$\frac{d^{2l}}{d\theta^{2l}} \mathscr{P}_k^{(\alpha,\beta)}(\cos\theta)\Big|_{\theta=0} = \sum_{j=1}^{l} c_j^l(\alpha, \beta)[k(k + \alpha + \beta + 1)]^j, \qquad l \geq 1. \tag{2.6}$$

Moreover the scalars $c_j^l(\alpha, \beta)$ *with* $1 \leq j \leq l$ *are given explicitly by the formula*

$$c_j^l(\alpha, \beta) = \sum_{m=j}^{l} \mathsf{a}_m^l \mathsf{b}_j^m, \tag{2.7}$$

where b_j^m *are defined recursively as:* $\mathsf{b}_m^m = 1$, $\mathsf{b}_1^{m+1} = -m(m + \alpha + \beta + 1)\mathsf{b}_1^m$ *for* $m \geq 1$ *and* $\mathsf{b}_j^{m+1} = \mathsf{b}_{j-1}^m - m(m + \alpha + \beta + 1)\mathsf{b}_j^m$ *for* $2 \leq j \leq m$ *while* a_m^l *are given by*

$$\mathsf{a}_m^l = \frac{2^{-m}\Gamma(\alpha + 1)}{\Gamma(\alpha + m + 1)} B_{2l,m}(0, -1, 0, +1, 0, \dots). \tag{2.8}$$

Here $B_{k,m}$ *are the partial exponential Bell polynomials as defined by* (2.1).

Proof. Let us start by justifying (2.6). Indeed upon utilising the Faà di Bruno formula we can write for any fixed $l \geq 1$

$$\frac{d^{2l}}{d\theta^{2l}} \mathscr{P}_k^{(\alpha,\beta)}(\cos\theta)\Big|_{\theta=0} = \sum_{m=1}^{2l} \frac{d^m}{dt^m} \mathscr{P}_k^{(\alpha,\beta)}(t)\Big|_{t=1} B_{2l,m}(\cos'\theta, \cos''\theta, \cdots)\Big|_{\theta=0}. \tag{2.9}$$

Now using the following differential-recursive relation satisfied by the Jacobi polynomials $(m \geq 1)$ [*see* the Appendix and in particular (3.7)]

$$\frac{d^m}{dt^m} \mathscr{P}_k^{(\alpha,\beta)}(t) = \left[\frac{2^{-m}\Gamma(k + m + \alpha + \beta + 1)\Gamma(\alpha + 1)k!}{\Gamma(k + \alpha + \beta + 1)\Gamma(\alpha + m + 1)(k - m)!}\right] \mathscr{P}_{k-m}^{(\alpha+m,\beta+m)}(t). \tag{2.10}$$

Therefore by invoking Lemma 2.1 and using (2.9) above we have

$$\begin{aligned}\frac{d^{2l}}{d\theta^{2l}}\mathscr{P}_k^{(\alpha,\beta)}(\cos\theta)\Big|_{\theta=0} &= \sum_{m=1}^{2l}\left[\frac{2^{-m}\Gamma(k+m+\alpha+\beta+1)\Gamma(\alpha+1)k!}{\Gamma(k+\alpha+\beta+1)\Gamma(\alpha+m+1)(k-m)!}\right]\times\\ &\quad\times B_{2l,m}(0,-1,0,+1,\dots)\mathscr{P}_{k-m}^{(\alpha+m,\beta+m)}(1)\\ &= \sum_{m=1}^{l}\mathsf{a}_m^l\frac{\Gamma(k+\alpha+\beta+m+1)k!}{\Gamma(k+\alpha+\beta+1)(k-m)!},\end{aligned} \tag{2.11}$$

where in deducing the second identity we have used (2.8) along with the second equation in (3.8) and (3.9). Next it is straightforward to deduce by induction that the coefficients b_j^m (with $1 \le j \le m$) satisfy the relation

$$\prod_{p=0}^{m-1}(x-p(p+\alpha+\beta+1)) = \sum_{j=1}^{m}\mathsf{b}_j^m x^j. \tag{2.12}$$

Likewise a further set of straightforward algebraic manipulations enable us to write the coefficients of a_m^l in (2.11) as

$$\begin{aligned}\frac{\Gamma(k+\alpha+\beta+m+1)k!}{\Gamma(k+\alpha+\beta+1)(k-m)!} &= \prod_{p=0}^{m-1}\big(k(k+\alpha+\beta+1)-p(p+\alpha+\beta+1)\big)\\ &= \sum_{j=1}^{m}\mathsf{b}_j^m[k(k+\alpha+\beta+1)]^j.\end{aligned} \tag{2.13}$$

Therefore by putting all the above ingredients together we arrive at the identity

$$\frac{d^{2l}}{d\theta^{2l}}\mathscr{P}_k^{(\alpha,\beta)}(\cos\theta)\Big|_{\theta=0} = \sum_{m=1}^{l}\mathsf{a}_m^l\sum_{j=1}^{m}\mathsf{b}_j^m[k(k+\alpha+\beta+1)]^j$$

which leads to the desired conclusion. □

The list below presents the first few elements in the scale of polynomials $\mathscr{R}_l$ [*cf.* (1.5)] by explicitly calculating the associated Jacobi coefficients $c_j^l(\alpha,\beta)$ (with $1 \le j \le l \le 4$) upon invoking Theorem 2.2. The cases $5 \le l \le 6$ are further discussed and presented in Appendix B at the end. Indeed for $1 \le l \le 3$ we have

$$\mathscr{R}_1(X) = \frac{-X}{2(\alpha+1)}, \qquad \mathscr{R}_2(X) = \frac{3X^2-(\alpha+3\beta+2)X}{4(\alpha+1)(\alpha+2)}, \tag{2.14}$$

$$\mathscr{R}_3(X) = \frac{-15X^3+15(\alpha+3\beta+2)X^2-(4\alpha^2+30\alpha\beta+30\beta^2+20\alpha+60\beta+24)X}{8(\alpha+1)(\alpha+2)(\alpha+3)} \tag{2.15}$$

while for $l = 4$ we can proceed by first writing

$$\mathscr{R}_4(X) = \sum_{j=1}^{4}(-1)^j\frac{Q_j^4(\alpha,\beta)}{2^4\mathscr{A}_4(\alpha)}X^j, \tag{2.16}$$

where

$$\mathscr{A}_4 = (\alpha+4)(\alpha+3)(\alpha+2)(\alpha+1), \tag{2.17}$$

and then

$$Q_1^4 = 34\alpha^3 + 462\alpha^2\beta + 1050\alpha\beta^2 + 630\beta^3 + 306\alpha^2 + \\ + 2184\alpha\beta + 2310\beta^2 + 884\alpha + 2604\beta + 816 \tag{2.18}$$

$$Q_2^4 = 147\alpha^2 + 1050\alpha\beta + 1155\beta^2 + 714\alpha + 2310\beta + 924 \tag{2.19}$$

$$Q_3^4 = 210\alpha + 630\beta + 420, \quad Q_4^4 = 105. \tag{2.20}$$

3. Appendix A: The orthogonal family of Jacobi polynomials $P_k^{(\alpha,\beta)}$ ($k \geq 0$ and $\alpha, \beta > -1$)

The purpose of this appendix is to gather together some of the main results and calculations relating to Jacobi polynomials as used earlier, the Jacobi coefficients as formulated above and the associated polynomials $\mathscr{R}_l = \mathscr{R}_l(X)$. For more information and detail on these and related scales of orthogonal polynomials the interested reader is referred to [5, 7, 8] and [11, 12].

Recall that the scale of Jacobi polynomials $P_k^{(\alpha,\beta)}(t)$ (with integer $k \geq 0$ and real $\alpha, \beta > -1$) is defined by the generating function relation

$$\frac{2^{\alpha+\beta}/R}{(1-w+R)^\alpha(1+w+R)^\beta} = \sum_{k=0}^{\infty} P_k^{(\alpha,\beta)}(t) w^k, \qquad |w| < 1, \tag{3.1}$$

where $R = \sqrt{1-2tw+w^2}$. The Jacobi polynomial $y = P_k^{(\alpha,\beta)}(t)$ satisfies the second-order differential equation

$$\left(1-t^2\right)\frac{d^2y}{dt^2} - (\alpha - \beta + (\alpha+\beta+2)t)\frac{dy}{dt} + k(k+\alpha+\beta+1)y = 0, \tag{3.2}$$

that in turn constitute a regular Sturm-Liouville system with the associated Jacobi operator a positive selfadjoint second order linear differential operator in the weighted space $L^2_{d\mu}(-1,1)$ with $d\mu = (1-t)^\alpha(1+t)^\beta dt$. The spectrum here is purely discrete and given by the sequence of eigenvalues and associated eigenfunctions

$$\lambda_k^{(\alpha,\beta)} = k(k+\alpha+\beta+1), \tag{3.3}$$

$$y = P_k^{(\alpha,\beta)}(t), \qquad k \geq 0,$$

respectively. In particular and as a consequence the Jacobi polynomials satisfy the orthogonality relations:

$$\langle P_k^{(\alpha,\beta)}, P_m^{(\alpha,\beta)} \rangle_{L^2_{d\mu}} = \int_{-1}^{1} P_k^{(\alpha,\beta)}(t) P_m^{(\alpha,\beta)}(t)\,(1-t)^\alpha(1+t)^\beta dt = 0, \tag{3.4}$$

for $0 \leq k \neq m$ whilst for the remaining cases we have

$$||P_k^{(\alpha,\beta)}||^2_{L^2_{d\mu}} = 2^{\alpha+\beta+1}\frac{(\alpha+1)_k(\beta+1)_k(\alpha+\beta+k+1)}{(\alpha+\beta+2)_k(\beta+\alpha+2k+1)}\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)k!}. \tag{3.5}$$

Note that here and below we write $(x)_k = \Gamma(x+k)/\Gamma(x)$ to denote the rising factorial. It can be shown that the Jacobi polynomials admit the power series representation

$$P_k^{(\alpha,\beta)}(t) = \frac{\Gamma(\alpha+k+1)}{\Gamma(\alpha+\beta+k+1)} \sum_{l=0}^{k} \binom{k}{l} \frac{\Gamma(\alpha+\beta+k+l+1)}{2^l \Gamma(\alpha+l+1)k!}(t-1)^l, \tag{3.6}$$

and that for $m \geq 1$ satisfy the useful differential-recursive formula

$$\frac{d^m}{dt^m} P_k^{(\alpha,\beta)}(t) = \frac{\Gamma(k+m+\alpha+\beta+1)}{2^m \Gamma(k+\alpha+\beta+1)} P_{k-m}^{(\alpha+m,\beta+m)}(t). \tag{3.7}$$

Here we also have the reflection-symmetry as well as the pointwise identities

$$P_k^{(\alpha,\beta)}(-t) = (-1)^k P_k^{(\beta,\alpha)}(t), \quad P_k^{(\alpha,\beta)}(1) = \frac{(\alpha+1)_k}{k!}. \tag{3.8}$$

The normalised form of the Jacobi polynomial as used throughout is defined as the quotient

$$\mathscr{P}_k^{(\alpha,\beta)}(t) = \frac{P_k^{(\alpha,\beta)}(t)}{P_k^{(\alpha,\beta)}(1)} = \frac{k!}{(\alpha+1)_k} P_k^{(\alpha,\beta)}(t). \tag{3.9}$$

Note that in particular $\mathscr{P}_k^{(\alpha,\beta)}(1) = 1$.

4. Appendix B: The Jacobi coefficients $c_j^l(\alpha,\beta)$ for the parameter range $5 \leq l \leq 6$

- Here we list the coefficients of the polynomial $\mathscr{R}_5(X) = \sum_{j=1}^5 c_j^5(\alpha,\beta)X^j$, computed using Theorem 2.2. Indeed for $1 \leq j \leq 5$ these can be described as

$$c_j^5(\alpha,\beta) = (-1)^j \frac{Q_j^5(\alpha,\beta)}{2^5 \mathscr{A}_5(\alpha)}, \tag{4.1}$$

where $\mathscr{A}_5 = \mathscr{A}_5(\alpha)$ is the polynomial

$$\mathscr{A}_5(\alpha) = (\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)(\alpha+5) \tag{4.2}$$

and $Q_j^5 = Q_j^5(\alpha,\beta)$ are the degree $5-j$ polynomials given respectively by

$$\begin{aligned} Q_1^5(\alpha,\beta) = 8(&62\,\alpha^4 + 1320\,\alpha^3\beta + 5040\,\alpha^2\beta^2 + 6615\,\alpha\beta^3 + 2835\,\beta^4 + \\ &+ 868\,\alpha^3 + 10800\,\alpha^2\beta + 25515\,\alpha\beta^2 + 16065\,\beta^3 + \\ &+ 4402\,\alpha^2 + 29910\,\alpha\beta + 32760\,\beta^2 + \\ &+ 9548\,\alpha + 27540\,\beta + 7440) \end{aligned}$$

$$\begin{aligned} Q_2^5(\alpha,\beta) = 2(&1185\,\alpha^3 + 14805\,\alpha^2\beta + 36225\,\alpha\beta^2 + 23625\,\beta^3 + \\ &+ 10125\alpha^2 + 75600\,\alpha\beta + 88515\,\beta^2 + \\ &+ 30810\alpha + 101430\,\beta + 32040) \end{aligned}$$

$$\begin{aligned} Q_3^5(\alpha,\beta) = &\,4095\,\alpha^2 + 28350\,\alpha\beta + 33075\,\beta^2 + \\ &+ 19530\,\alpha + 66150\,\beta + 26460 \end{aligned}$$

$$Q_4^5(\alpha,\beta) = 2(1575\,\alpha + 4725\,\beta + 3150), \qquad Q_5^5(\alpha,\beta) = 945.$$

- Likewise the coefficients of the polynomial $\mathscr{R}_6(X) = \sum_{j=1}^{6} c_j^6(\alpha,\beta)X^j$, again computed using Theorem 2.2, can be described for $1 \le j \le 6$ as
$$c_j^6(\alpha,\beta) = (-1)^j \frac{Q_j^6(\alpha,\beta)}{2^6 \mathscr{A}_6(\alpha)}, \tag{4.3}$$
where $\mathscr{A}_6 = \mathscr{A}_6(\alpha)$ is the polynomial
$$\mathscr{A}_6 = (\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)(\alpha+5)(\alpha+6) \tag{4.4}$$
and $Q_j^6 = Q_j^6(\alpha,\beta)$ are the degree $6-j$ polynomials given respectively by

$$\begin{aligned}
Q_1^6 = 8(&1382\,\alpha^5 + 42306\,\alpha^4\beta + 238425\,\alpha^3\beta^2 + 509355\,\alpha^2\beta^3 + 467775\,\alpha\beta^4 + \\
&+ 155925\,\beta^5 + 27640\,\alpha^4 + 523083\,\alpha^3\beta + 2048310\,\alpha^2\beta^2 + \\
&+ 2796255\,\alpha\beta^3 + 1247400\,\beta^4 + 214210\,\alpha^3 + 2476749\,\alpha^2\beta + \\
&+ 5980260\,\alpha\beta^2 + 3898125\,\beta^3 + 801560\,\alpha^2 + 5224362\,\alpha\beta + \\
&+ 5845950\,\beta^2 + 1442808\,\alpha + 4082760\,\beta + 995040) \\
Q_2^6 = 2(&28479\,\alpha^4 + 543510\,\alpha^3\beta + 2196810\,\alpha^2\beta^2 + 3097710\,\alpha\beta^3 + 1424115\,\beta^4 + \\
&+ 371877\,\alpha^3 + 4732695\,\alpha^2\beta + 12179475\,\alpha\beta^2 + 8347185\,\beta^3 + \\
&+ 1950036\,\alpha^2 + 14481720\,\alpha\beta + 17588340\beta^2 + \\
&+ 4660788\,\alpha + 15200460\,\beta + 4173840) \\
Q_3^6 = \ &111705\,\alpha^3 + 1320165\,\alpha^2\beta + 3378375\,\alpha\beta^2 + 2338875\,\beta^3 + \\
&+ 923670\,\alpha^2 + 7068600\,\alpha\beta + 8877330\,\beta^2 + \\
&+ 2895420\,\alpha + 10270260\,\beta + 3259080 \\
Q_4^6 = \ &107415\,\alpha^2 + 727650\,\alpha\beta + 883575\,\beta^2 + \\
&+ 505890\,\alpha + 1767150\,\beta + 706860 \\
Q_5^6 = \ &51975\,\alpha + 155925\,\beta + 103950, \qquad Q_6^6 = 10395.
\end{aligned}$$

The higher order Jacobi coefficients $c_j^l(\alpha,\beta)$ and polynomials $\mathscr{R}_l$ (with $l \ge 7$) follow a similar pattern but are naturally lengthier to calculate.

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Department of Mathematics, University of Sussex, Falmer, Brighton, UK.
E-mail address: s.day@sussex.ac.uk
E-mail address: a.taheri@sussex.ac.uk