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STABILITY OF THE COSINE-SINE FUNCTIONAL EQUATION WITH INVOLUTION

JEONGWOOK CHANG,[1] CHANG-KWON CHOI,[2*] JONGJIN KIM,[3] and PRASANNA K. SAHOO[4]

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Abstract. Let S and G be a commutative semigroup and a commutative group respectively, $\mathbb{C}$ and $\mathbb{R}^+$ the sets of complex numbers and nonnegative real numbers respectively, $\sigma : S \to S$ or $\sigma : G \to G$ an involution and $\psi : G \to \mathbb{R}^+$ be fixed. In this paper, we first investigate general solutions of the equation

$$g(x + \sigma y) = g(x)g(y) + f(x)f(y)$$

for all $x, y \in S$, where $f, g : S \to \mathbb{C}$ are unknown functions to be determined. Secondly, we consider the Hyers-Ulam stability of the equation, i.e., we study the functional inequality

$$|g(x + \sigma y) - g(x)g(y) - f(x)f(y)| \le \psi(y)$$

for all $x, y \in G$, where $f, g : G \to \mathbb{C}$.

1. Introduction

The cosine function admits the following decomposition

$$\cos(x - y) = \cos x \cos y + \sin x \sin y$$

and $g(x) = \cos x,\ f(x) = \sin x$ satisfies the functional equation

$$g(x - y) = g(x)g(y) + f(x)f(y). \tag{1.2}$$

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* Corresponding author.
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The functional equation (1.1) was treated by Gerretsen [11] and Vaughan [22] among others. The general solutions of (1.1) are described in [2, pp. 216–217] when the unknown functions g, f are functions from a group G into a field $\mathbb{F}$. For some related equations we refer the reader to [1, p. 177] and [2, pp. 209–217].

The Hyers-Ulam stability problems of functional equations go back to 1940 when S. M. Ulam proposed a question concerning the approximate homormorphisms from a group to a metric group(see [21]). A partial answer was given by D.H. Hyers[12] under the assumption that the target space of the involved mappings is a Banach space. After the result of Hyers, T. Aoki [3] and D.G. Bourgin [5] dealt with this problem, however, there were no other results on this problem until 1978 when Th.M. Rassias [17] dealt again with the inequality of Aoki [3]. Following Rassias' result a great number of papers on the subject have been published concerning numerous functional equations in various directions [4, 6, 7, 12, 14, 15, 17, 18, 20]. In particular, Székelyhidi [20] investigated the Hyers-Ulam stability of the trigonometric functional equations

$$f(x+y) = f(x)g(y) + g(x)f(y) \tag{1.3}$$

and

$$g(x+y) = g(x)g(y) - f(x)f(y) \tag{1.4}$$

for all $x, y \in G$, where $f, g : G \to \mathbb{C}$ without the commutativity of G. Using the elegant method of Székelyhidi, Chung and Chang [6, 7] obtained the Hyers-Ulam stability of functional equations

$$f(x-y) = f(x)g(y) - g(x)f(y) \tag{1.5}$$

and

$$g(x-y) = g(x)g(y) + f(x)f(y) \tag{1.6}$$

for all $x, y \in G$.

Recently, several authors [9, 16, 19] have studied functional equations with involutions which generalize previous results on some classical functional equations such as the d'Alembert's functional equation [10] and the Wilson's functional equation [23]. In particular, generalizing (1.4), Chung, Choi and Kim [8] determined the general solutions and the Hyers-Ulam stability of the functional equation

$$f(x+\sigma y) = f(x)g(y) - g(x)f(y) \tag{1.7}$$

for all $x, y \in S$, where $f, g : S \to \mathbb{C}$ and σ is an involution on the semigroup S. In this paper, generalizing the functional equation (1.5), we first determine all general solutions of the functional equation

$$g(x+\sigma y) = g(x)g(y) + f(x)f(y) \tag{1.8}$$

for all $x, y \in S$. Secondly, generalizing the results in [6, 7] we prove the Hyers-Ulam stability for (1.7), i.e., we study the behavior of functions g and f satisfying functional inequality

$$|g(x+\sigma y) - g(x)g(y) - f(x)f(y)| \le \psi(y) \tag{1.9}$$

for all $x, y \in G$, where $f, g : G \to \mathbb{C}$ and $\psi : G \to \mathbb{R}^+$ (the set of nonnegative real numbers).

2. General solutions of the functional equation (1.7)

In this section we present the general solutions (g, f) of the functional equations (1.7) on semigroups. Throughout this section we denote by S a commutative semigroup with an identity element. A function $\sigma : S \to S$ is said to be *an involution* if $\sigma(x+y) = \sigma(x)+\sigma(y)$ for all $x, y \in S$ and $\sigma(\sigma(x)) = x$ for all $x \in S$. For simplicity we write σx instead of $\sigma(x)$. A function $m : S \to \mathbb{C}$ is called *an exponential function* provided that $m(x + y) = m(x)m(y)$ for all $x, y \in S$ and $a : S \to \mathbb{C}$ is called *an additive function* provided that $a(x + y) = a(x) + a(y)$ for all $x, y \in S$.

As a direct consequence of a theorem of Sinopoulos [19] we have the following lemma.

Lemma 2.1. *Let $g : S \to \mathbb{C}$ satisfy the functional equation*

$$g(x + y) + g(x + \sigma y) = 2g(x)g(y) \tag{2.1}$$

for all $x, y \in S$. Then there exists an exponential function $m : S \to \mathbb{C}$ such that

$$g(x) = \frac{m(x) + m(\sigma x)}{2} \tag{2.2}$$

for all $x \in S$.

In the following, we exclude the trivial cases when $f(x) = g(x) \equiv 0$.

Theorem 2.2. *Let $f, g : S \to \mathbb{C}$ satisfy the functional equation*

$$g(x + \sigma y) = g(x)g(y) + f(x)f(y) \tag{2.3}$$

for all $x, y \in S$. Then either (g, f) has the form

$$g(x) = \frac{m(x) + m(\sigma x)}{2}, \; f(x) = c_1\frac{m(x) - m(\sigma x)}{2} \tag{2.4}$$

for all $x \in S$, where $m : S \to \mathbb{C}$ is an arbitrary exponential function and $c_1 \in \mathbb{C}$ with $c_1^2 = -1$, or

$$\begin{cases} g(x) = c_2E(x) \\ f(x) = c_3E(x) \end{cases} \quad \text{or} \quad \begin{cases} g(x) = \mu(x)(1 - a(x)) \\ f(x) = c_1\mu(x)a(x) \end{cases}$$

$$\text{or} \quad \begin{cases} g(x) = (1 - c_2)\mu(x) + c_2\nu(x) \\ f(x) = c_3(\mu(x) - \nu(x)) \end{cases} \tag{2.6}$$

for all $x, y \in S$, where $E, \mu : S \to \mathbb{C}$ are exponential functions satisfying $E \circ \sigma = E$, $\mu \circ \sigma = \mu$ and a, ν are an additive function and an exponential function on $S^* := \{x \in S : \mu(x) \neq 0\}$ satisfying $a \circ \sigma = a$ on S^* with arbitrary values on $S \setminus S^*$, $\nu \circ \sigma = \nu$ on S^* and $\nu = 0$ on $S \setminus S^*$, and $c_1, c_2, c_3 \in \mathbb{C}$ are arbitrary constants satisfying $c_1^2 = -1$, $c_2^2 + c_3^2 = c_2$ with $c_2 \neq 0, 1$.

Proof. Replacing (x, y) by (y, x) in (2.3) we have

$$g(y + \sigma x) - g(y)g(x) - f(x)f(y) = 0 \tag{2.7}$$

for all $x, y \in S$. Subtracting (2.6) from (2.3) we have

$$g(y + \sigma x) = g(x + \sigma y) \tag{2.8}$$

for all $x, y \in S$. Putting $y = 0$ in (2.7) we have

$$g(\sigma x) = g(x) \tag{2.9}$$

for all $x \in S$. Replacing x by σx and y by σy in (2.3) we have

$$g(\sigma x + y) - g(\sigma x)g(\sigma y) - f(\sigma x)f(\sigma y) = 0 \tag{2.10}$$

for all $x, y \in S$. From (2.8) we have

$$g(\sigma x + y) - g(x)g(y) - f(\sigma x)f(\sigma y) = 0 \tag{2.11}$$

for all $x, y \in S$. Subtracting (2.10) from (2.3) and by (2.7) we have

$$f(\sigma x)f(\sigma y) = f(x)f(y) \tag{2.12}$$

for all $x, y \in S$. Letting $y = x$ in (2.11), for each $x \in S$ we have $f(\sigma x) = f(x)$ or $f(\sigma x) = -f(x)$. Assume that there exists a $y_0 \in S$ such that $f(\sigma y_0) \neq f(y_0)$. Then we have $f(\sigma y_0) = -f(y_0)$. Putting $y = y_0$ in (2.11) we obtain $f(x) = -f(\sigma x)$ for all $x \in S$. Thus we have

$$f(\sigma x) = -f(x) \tag{2.13}$$

for all $x \in S$, or

$$f(\sigma x) = f(x) \tag{2.14}$$

for all $x \in S$.

Case (i). Suppose that (2.12) holds. Interchanging y with σy in (2.3) and using the fact that $g(\sigma y) = g(y)$ and $f(\sigma y) = -f(y)$ we have

$$g(x + y) = g(x)g(y) - f(x)f(y) \tag{2.15}$$

for all $x, y \in S$. Adding (2.3) and (2.14) we obtain

$$g(x + y) + g(x + \sigma y) = 2g(x)g(y) \tag{2.16}$$

for all $x, y \in S$. The general solution of the functional equation can be obtained from Lemma 2.1 as

$$g(x) = \frac{m(x) + m(\sigma x)}{2} \tag{2.17}$$

for all $x \in S$, where $m : S \to \mathbb{C}$ is an exponential function. Using (2.16) in (2.14) and simplifying we obtain

$$f(x)f(y) = -\left(\frac{m(x) - m(\sigma x)}{2}\right)\left(\frac{m(y) - m(\sigma y)}{2}\right) \tag{2.18}$$

for all $x, y \in S$. Hence

$$f(x) = c_1 \frac{m(x) - m(\sigma x)}{2} \tag{2.19}$$

for all $x \in S$, where $c_1 \in \mathbb{C}$ such that $c_1^2 = -1$. Hence for this case we have the asserted solution (2.4).

Case (ii). Suppose (2.13) holds. Letting σy for y in (2.3) and using the fact $g(\sigma y) = g(y)$, $f(\sigma y) = f(y)$ we get

$$g(x+y) = g(x)g(y) + f(x)f(y) \tag{2.20}$$

for all $x, y \in S$. Computing $g(x+y+z)$ first as $g(x+(y+z))$ and then as $g((x+y)+z)$, using (2.19) we obtain

$$\begin{aligned} g(x+y+z) &= g(x)g(y+z) + f(x)f(y+z) \\ &= g(x)[g(y)g(z) + f(y)f(z)] + f(x)f(y+z) \\ &= g(x)g(y)g(z) + g(x)f(y)f(z) + f(x)f(y+z), \end{aligned}$$

$$\begin{aligned} g(x+y+z) &= g(x+y)g(z) + f(x+y)f(z) \\ &= [g(x)g(y) + f(x)f(y)]g(z) + f(x+y)f(z) \\ &= g(x)g(y)g(z) + f(x)f(y)g(z) + f(x+y)f(z) \end{aligned}$$

for all $x, y, z \in S$. Comparing the last two expressions we have

$$f(x+y)f(z) - g(x)f(y)f(z) = f(x)f(y+z) - f(x)f(y)g(z) \tag{2.21}$$

for all $x, y, z \in S$. Subtracting $f(x)g(y)f(z)$ from both sides of (2.20), we get

$$[f(x+y) - g(x)f(y) - g(y)f(x)]f(z) = [f(y+z) - g(y)f(z) - g(z)f(y)]f(x) \tag{2.22}$$

for all $x, y, z \in S$. We fix $z = z_0$ with $f(z_0) \neq 0$ and obtain

$$f(x+y) - g(x)f(y) - g(y)f(x) = f(x)k(y) \tag{2.23}$$

for all $x, y \in S$, where $k(y) := f(z_0)^{-1}[f(y+z_0) - g(y)f(z_0) - g(z_0)f(y)]$. Replacing (x, y) by (y, x) in (2.20) we see that

$$f(x)k(y) = f(y)k(x) \tag{2.24}$$

for all $x, y, z \in S$. Put $y = z_0$ we have

$$k(x) = \beta f(x) \tag{2.25}$$

for all $x \in S$, where $\beta = \frac{k_(z_0)}{f(z_0)}$. Hence (2.24) in (2.22) yields

$$f(x+y) = g(x)f(y) + g(y)f(x) + \beta f(x)f(y) \tag{2.26}$$

for all $x, y \in S$. Multiplying (2.25) by λ and adding the resulting expression to (2.19) we have

$$\begin{aligned} &g(x+y) + \lambda f(x+y) \\ &\quad = g(x)g(y) + f(x)f(y) + \lambda g(x)f(y) + \lambda f(x)g(y) + \beta\lambda f(x)f(y) \end{aligned} \tag{2.27}$$

for all $x, y \in S$. The functional equation can be written as

$$g(x+y) + \lambda f(x+y) = [g(x) + \lambda f(x)][g(y) + \lambda f(y)] \tag{2.28}$$

if and only if λ satisfies

$$\lambda^2 - \beta\lambda - 1 = 0. \tag{2.29}$$

By fixing λ to be such a constant, we get

$$g(x) + \lambda f(x) = \mu(x) \tag{2.30}$$

for all $x \in S$, where $\mu : S \to \mathbb{C}$ is an exponential map. From (2.28) it is easy to see that $\lambda \neq 0$. From (2.29) we have

$$g(x) = \mu(x) - \lambda f(x) \tag{2.31}$$

for all $x \in S$ and letting this into (2.19) and simplifying we obtain

$$\lambda f(x+y) = \lambda f(x)\mu(y) + \lambda f(y)\mu(x) - (\lambda^2+1)f(x)f(y) \tag{2.32}$$

for all $x, y \in S$. There are two possibilities: (1) $\mu = 0$ and (2) $\mu \neq 0$. If $\mu = 0$, then from (2.31), we get

$$f(x+y) = -\lambda^{-1}(\lambda^2+1)f(x)f(y) \tag{2.33}$$

for all $x, y \in S$. We define $E : S \to \mathbb{C}$ given by

$$E(x) = -\lambda^{-1}(\lambda^2+1)f(x) \tag{2.34}$$

for all $x \in S$. Then by (2.33), the equation (2.34) reduces to

$$E(x+y) = E(x)E(y) \tag{2.35}$$

for all $x, y \in S$. From (2.13), (2.33) and (2.34), $E : S \to \mathbb{C}$ is an exponential function satisfying $E(x) = E(\sigma x)$.

Hence from (2.30) and (2.33) we get

$$g(x) = c_2 E(x) \quad \textit{and} \quad f(x) = c_3 E(x) \tag{2.36}$$

for all $x \in S$, where $c_2 := \frac{\lambda^2}{\lambda^2+1}$ and $c_3 := -\frac{\lambda}{\lambda^2+1}$ with $\lambda \neq 0$. Note that the constants (c_2, c_3) represents all solutions of the equation $c_2^2 + c_3^2 = c_2$ such that $c_2 \neq 0, 1$. Thus we have the first case of the asserted solutions (2.5).

The other possibility is $\mu \neq 0$. Let $S^* = \{x \in S : \mu(x) \neq 0\}$. Then $S \setminus S^*$ is an ideal in S and S^* is a subsemigroup of S. Dividing (2.31) by

$$\mu(x+y) = \mu(x)\mu(y) \tag{2.37}$$

side by side, we obtain

$$\frac{\lambda f(x+y)}{\mu(x+y)} = \frac{\lambda f(x)}{\mu(x)} + \frac{\lambda f(y)}{\mu(y)} - \frac{\lambda^2+1}{\lambda^2}\left(\frac{\lambda f(x)}{\mu(x)}\right)\left(\frac{\lambda f(y)}{\mu(y)}\right) \tag{2.38}$$

for all $x, y \in S^*$. When $\lambda^2 + 1 = 0$, we have

$$\frac{\lambda f(x+y)}{\mu(x+y)} = \frac{\lambda f(x)}{\mu(x)} + \frac{\lambda f(y)}{\mu(y)} \tag{2.39}$$

for all $x, y \in S^*$. Hence

$$\frac{\lambda f(x)}{\mu(x)} = a(x) \tag{2.40}$$

for all $x \in S^*$, where $a : S^* \to \mathbb{C}$ is an additive function. Therefore

$$f(x) = \lambda^{-1}\mu(x)a(x) \tag{2.41}$$

for all $x \in S^*$ and by (2.30) and (2.40), we get

$$g(x) = \mu(x) - \mu(x)a(x) \tag{2.42}$$

for all $x \in S^*$. Letting $c_1 = \lambda^{-1}$ from (2.40) and (2.41) we have the second case of the asserted solutions (2.5). It is easy to check that the constant c_1 satisfies $c_1^2 = -1$ because of $\lambda^2 + 1 = 0$.

When $\lambda^2 + 1 \neq 0$, (2.37) yields

$$\nu(x + y) = \nu(x)\nu(y) \tag{2.43}$$

for all $x, y \in S^*$, where $\nu(x) = 1 - \frac{\lambda^2+1}{\lambda^2}\left(\frac{\lambda f(x)}{\mu(x)}\right)$. Hence $\nu : S^* \to \mathbb{C}$ is an exponential function. Therefore

$$f(x) = c_3\mu(x) - c_3\nu(x)\mu(x) \tag{2.44}$$

for all $x \in S^*$ and by (2.30) and (2.43), we get

$$g(x) = (1 - c_2)\mu(x) + c_2\nu(x)\mu(x) \tag{2.45}$$

for all $x \in S^*$, where $c_2 := \frac{\lambda^2}{\lambda^2+1}$ and $c_3 := -\frac{\lambda}{\lambda^2+1}$. Replacing $\mu(x)\nu(x)$ by $\nu(x)$ in (2.43) and (2.44) we get the third case of the asserted solutions (2.5). It follows from (2.30), (2.33), (2.39) and (2.43) that $\mu(\sigma x) = \mu(x), E(\sigma x) = E(x), a(\sigma x) = a(x), \nu(\sigma x) = \nu(x)$ and the proof of the theorem is now complete. $\square$

Remark 2.3. Let $\sigma = I$ be the identity involution. Then as a direct consequence of Theorem 2.2 we obtain the solutions of hyperbolic cosine-sine functional equation

$$g(x + y) = g(x)g(y) + f(x)f(y) \tag{2.46}$$

for all $x, y \in S$. Indeed, all solutions of (2.45) are given by (2.5) with exponential functions $E, \mu : S \to \mathbb{C}$, $\nu : S^* \to \mathbb{C}$, an additive function $a : S^* \to \mathbb{C}$, and constants $c_2, c_3 \in \mathbb{C}$ satisfying $c_2^2 + c_3^2 = c_2$, $c_2 \neq 0$.

Let $(H, +)$ be a commutative semigroup and $f, g : H \times H \to \mathbb{C}$. As a consequence of Theorem 2.2, we determine all general solutions of the functional equation

$$g(x_1 + y_2, x_2 + y_1) = g(x_1, x_2)g(y_1, y_2) + f(x_1, x_2)f(y_1, y_2) \tag{2.47}$$

for all $x_1, x_2, y_1, y_2 \in H$. We exclude the trivial cases when g is constant.

Letting $\sigma(x_1, x_2) = (x_2, x_1)$ for all $x_1, x_2 \in H$ and using the same argument as in [8, Theorem 5] we obtain the following.

Corollary 2.4. *Let $f, g : H \times H \to \mathbb{C}$ satisfy the functional equation* (2.46). *Then either (g, f) has the form*

$$g(x_1, x_2) = \frac{m_1(x_1)m_2(x_2) + m_2(x_1)m_1(x_2)}{2},$$
$$f(x_1, x_2) = c_1\frac{m_1(x_1)m_2(x_2) - m_2(x_1)m_1(x_2)}{2}$$

for all $x_1, x_2 \in H$, where $m_1, m_2 : H \to \mathbb{C}$ are arbitrary exponential functions and $c_1 \in \mathbb{C}$ with $c_1^2 = -1$, or

$$\begin{cases} g(x_1, x_2) = c_2E(x_1 + x_2) \\ f(x_1, x_2) = c_3E(x_1 + x_2) \end{cases} \quad \textit{or} \quad \begin{cases} g(x_1, x_2) = \mu(x_1 + x_2)(1 - a(x_1 + x_2)) \\ f(x_1, x_2) = c_1\mu(x_1 + x_2)a(x_1 + x_2) \end{cases}$$

$$\text{or}\ \begin{cases} g(x_1,x_2) = (1-c_2)\mu(x_1+x_2) + c_2\nu(x_1+x_2) \\ f(x_1,x_2) = c_3(\mu(x_1+x_2) - \nu(x_1+x_2)) \end{cases}$$

for all $x_1, x_2 \in H$, where $E, \mu : H \to \mathbb{C}$ are exponential functions on H, a is an additive function on $H^* := \{x \in H : \mu(x) \neq 0\}$ with arbitrary values on $H \setminus H^*$, ν is an exponential function on H^* and $\nu = 0$ on $H \setminus H^*$, and $c_1, c_2, c_3 \in \mathbb{C}$ are arbitrary constants satisfying $c_1^2 = -1,\ c_2^2 + c_3^2 = c_2$ with $c_2 \neq 0, 1$.

If $S = G$ is a commutative group and $\sigma(x) = -x$ for all $x \in G$, we have the following.

Corollary 2.5. *Let $f, g : G \to \mathbb{C}$ satisfy the functional equation*

$$g(x-y) = g(x)g(y) + f(x)f(y) \tag{2.50}$$

for all $x, y \in G$. Then either (g, f) has the form

$$g(x) = \frac{m(x)+m(-x)}{2},\ f(x) = c_1\frac{m(x)-m(-x)}{2} \tag{2.51}$$

for all $x \in G$, where $m : G \to \mathbb{C}$ is an arbitrary exponential function and $c_1 \in \mathbb{C}$ with $c_1^2 = -1$, or

$$\begin{cases} g(x) = c_2E(x) \\ f(x) = c_3E(x) \end{cases} \quad \text{or} \quad \begin{cases} g(x) = (1-c_2)\mu(x) + c_2\nu(x) \\ f(x) = c_3(\mu(x) - \nu(x)) \end{cases} \tag{2.52}$$

for all $x, y \in G$, where $E, \mu, \nu : G \to \mathbb{C}$ are exponential functions satisfying $(E(x))^2 = (\mu(x))^2 = (\nu(x))^2 = 1$ for all $x \in G$ and $c_1, c_2, c_3 \in \mathbb{C}$ are arbitrary constants satisfying $c_1^2 = -1,\ c_2^2 + c_3^2 = c_2$ with $c_2 \neq 0, 1$.

Proof. If $S = G$ is a group, then we have $G^* = \{x \in G : \mu(x) \neq 0\} = G$. Now, since the functions $a, E, \mu, \nu : G \to \mathbb{C}$ satisfy $a(-x) = a(x), E(-x) = E(x), \mu(-x) = \mu(x), \nu(-x) = \nu(x)$ for all $x \in G$, we have $a(x) = 0$ for all $x \in G$ and $(E(x))^2 = (\mu(x))^2 = (\nu(x))^2 = 1$ for all $x \in G$, and the second case of (2.5) is reduced to the case $m = m \circ \sigma$ of (2.4). This completes the proof. □

In particular if $S = G$ is a 2-divisible commutative group and $\sigma(x) = -x$ for all $x \in G$, then since $E = \mu = \nu = 1$ we have the following.

Corollary 2.6. *Let $f, g : G \to \mathbb{C}$ satisfy the functional equation*

$$g(x-y) = g(x)g(y) + f(x)f(y)$$

for all $x, y \in G$. Then either (g, f) has the form

$$g(x) = c_2, \quad f(x) = c_3$$

for all $x \in G$, where $c_2^2 + c_3^2 = c_2$, or

$$g(x) = \frac{m(x)+m(-x)}{2},\ f(x) = c_1\frac{m(x)-m(-x)}{2}$$

for all $x \in G$, where $m : G \to \mathbb{C}$ is an arbitrary exponential function and $c_1 \in \mathbb{C}$ with $c_1^2 = -1$.

Remark 2.7. If G is not 2-divisible, we can find a nonconstant solution (g, f) of (2.47) of the form (2.49). Indeed, let $G = \mathbb{Z}$ be the set of integers. Define $E : \mathbb{Z} \to \mathbb{C}$ by $E(2k) = 1$, $E(2k-1) = -1$ for all $k \in \mathbb{Z}$. Then E is a nonconstant exponential function. Letting $\mu = E$, $\nu = 1$ and $c_2^2 + c_3^2 = c_2$ we obtain the following nonconstant solutions of the form (2.49)

$$\begin{cases} g(x) = c_2 E(x) \\ f(x) = c_3 E(x) \end{cases} \quad or \quad \begin{cases} g(x) = (1-c_2)E(x) + c_2 \\ f(x) = c_3 E(x) - c_3 \end{cases}$$

3. Stability of the functional equation (1.7)

Throughout this section, let G be a commutative group, $\psi : G \to [0, \infty)$ be fixed and $f, g : G \to \mathbb{C}$. In this section we consider the stability of the functional equation (1.7), i.e., we deal with the functional inequality

$$|g(x + \sigma y) - g(x)g(y) - f(x)f(y)| \le \psi(y) \tag{3.1}$$

for all $x, y \in G$. For the proof of the stability of (3.1) we need the following.

Lemma 3.1. *Let $f, g : G \to \mathbb{C}$ satisfy the inequality* (3.1) *for all $x, y \in G$. Then there exist $\mu_1, \mu_2 \in \mathbb{C}$(not both zero) and $M > 0$ such that*

$$|\mu_1 f(x) - \mu_2 g(x)| \le M \tag{3.2}$$

for all $x \in G$, or else

$$g(x + \sigma y) - g(x)g(y) - f(x)f(y) = 0 \tag{3.3}$$

for all $x, y \in G$.

Proof. Suppose that $\mu_1 f(x) - \mu_2 g(x)$ is bounded only when $\mu_1 = \mu_2 = 0$. Let

$$l(x, y) = g(x + y) - g(x)g(\sigma y) - f(x)f(\sigma y) \tag{3.4}$$

for all $x, y \in G$. Choose y_1 satisfying $f(\sigma y_1) \ne 0$. Then from (3.4), we have

$$f(x) = \omega_1 g(x) + \omega_2 g(x + y_1) - \omega_2 l(x, y_1) \tag{3.5}$$

for all $x \in G$, where $\omega_1 = -\frac{g(\sigma y_1)}{f(\sigma y_1)}$ and $\omega_2 = \frac{1}{f(\sigma y_1)}$. From (3.4) and (3.5) we have

$$\begin{aligned} g(x + y + z) &= g(x + y)g(\sigma z) + f(x + y)f(\sigma z) + l(x + y, z) \\ &= \big(g(x)g(\sigma y) + f(x)f(\sigma y) + l(x, y)\big)g(\sigma z) \\ &\quad + \big(\omega_1 g(x + y) + \omega_2 g(x + y + y_1) - \omega_2 l(x + y, y_1)\big)f(\sigma z) \\ &\quad + l(x + y, z) \\ &= \big(g(x)g(\sigma y) + f(x)f(\sigma y) + l(x, y)\big)g(\sigma z) \\ &\quad + \omega_1\big(g(x)g(\sigma y) + f(x)f(\sigma y) + l(x, y)\big)f(\sigma z) \\ &\quad + \omega_2\big(g(x)g(\sigma(y + y_1)) + f(x)f(\sigma(y + y_1)) \\ &\quad + l(x, y + y_1)\big)f(\sigma z) - \omega_2 l(x + y, y_1)f(\sigma z) + l(x + y, z) \end{aligned} \tag{3.6}$$

for all $x, y, z \in G$. Also, from (3.4) we have

$$g(x+y+z) = g(x)g(\sigma(y+z)) + f(x)f(\sigma(y+z)) + l(x, y+z) \tag{3.7}$$

for all $x, y, z \in G$. Equating (3.6) and (3.7) and then isolating $l(\cdot,\cdot)$ terms into the right hand sides, we have

$$\begin{aligned}
&\big(g(\sigma y)g(\sigma z) + \omega_1 g(\sigma y)f(\sigma z) + \omega_2 g(\sigma(y+y_1))f(\sigma z) - g(\sigma(y+z))\big)g(x) \\
&\quad + \big(f(\sigma y)g(\sigma z) + \omega_1 f(\sigma y)f(\sigma z) + \omega_2 f(\sigma(y+y_1))f(\sigma z) - f(\sigma(y+z))\big)f(x) \\
&\qquad = -l(x,y)g(\sigma z) - \omega_1 l(x,y)f(\sigma z) - \omega_2 l(x, y+y_1)f(\sigma z) \\
&\qquad\quad + \omega_2 l(x+y, y_1)f(\sigma z) - l(x+y, z) + l(x, y+z)
\end{aligned} \tag{3.8}$$

for all $x, y, z \in G$. So the left side of (3.8) is the of the form $\mu_1(y,z)\, f(x) - \mu_2(y,z)\, g(x)$. Next we show that the right hand side of (3.8) is bounded as a function of x. Taking the absolute value of right hand sides of (3.8) and using triangle inequality and (3.1), we have

$$\begin{aligned}
&|-l(x,y)g(\sigma z) - \omega_1 l(x,y)f(\sigma z) - \omega_2 l(x, y+y_1)f(\sigma z) \\
&\qquad\qquad + \omega_2 l(x+y, y_1)f(\sigma z) - l(x+y,z) + l(x, y+z)| \\
&\quad \le |l(x,y)||g(\sigma z)| + |l(x,y)||\omega_1 f(\sigma z)| + |l(x, y+y_1)||\omega_2 f(\sigma z)| \\
&\qquad + |l(x+y, y_1)||\omega_2 f(\sigma z)| + |l(x+y, z)| + |l(x, y+z)| \\
&\quad \le \psi(\sigma y)|g(\sigma z)| + \psi(\sigma y)|\omega_1 f(\sigma z)| + \psi(\sigma(y+y_1))|\omega_2 f(\sigma z)| \\
&\qquad + \psi(\sigma y_1)|\omega_2 f(\sigma z)| + \psi(\sigma z) + \psi(\sigma(y+z))
\end{aligned} \tag{3.9}$$

for all $x, y, z \in G$. In view of (3.9), for fix y, z, the right hand side of (3.8) is bounded as a function of x. So by our assumption, the left hand side of (3.8) vanishes, so does its right hand side yielding

$$\begin{aligned}
l(x,y)\, g(\sigma z) + \big(\omega_1\, l(x,y) + \omega_2\, l(x, y+y_1) - \omega_2\, l(x+y, y_1)\big)\, f(\sigma z) \\
= l(x, y+z) - l(x+y, z)
\end{aligned} \tag{3.10}$$

for all $x, y, z \in G$. From (3.4) we can write

$$\begin{aligned}
&l(x, y+z) - l(x+y, z) \\
&= g(x+y+z) - g(x)g(\sigma(y+z)) - f(x)f(\sigma(y+z)) \\
&\quad - g(x+y+z) + g(x+y)g(\sigma z) + f(x+y)f(\sigma z) \\
&= g(\sigma(x+y+z)) - g(x)g(\sigma(y+z)) - f(x)f(\sigma(y+z)) \\
&\quad - g(\sigma(x+y+z)) + g(x+y)g(\sigma z) + f(x+y)f(\sigma z) \\
&= g(\sigma(y+z) + \sigma x) - g(\sigma(y+z))g(x) - f(\sigma(y+z))f(x) \\
&\quad - g(\sigma z + \sigma(x+y)) + g(\sigma z)g(x+y) + f(\sigma z)f(x+y) \\
&= l(\sigma(y+z), \sigma x) - l(\sigma z, \sigma(x+y))
\end{aligned} \tag{3.11}$$

for all $x, y, z \in G$. Using (3.11) and the triangle inequality we have

$$\begin{aligned}
|l(x, y+z) - l(x+y, z)| &= |l(\sigma(y+z), \sigma x) - l(\sigma z, \sigma(x+y))| \\
&\le |l(\sigma(y+z), \sigma x)| + |l(\sigma z, \sigma(x+y))| \\
&\le \psi(x) + \psi(x+y)
\end{aligned} \tag{3.12}$$

for all $x, y, z \in G$. Thus, if we fix x, y in (3.10) the left hand side of (3.10) is a bounded function of z. Hence by our assumption, we have $l(x, y) = 0$ for all $x, y \in G$. This completes the proof. $\square$

For the proof of the main result we also need the following three lemmas.

Lemma 3.2. [13] *Let $\Psi : G \to [0, \infty)$ be a function. Assume that $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x+y) - f(x)g(y)| \le \Psi(y) \tag{3.13}$$

for all $x, y \in G$, then either f is a bounded function or g is an exponential function.

Lemma 3.3. *Let $m : G \to \mathbb{C}$ be a bounded exponential function satisfying $m(y) \neq m(\sigma y)$ for some $y \in G$. Then there exists $y_0 \in G$ such that*

$$|m(y_0) - m(\sigma y_0)| \ge \sqrt{3}.$$

Furthermore, $\sqrt{3}$ is the best constant in general.

Proof. Since m is a bounded exponential, there exists $C > 0$ such that $|m(x)|^k = |m(kx)| \le C$ for all $k \in \mathbb{Z}$ and $x \in G$, which implies $|m(x)| = 1$ for all $x \in G$. Assume that $m(\sigma y) \neq m(y)$. Then we have $m(\sigma y) = e^{i\theta_1}$, $m(y) = e^{i\theta_2}$ for some $\theta_1, \theta_2 \in [0, 2\pi]$. We may assume that $\theta_1 < \theta_2$. If $\theta_2 - \theta_1 \in [\frac{2\pi}{3}, \frac{4\pi}{3}]$, we have $|m(y) - m(\sigma y)| = |e^{i\theta_2} - e^{i\theta_1}| \ge \sqrt{3}$. If $\theta_2 - \theta_1 \in [0, \frac{2\pi}{3}]$ or $\theta_2 - \theta_1 \in [\frac{4\pi}{3}, 2\pi]$, then there exists an integer k such that $k\theta_2 - k\theta_1 \in [\frac{2\pi}{3} + 2n\pi, \frac{4\pi}{3} + 2n\pi]$ for some integer n. Thus we have $|m(ky) - m(\sigma(ky))| = |m(ky) - m(k\sigma y)| = |e^{ik\theta_2} - e^{ik\theta_1}| \ge \sqrt{3}$. Now define $m : \mathbb{Z} \to \mathbb{C}$ by $m(k) = e^{\frac{ik\pi}{3}}$ and let $\sigma(x) = -x$. Then we have $|m(3k+1) - m(-3k-1)| = \sqrt{3}$ for all $k \in \mathbb{Z}$. Thus $\sqrt{3}$ is the biggest one. This completes the proof. $\square$

From now on we assume that

$$\Phi_1(x) := \sum_{k=0}^{\infty} 2^{-k-1} \psi\left(2^k x\right) < \infty \tag{3.14}$$

for all $x \in G$, or else

$$\Phi_2(x) := \sum_{k=0}^{\infty} 2^{k} \psi\left(2^{-k-1} x\right) < \infty \tag{3.15}$$

for all $x \in G$.

Lemma 3.4. [3] *Assume that $f : G \to \mathbb{C}$ satisfies the functional inequality*

$$|f(x+y) - f(x) - f(y)| \le \psi(y)$$

for all $x, y \in G$. Then there exists a unique additive function a_1 given by

$$a_1(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$$

such that

$$|f(x) - a_1(x)| \le \Phi_1(x)$$

for all $x \in G$ provided that (3.14) *holds, and there exists a unique additive function a_2 given by*

$$a_2(x) = \lim_{n\to\infty} 2^n f(2^{-n}x)$$

such that

$$|f(x) - a_2(x)| \le \Phi_2(x)$$

for all $x \in G$ provided that (3.15) *holds.*

Next we present the second main results of this paper.

Theorem 3.5. *Let $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$|g(x+\sigma y) - g(x)g(y) - f(x)f(y)| \le \psi(y) \tag{3.21}$$

for all $x, y \in G$, then (g, f) satisfies one of the following:

(i) *g and f are bounded functions,*

(ii) *f is a bounded function and $g = m$ is an unbounded exponential function such that $m = m \circ \sigma$,*

(iii) *there exist an unbounded exponential function m satisfying $m = m \circ \sigma$ and a bounded function r such that*

$$f(x) = \frac{\lambda m(x) + r(x)}{\lambda^2+1},\ g(x) = \frac{m(x) - \lambda r(x)}{\lambda^2+1}$$

for all $x \in G$,

(iv) *$g(x) = \frac{m(x)+m(\sigma x)}{2}$ and $f(x) = c_1\frac{m(x)-m(\sigma x)}{2}$, where $m : G \to \mathbb{C}$ is an exponential function,*

(v) *there is an additive function $a : G \to \mathbb{C}$ and exponential functions $E, \mu, \nu : G \to \mathbb{C}$ satisfying $a \circ \sigma = a$, $E \circ \sigma = E$, $\mu \circ \sigma = \mu$, $\nu \circ \sigma = \nu$, and $c_1, c_2, c_3 \in \mathbb{C}$ such that $c_1^2 = -1$, $c_2^2 + c_3^2 = c_2$ with $c_2 \ne 0, 1$,*

$$\begin{cases} g(x) = c_2E(x) \\ f(x) = c_3E(x) \end{cases} \quad \text{or} \quad \begin{cases} g(x) = \mu(x)(1-a(x)) \\ f(x) = c_1\mu(x)a(x) \end{cases}$$

$$\text{or} \quad \begin{cases} g(x) = (1-c_2)\mu(x) + c_2\nu(x) \\ f(x) = c_3(\mu(x) - \nu(x)) \end{cases}$$

for all $x, y \in G$,

(vi) *there exist $\lambda \in \mathbb{C}$ with $\lambda^2 = -1$, a bounded exponential function m satisfying $m \ne m \circ \sigma$ and $d \ge 0$ such that*

$$f(x) = \lambda(g(x) - m(x)), \quad |g(x)| \le \frac{2\sqrt{3}}{3}(\psi(x) + d)$$

for all $x \in G$,

(vii) *there exist $\lambda \in \mathbb{C}$ with $\lambda^2 = -1$ and a bounded exponential function m satisfying $m = m \circ \sigma$ such that*

$$f(x) = \lambda(g(x) - m(x))$$

for all $x \in G$, and g satisfies one of the following; there exists an additive function $a_1 : G \to \mathbb{C}$ such that

$$|g(x) - (a_1(x) + g(0))m(x)| \le 2\Phi_1(x)$$

for all $x \in G$*, or there exists an additive function* $a_2 : G \to \mathbb{C}$ *such that*

$$|g(x) - (a_2(x) + g(0))m(x)| \leq 2\Phi_2(x)$$

for all $x \in G$*, where* Φ_1 *and* Φ_2 *are the functions given in* (3.14) *and* (3.15) *and* $g(0) = 1$ *if* $\psi(0) = 0$*.*

Proof. In view of Lemma 3.1, we first consider the case when f, g satisfies (3.2). If f is bounded, then in view of the inequality (3.16), $g(x+y) - g(x)g(\sigma y)$ is also bounded for each y. By Lemma 3.2, g is bounded or $g \circ \sigma$ is an unbounded exponential function and so is g. If g is bounded, the case (i) follows. If g is an unbounded exponential function, say $g = m$, then from (3.16), using the triangle inequality we have for some $d \geq 0$,

$$|m(x)(m(\sigma y) - m(y))| \leq \psi(y) + d \tag{3.28}$$

for all $x, y \in G$. Thus, $m(y) = m(\sigma y)$ for all $y \in G$ is bounded, which gives the case (ii).

If f is unbounded, then in view of (3.16), g is also unbounded and we can write

$$f(x) = \lambda g(x) + r(x) \tag{3.29}$$

for all $x \in G$, where $\lambda \neq 0$ and r is a bounded function. Putting (3.18) in (3.16), replacing y by σy and using the triangle inequality we have

$$\begin{aligned} &|g(x+y) - g(x)((\lambda^2+1)g(\sigma y) + \lambda r(\sigma y))| \\ &\qquad \leq |(\lambda g(\sigma y) + r(\sigma y))r(x)| + \psi(\sigma y) \leq \psi^*(y) \end{aligned} \tag{3.30}$$

for all $x, y \in G$ and for some ψ^*. From (3.19), using Lemma 3.2 we have

$$(\lambda^2+1)g(y) + \lambda r(y) = m(y) \tag{3.31}$$

for all $y \in G$ and for some exponential function m. If $\lambda^2 \neq -1$, we have

$$f(x) = \frac{\lambda m(x) + r(x)}{\lambda^2+1}, \;\; g(x) = \frac{m(x) - \lambda r(x)}{\lambda^2+1} \tag{3.32}$$

for all $x \in G$. Putting (3.21) in (3.16), multiplying $|\lambda^2+1|$ in the result and using the triangle inequality we have for some $d \geq 0$,

$$|m(x)(m(\sigma y) - m(y))| \leq |\lambda^2+1|\psi(y) + d \tag{3.33}$$

for all $x, y \in G$. Since m is an unbounded function, from (3.22) we have $m = m \circ \sigma$. If $\lambda^2 = -1$, then from (3.18) and (3.20) we have

$$f(x) = \lambda(g(x) - m(x)) \tag{3.34}$$

for all $x \in G$, where $\lambda^2 = -1$ and m is a bounded exponential function and hence $|m(x)| = 1$ for all $x \in G$. Putting (3.23) in (3.16), we have

$$|g(x+\sigma y) - g(x)m(y) - m(x)g(y) + m(x)m(y)| \leq \psi(y) \tag{3.35}$$

for all $x, y \in G$. Since g is unbounded, we have $m \neq 0$ and hence $m(0) = 1$. Putting $x = y = 0$ in (3.24) we see that $g(0) = 1$ if $\psi(0) = 0$. Replacing y by σy in (3.24) we have

$$|g(x+y) - g(x)m(\sigma y) - m(x)g(\sigma y) + m(x)m(\sigma y)| \leq \psi(\sigma y) \tag{3.36}$$

for all $x, y \in G$. Putting $x = 0$ in (3.25) and multiplying $|m(x)|$ in the result we have

$$|m(x)g(y) - g(0)m(x)m(\sigma y) - m(x)g(\sigma y) + m(x)m(\sigma y)| \le \psi(\sigma y) \tag{3.37}$$

for all $x, y \in G$.

From (3.25) and (3.26), using the triangle inequality we have

$$|g(x+y) - g(x)m(\sigma y) - m(x)g(y) + g(0)m(x)m(\sigma y)| \le 2\psi(\sigma y) \tag{3.38}$$

for all $x, y \in G$.

First, we consider the case $m(y_0) \ne m(\sigma y_0)$ for some $y_0 \in G$. Replacing x by y and y by x in (3.27) we have

$$|g(y+x) - m(\sigma x)g(y) - g(x)m(y) + g(0)m(\sigma x)m(y)| \le 2\psi(\sigma x) \tag{3.39}$$

for all $x, y \in G$. From (3.27) and (3.28), using the triangle inequality we have

$$\begin{aligned}|g(x)(m(\sigma y) - m(y)) - g(y)(m(\sigma x) - m(x)) - g(0)(m(x)m(\sigma y) - m(\sigma x)m(y))| \\ \le 2(\psi(\sigma x) + \psi(\sigma y))\end{aligned} \tag{3.40}$$

for all $x, y \in G$. By Lemma 3.3, there exists a $y_0 \in G$ such that $|m(\sigma y_0) - m(y_0)| \ge \sqrt{3}$, putting $y = y_0$ in (3.29), using the triangle inequality and dividing the result by $|m(\sigma y_0) - m(y_0)|$ we have

$$|g(x)| \le \frac{2\sqrt{3}}{3}(\psi(\sigma x) + d) \tag{3.41}$$

for all $x \in G$, where $d = \psi(y_0) + |g(y_0)| + |g(0)|$, which gives (vi). Now, we consider the case when $m(x) = m(\sigma x)$ for all $x \in G$. Dividing both the sides (3.27) by $|m(x+y)| = |m(x)m(y)|$ we have

$$|F(x+y) - F(x) - F(y)| \le 2\psi(y) \tag{3.42}$$

for all $x, y \in G$, where $F(x) = \frac{g(x)}{m(x)} - g(0)$. Using Lemma 3.4 and multiplying $|m(x)|$ in the result we get (vii). If f, g satisfies (3.3), then by Theorem 2.2, all solutions of (3.3) are given by (iv) or (v). This completes the proof. □

Remark 3.6. Let $\sigma = I$ be the identity. Then as a direct consequence of Theorem 3.5 we obtain the Hyers-Ulam stability of the hyperbolic cosine-sine functional equation

$$g(x+y) = g(x)g(y) + f(x)f(y).$$

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[1]Department of Mathematics Education, Dankook University, Yongin 16890, Republic of Korea

E-mail address: jchang@dankook.ac.kr

[2]Department of Mathematics and Liberal Education Institute, Kunsan National University, Gunsan 54150, Republic of Korea
E-mail address: ck38@kunsan.ac.kr

[3]Department of Mathematics and Institute of Pure and Applied Mathematics, Jeonbuk National University, Jeonju 54896, Republic of Korea
E-mail address: jjkim@jbnu.ac.kr

[4]Department of Mathematics, University of Louisville, Kentucky 40292, USA
E-mail address: sahoo@louisville.edu