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L^p FOURIER TRANSFORMATION ON NON-UNIMODULAR LOCALLY COMPACT GROUPS

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ABSTRACT. Let G be a locally compact group with modular function Δ and left regular representation λ. We define the L^p Fourier transform of a function $f \in L^p(G)$, $1 \leq p \leq 2$, to be essentially the operator $\lambda(f)\Delta^{\frac{1}{q}}$ on $L^2(G)$ (where $\frac{1}{p} + \frac{1}{q} = 1$) and show that a generalized Hausdorff–Young theorem holds. To do this, we first treat in detail the spatial L^p spaces $L^p(\psi_0)$, $1 \leq p \leq \infty$, associated with the von Neumann algebra $M = \lambda(G)''$ on $L^2(G)$ and the canonical weight ψ_0 on its commutant. In particular, we discuss isometric isomorphisms of $L^2(\psi_0)$ onto $L^2(G)$ and of $L^1(\psi_0)$ onto the Fourier algebra $A(G)$. Also, we give a characterization of positive definite functions belonging to $A(G)$ among all continuous positive definite functions.

INTRODUCTION

Suppose that G is an abelian locally compact group with dual group $\hat{G}$. Then the Hausdorff–Young theorem states that if $f \in L^p(G)$, where $1 \leq p \leq 2$, then its Fourier transform $\mathcal{F}(f)$ belongs to $L^q(\hat{G})$, where $\frac{1}{p} + \frac{1}{q} = 1$ (cf. [23, p. 117]). In the case of Fourier series, i.e. when G is the circle group and $\hat{G}$ the integers, this is a classical result due to F. Hausdorff and W. H. Young. [24, p. 101]. An extension of this theorem to all unimodular locally compact groups was given by R. A. Kunze [14]. In this paper we shall treat the case of general, i.e. not necessarily unimodular, locally compact group.

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In order to describe our results, we first briefly recall those of [14]. Suppose that f is an integrable function on a unimodular group G. Then we consider the Fourier transform $\mathcal{F}(f)$ to be the operator $\lambda(f)$ of left convolution by f on $L^2(G)$. (As pointed out by Kunze [14], this point of view is justified by the fact that in the abelian case $\lambda(f)$ is unitarily equivalent to the operator on $L^2(\hat{G})$ of multiplication by the (ordinary) Fourier transform $\hat{f}$. The Fourier transformation maps $L^1(G)$ into the space $L^\infty(G')$, defined as the von Neumann algebra M generated by $\lambda(L^1(G))$. More generally, one can define $\lambda(f)$ as an (unbounded) operator on $L^2(G)$ even for functions f not in $L^1(G)$. It then turns out that λ maps each $L^p(G)$, $1 \leq p \leq 2$, norm-decreasingly into a certain space $L^q(G')$ of closed densely defined operators on $L^2(G)$ (where $\frac{1}{p} + \frac{1}{q} = 1$). This is the Hausdorff–Young theorem. Kunze introduced the spaces $L^q(G')$ as spaces of measurable operators (in the sense of [21]) with respect to the canonical gage on M [14, p. 533]. An equivalent but simpler way of introducing the $L^q(G')$ is to consider the trace φ_0 on M characterized by $\varphi_0(\lambda(h) * \lambda(h)) = \|h\|_2^2$ for certain functions h, and then take $L^q(G')$ to be $L^q(M, \varphi_0)$ as defined by E. Nelson [15], viewing it as a space of "φ_0-measurable" operators [15, Theorem 5]. (In either case, the L^q spaces obtained are isomorphic to the abstract L^q spaces of J. Dixmier [5] associated with a trace on a von Neumann algebra.)

In the general (non-unimodular) case, φ_0 is no longer a trace, and the lack of adequate spaces L^q into which the $L^p(G)$ were to be mapped for a long time prevented the formulation of a Hausdorff–Young theorem, except for some special cases ([7, §8], [20, Proposition 15]). In [10], however, U. Haagerup constructed abstract L^p spaces corresponding to an arbitrary von Neumann algebra, and combining methods from [10] with the recent theory of spatial derivatives by A. Connes [2], M. Hilsum has developed a spatial theory of L^p spaces [12]. If M is a von Neumann algebra acting on a Hilbert space H and ψ is a weight on its commutant M', then the elements of $L^p(M, H, \psi)$ are (in general unbounded) operators on H satisfying a certain homogeneity property with respect to ψ. We shall see that when using these spaces (in the particular case of $M = \lambda(G)''$, $H = L^2(G)$, and $\psi =$ the canonical weight on M') and when defining the L^p Fourier transform of an L^p function f to be the operator $\xi \to f^* \Delta^{\frac{1}{q}} \xi$ on $L^2(G)$ (where Δ is the modular function of the group), one gets a nice L^p Fourier transformation theory and in particular a Hausdorff–Young theorem.

The paper is organized as follows. In Section 1 we fix the notations and describe our set-up. In Section 2, we study the L^p spaces of [12] in our particular case; we give a reformulation of the α-homogeneity property appearing in [2] that does not involve modular automorphism groups and we characterize $L^p(\psi_0)$ operators among all $(-\frac{1}{p})$-homogeneous operators. In Section 3, we treat the case $p = 2$ and obtain explicit expressions for the L^2 Fourier transformation $\mathcal{F}_2 = \mathcal{P}$, called the Plancherel transformation, as well as for its inverse.

Next, in Section 4, we deal with the case of a general $p \in [1, 2]$; we define the L^p Fourier transformation $\mathcal{F}_p$, and using interpolation (specifically, the three lines theorem) we prove our version of the Hausdorff–Young theorem.

Finally, in Section 5, we define an L^p Fourier cotransformation $\overline{\mathcal{F}}_p$ taking $L^p(\psi_0)$, $1 \leq p \leq 2$, into $L^q(G)$ and we investigate the relations between cotransformation and Fourier inversion. A detailed study of the $p = 1$ case gives a new characterization of $A(G)_+$ functions among all continuous positive definite functions on G.

1. Preliminaries and notation

Let G be a locally compact group with left Haar measure dx. We denote by $\mathcal{K}(\mathrm{G})$ the set of continuous functions on G with compact support and by $L^p(G)$, $1 \leq p \leq \infty$, the ordinary Lebesgue spaces with respect to dx. The modular function Δ on G is given by

$$\int f(xa^{-1})dx = \Delta(a) \int f(x)dx$$

for all $f \in \mathcal{K}(G)$ and $a \in G$. For functions f on G we put

$$\check{f}(x) = f(x^{-1}), \quad \tilde{f}(x) = \overline{f(x^{-1})}, \quad f^*(x) = \Delta^{-1}(x)\overline{f(x^{-1})}$$

and

$$(Jf)(x) = \Delta^{-\frac{1}{2}}(x)\overline{f(x^{-1})}$$

for all $x \in G$. More generally, for each $p \in [1, \infty]$, we define

$$(J_pf)(x) = \Delta^{-1/p}(x)\overline{f(x^{-1})}, \quad x \in G.$$

Then in particular $J_1f = f^*$, $J_2f = Jf$, $J_\infty f = \tilde{f}$. Note that for each $p \in [1, \infty]$, the operation J_p is a conjugate linear isometric involution of $L^p(G)$.

We shall often make use of the following non-unimodular version of Young's inequalities for convolution:

Lemma 1.1. *(Young's convolution inequalities.) Let $p_1, p_2, p \in [1, \infty]$ and $\frac{1}{p_1} + \frac{1}{q_1} = 1$. Assume that $\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p} = 1$. Then for all $f_1 \in L^{p_1}(G)$ and $f_2 \in L^{p_2}(G)$ the convolution product $f_1 * \Delta^{\frac{1}{q_1}} f_2$ exists and belongs to $L^p(G)$, and*

$$\|f_1 * \Delta^{\frac{1}{q_1}} f_2\|_p \leq \|f_1\|_{p_1} \|f_2\|_{p_2}.$$

This theorem is well-known in the unimodular case as well as in the special cases $(p_1, p_2, p) = (p_1, q_1, \infty)$ (where it follows from Hölder's inequality), $(p_1, p_2, p) = (1, p, p)$ or $(p_1, p_2, p) = (p, 1, p)$ [11, (20.14)], The general case has also been noted [13, Remark 2.2]. It can be proved by modifying the proof of [11, (20.18)] or by interpolation from the special cases mentioned above.

For operators T on the Hilbert space $L^2(G)$ we use the notation $D(T)$ (domain of T), $R(T)$ (range of T), $N(T)$ (kernel of T). If T is preclosed, we denote by $[T]$ the closure of T. If T is a positive self-adjoint operator and P the projection onto $N(T)^\perp$, then by definition T^{it}, $t \in \mathbb{R}$, is the partial isometry coinciding with the unitary $(TP)^{it}$ on $N(T)^\perp$ and O on $N(T)$. By convention, when speaking of operators, "bounded" always means "bounded and everywhere defined".

We denote by λ and ρ the left and right regular representations of G on $L^2(G)$, i.e. the unitary representations given by

$$(\lambda(x)f)(y) = f(x^{-1}y),$$

$$(\rho(x)f)(y) = \Delta^{\frac{1}{2}}(x)f(yx),$$

for all $x, y \in G$ and $f \in L^2(G)$. The corresponding representations of the algebra $L^1(G)$ (as in [4, 13.3]) are given by

$$\lambda(h)f = h * f \quad \text{and} \quad \rho(h)f = f * \Delta^{-\frac{1}{2}}\check{h}$$

for all $h \in L^1(G)$ and $f \in L^2(G)$.

We denote by M the von Neumann algebra of operators on $L^2(G)$ generated by $\lambda(G)$ (or $\lambda(\mathcal{K}(G))$, or $\lambda(L^1(G))$). In other words, M is the left von Neumann algebra of $\mathcal{K}(G)$, where $\mathcal{K}(G)$ is considered as a left Hilbert algebra [3, Definition 2.1] with convolution, involution *, and the ordinary inner product in $L^2(G)$. The commutant M' of M is the von Neumann algebra generated by $\rho(G)$, and $M' = JMJ$.

A function $\xi \in L^2(G)$ is called left (resp. right) bounded if left (resp. right) convolution with ξ on $\mathcal{K}(G)$ extends to a bounded operator on $L^2(G)$, i.e. if there exists a bounded operator $\lambda(\xi)$ (resp. $\lambda'(\xi)$) such that $\forall k \in \mathcal{K}(G) : \lambda(\xi)k = \xi * k$ (resp. $\lambda'(\xi)k = k * \xi$). The set of left (resp. right) bounded $L^2(G)$-functions is denoted $\mathfrak{A}_l$ (resp. $\mathfrak{A}_r$). Obviously, $\mathcal{K}(G) \subseteq \mathfrak{A}_l, \mathcal{K}(G) \subseteq \mathfrak{A}_r$, and for $\xi \in \mathcal{K}(G)$ we have $\lambda'(\xi) = \rho(\Delta^{-\frac{1}{2}}\check{\xi})$. Note that $\xi \in L^2(G)$ is left bounded if and only if the operator $\eta \to \lambda'(\eta)\xi : \mathfrak{A}_r \to L^2(G)$ extends to a bounded operator on $L^2(G)$; if this is the case, we have $\lambda(\xi)\eta = \lambda'(\eta)\xi$ for all $\eta \in \mathfrak{A}_r$. (Our definition of left-boundedness therefore agrees with [1, Définition 2.1]). If $\xi \in \mathfrak{A}_l$ and $T \in M$, then $T\xi \in \mathfrak{A}_l$ and $\lambda(T\xi) = T\lambda(\xi)$.

We denote by φ_0 the canonical weight on M [1, Définition 2.12]. Then the weight ψ_0 on M' given by $\psi_0(y) = \varphi_0(JyJ)$ for all $y \in (M')_+$ is called the canonical weight on M'. The corresponding modular automorphism groups are given by

$$\sigma_t^{\varphi_0}(x) = \Delta^{it}x\Delta^{-it}, x \in M,$$

$$\sigma_t^{\psi_0}(y) = \Delta^{-it}y\Delta^{it}, y \in M',$$

for all $t \in \mathbb{R}$. Here, Δ denotes the multiplication operator on $L^2(G)$ by the function Δ (note that we shall not distinguish in our notation between the function Δ and the corresponding multiplication operator). With this definition, Δ is in fact the modular operator of $\mathcal{K}(G)$ (as defined in [3, Lemma 2.2]).

It follows from the defining property of φ_0 [1, Théorème 2.11] that for all $y \in M'$ we have

$$\psi_0(y * y) = \begin{cases} \|\eta\|_2^2 & \text{if } y = \lambda'(\eta) \text{ for some } \eta \in \mathfrak{A}_r, \\ \infty & \text{otherwise} \end{cases}$$

We identify the Hilbert space completion H_{ψ_0} of $n_{\psi_0} = \{y \in M' | \psi_0(y * y) < \infty\}$ with $L^2(G)$ via $\eta \to \lambda'(\eta)$.

Now recall that by definition [2, Definition 1], $D(L^2(G), \psi_0)$ is the set of $\xi \in L^2(G)$ such that $y \mapsto y\xi : n_{\psi_0} \to L^2(G)$ extends to a bounded operator $R^{\psi_0}(\xi) : H_{\psi_0} \to L^2(G)$, i.e., in view of the identification of H_{ψ_0} with $L^2(G)$, such that $\eta \mapsto \lambda'(\eta)\xi : \mathfrak{A}_r \to L^2(G)$ extends to a bounded operator on $L^2(G)$. Thus $D(L^2(G), \psi_0) = \mathfrak{A}_l$, and for all $\xi \in D(L^2(G), \psi_0)$ we have $R^{\psi_0}(\xi) = \lambda(\xi)$.

If φ is a normal semi-finite weight on M, then by definition [2], $\frac{d\varphi}{d\psi_0}$ is the unique positive self-adjoint operator T satisfying

$$\forall \xi \in \mathfrak{A}_l : \varphi(\lambda(\xi)\lambda(\xi)^*) = \begin{cases} \|T^{\frac{1}{2}}\xi\|^2 & \text{if } \xi \in D(T^{\frac{1}{2}}) \\ \infty & \text{otherwise} \end{cases}$$

and

$$T^{\frac{1}{2}} = [T^{\frac{1}{2}}|_{\mathfrak{A}_l \cap D(T^{\frac{1}{2}})}].$$

In particular, we have

$$\frac{d\varphi_0}{d\psi_0} = \Delta$$

(cf. [2, Lemma 10 (b)] together with the proof of [2, Lemma 10 (a)].

If φ is a functional, then by the definition of $\frac{d\varphi}{d\psi_0}$ we have $\mathfrak{A}_l \subseteq D\left((\frac{d\varphi}{d\psi_0})^{\frac{1}{2}}\right)$ and $(\frac{d\varphi}{d\psi_0})^{\frac{1}{2}} = \left[(\frac{d\varphi}{d\psi_0})^{\frac{1}{2}}|_{\mathfrak{A}_l}\right]$. Finally, we note that the predual space M_* of the von Neumann algebra M may be viewed as a space of functions on the group in the following manner: for each $\varphi \in M_*$, define $u : G \to \mathbb{C}$ by

$$u(x) = \varphi(\lambda(x)), x \in G.$$

Then u is a continuous function on the group determining φ completely. The linear space of such functions, normed by $\|u\| = \|\varphi\|$, is exactly the Fourier algebra $A(G)$ of G introduced by P. Eymard [6] (this follows from [6, Théorème (3.10)]).
The identification of $A(G)$ with M_* is such that

$$\langle \varphi, \lambda(f)\rangle = \int \varphi(x)f(x)dx$$

for all $\varphi \in M_* \simeq A(G)$ and all $f \in L^1(G)$.

Recall that by [4, 13.4.4] a continuous function φ on G is positive definite if and only if

$$\forall \xi \in \mathcal{K}(G) : \int \varphi(x)(\xi * \xi^*)(x)dx \geq 0,$$

i.e. if and only if

$$\forall \xi \in \mathcal{K}(G) : \int\int \varphi(yx^{-1})\xi(y)\overline{\xi(x)}dydx \geq 0.$$

If $\varphi \in A(G)$, then φ is positive definite if and only if the corresponding functional $\varphi \in M_*$ is positive. We denote by $A(G)_+$ the set of positive definite $\varphi \in A(G)$.

2. Homogeneous operators on $L^2(G)$ and the spaces $L^p(\psi_0)$

Definition 2.1. Let $\alpha \in \mathbb{R}$. An operator T on $L^2(G)$ is called α-homogeneous if

$$\forall x \in G : \rho(x)T \subseteq \Delta^{-\alpha}(x)T\rho(x).$$

Remark 2.2. (1) The O-homogeneous operators are precisely the operators affiliated with M.

(2) If T is α-homogeneous, then actually $\rho(x)T = \Delta^{-\alpha}(x)T\rho(x)$ for all $x \in G$ (to see this, replace x by x^{-1} in the definition).

(3) If T and S are both α-homogeneous, then $T + S$ is α-homogeneous. If T is α-homogeneous and S is β-homogeneous, then TS is $(\alpha+\beta)$-homogeneous. If T is densely defined and α-homogeneous, then T^* is also α-homogeneous. If T is positive self-adjoint and α-homogeneous and $\beta \in \mathbb{R}_+$, then T^β is $(\alpha\beta)$-homogeneous (use $\rho(x)T^\beta\rho(x^{-1}) = (\rho(x)T\rho(x^{-1}))^\beta$).

(4) If T is α-homogeneous for some $\alpha \in \mathbb{R}$, then the projection onto $N(T)^\perp$ belongs to M (since $N(T)$ is invariant under all $\rho(x), x \in G$).

(5) If a preclosed operator T is α-homogeneous, then its closure $[T]$ is also α-homogeneous.

(6) For each $\alpha \in \mathbb{R}$, $\Delta^{-\alpha}$ is α-homogeneous.

Lemma 2.3. *Let T be a closed densely defined operator on $L^2(G)$ with polar decomposition $T = U|T|$. Let $\alpha \in \mathbb{R}$. Then T is α-homogeneous if and only if $U \in M$ and $|T|$ is α-homogeneous.*

Proof. If T is α-homogeneous, then, by Remark 2.2(3), $|T| = (T^*T)^{\frac{1}{2}}$ is also α-homogeneous. Then for all $x \in G$ and $\xi \in D(|T|)$ we have $\rho(x)U|T|\xi = \rho(x)T\xi = \Delta^{-\alpha}(x)T\rho(x)\xi = \Delta^{-\alpha}(x)U|T|\rho(x)\xi = U\rho(x)|T|\xi$, i.e. $\rho(x)U \subseteq U\rho(x)$ on $R(|T|)$. Since the projection onto $R(|T|) = N(|T|)^\perp$ belongs to M, we conclude that U commutes with all $\rho(x)$; thus $U \in M$.

The "if"-part follows directly from Remarks 2.2(3) and 2.2(1). □

Lemma 2.4. *Let T be a closed densely defined operator on $L^2(G)$ and $\alpha \in \mathbb{C}$. If*

$$\forall x \in G : \rho(x)T \subseteq \Delta^{-\alpha}(x)T\rho(x),$$

then

$$\forall f \in \mathcal{K}(G) : \lambda'(f)T \subseteq T\lambda'(\Delta^\alpha f).$$

Proof. Let $f \in \mathcal{K}(G)$ and $\xi \in D(T)$. Then for all $\eta \in D(T^*)$ we have

$$\begin{aligned}
(\rho(f)T\xi|\eta) &= \int f(x)(\rho(x)T\xi|\eta)dx \\
&= \int f(x)\Delta^{-\alpha}(x)(T\rho(x)\xi|\eta)dx \\
&= \int \Delta^{-\alpha}(x)f(x)(\rho(x)\xi|T * \eta)dx \\
&= (\rho(\Delta^{-\alpha}f)\xi|T^*\eta).
\end{aligned}$$

This shows that $\rho(\Delta^{-\alpha}f)\xi \in D(T^{**}) = D(T)$, and $T\rho(\Delta^{-\alpha}f)\xi = \rho(f)T\xi$ for all $\xi \in D(T)$, i.e.

$$\rho(f)T \subseteq T\rho(\Delta^{-\alpha}f).$$

Hence for all $f \in \mathcal{K}(G)$ we have

$$\lambda'(f)T = \rho(\Delta^{-\frac{1}{2}}\check{f}) \subseteq T\rho(\Delta^{-\alpha}\Delta^{-\frac{1}{2}}\check{f}) = T\lambda'(\Delta^\alpha f).$$

□

Lemma 2.5. *Let T be a closed densely defined operator on $L^2(G)$, α-homogeneous for some $\alpha \in \mathbb{R}$. Let $\xi \in \mathfrak{A}_l$. Then for all $t \in \mathbb{R}$ we have $|T|^{it}\xi \in \mathfrak{A}_l$ and*

$$\|\lambda(|T|^{it}\xi)\| \leq \|\lambda(\xi)\|.$$

Proof. By Lemma 2.3, we have $\rho(x)|T|\rho(x^{-1}) = \Delta^{-\alpha}(x)|T|$ for all $x \in G$, whence $\rho(x)|T|^{it}\rho(x^{-1}) = \Delta^{-i\alpha t}(x)|T|^{it}$ for all $x \in G$ and all $t \in \mathbb{R}$. Then, applying the preceding lemma to $|T|^{it}$, we obtain for all $\eta \in \mathcal{K}(G)$ that

$$|T|^{it}\xi * \eta = \lambda'(\eta)|T|^{it}\xi = |T|^{it}\lambda'(\Delta^{i\alpha t}\eta)\xi = |T|^{it}\lambda(\xi)\Delta^{i\alpha t}\eta$$

and thus

$$\|\,|T|^{it}\xi * \eta\|_2 \leq \|\,|T|^{it}\|\,\|\lambda(\xi)\|\,\|\Delta^{i\alpha t}\eta\|_2 \leq \|\lambda(\xi)\|\,\|\eta\|_2.$$

We conclude that $|T|^{it}\xi$ is left bounded and that

$$\|\lambda(|T|^{it}\xi)\| \leq \|\lambda(\xi)\|.$$

□

Remark 2.6. In particular, $\Delta^{it}\xi \in \mathfrak{A}_l$ with $\|\lambda(\Delta^{it}\xi)\| \leq \|\lambda(\xi)\|$ for all $\xi \in \mathfrak{A}_l$ and $t \in \mathbb{R}$

Our next lemma shows that α-homogeneity as defined here is equivalent to homogeneity of degree α with respect to ψ_0 as defined in [2, Definition 17].

Lemma 2.7. *Let $\alpha \in \mathbb{R}$, and let T be a closed densely defined operator on $L^2(G)$ with polar decomposition $T = U|T|$. Then the following conditions are equivalent:*

(i) *T is α-homogeneous,*

(ii) *$U \in M$ and $\forall y \in M' \quad \forall t \in \mathbb{R} : \sigma_{\alpha t}^{\psi_0}(y)|T|^{it} = |T|^{it}y$.*

Proof. By Lemma 2.3, we may assume that T is positive self-adjoint.

Denote by P the projection onto $N(T)^\perp$. If either (i) or (ii) holds, then P is in M, and thus the subspace $PL^2(G)$ is invariant under all operators considered. Therefore, we may suppose that $P \in M$, and the lemma is proved when we have shown the equivalence of

$$\forall x \in G : \rho(x)T\rho(x^{-1})P = \Delta^{-\alpha}(x)TP \tag{2.1}$$

and

$$\forall t \in \mathbb{R} \quad \forall y \in M' : \sigma_{\alpha t}^{\psi_0}(y)P = T^{it}yT^{-it}P. \tag{2.2}$$

Now for all $x \in G$ we have

$$\sigma_{\alpha t}^{\psi_0}(\rho(x)) = \Delta^{-i\alpha t}\rho(x)\Delta^{i\alpha t} = \Delta^{i\alpha t}(x)\rho(x)$$

since

$$(\Delta^{-i\alpha t}\rho(x)\Delta^{i\alpha t}f)(z) = \Delta^{-it}(z)\Delta^{\frac{1}{2}}(x)\Delta^{it}(zx)f(zx) = \Delta^{-it}(x)(\rho(x)f)(z)$$

for all $f \in L^2(G)$ and all $x, z \in G$. Then, since M' is generated by the $\rho(x)$, the condition (2.2) is equivalent to

$$\forall x \in G \quad \forall t \in \mathbb{R} : \Delta^{i\alpha t}(x)\rho(x)P = T^{it}\rho(x)T^{-it}P$$

or (changing t into $-t$)

$$\forall x \in G \quad \forall t \in \mathbb{R} : \rho(x)T^{it}\rho(x)P = \Delta^{-i\alpha t}(x)T^{it}P,$$

which in turn is equivalent to (2.1). □

Now, by [2, Theorem 13] a positive self-adjoint operator on $L^2(G)$ is (-1)-homogeneous if and only if it has the form $\frac{d\varphi}{d\psi_0}$ for a (necessarily unique) normal semi-finite weight φ on M.

We define the "integral with respect to ψ_0" of a positive self-adjoint (-1)-homogeneous operator T as

$$\int T d\psi_0 = \varphi(1) \in [0, \infty],$$

where $T = \frac{d\varphi}{d\psi_0}$. If $\int T d\psi_0 < \infty$, i.e. if φ is a functional, we shall say that T is integrable. (These definitions agree with those given in [2, remarks following Corollary 18].)

For each $p \in [1, \infty)$, we denote by $L^p(\psi_0)$ the set of closed densely defined $(-\frac{1}{p})$-homogeneous operators T on $L^2(G)$ satisfying

$$\int |T|^p d\psi_0 < \infty.$$

(Note that $|T|^p$ is (-1)-homogeneous, so that $\int |T|^p d\psi_0$ is defined.) We put $L^\infty(\psi_0) = M$.

The spaces $L^p(\psi_0)$ introduced here are special cases of the spatial L^p-spaces of M. Hilsum [12]. We recall their main properties (note, however, that our notation differs from that of [12] in that we maintain throughout the distinction between operators and their closures):

If $T, S \in L^p(\psi_0)$, then $T + S$ is densely defined and preclosed, and the closure $[T + S]$ belongs to $L^p(\psi_0)$. With the obvious scalar multiplication and the sum $(T, S) \mapsto [T+S]$, $L^p(\psi_0)$ is a linear space, and even a Banach space with the norm $\|.\|_p$ defined by $\|T\|_p = (\int |T|^p d\psi_0)^{1/p}$ if $p \in [1, \infty)$ and $\|T\|_p = \|T\|$ (operator norm) if $p = \infty$. The operation $T \mapsto T^*$ is an isometry of $L^p(\psi_0)$ onto $L^p(\psi_0)$. We denote $L^p(\psi_0)_+$ the set of positive self-adjoint operators belonging to $L^p(\psi_0)$

By linearity, $T \mapsto \int T d\psi_0$ defined on $L^1(\psi_0)_+$ extends to a linear form on the whole of $L^1(\psi_0)$ satisfying $\int T^* d\psi_0 = \overline{\int T d\psi_0}$ and $|\int T d\psi_0| \leq \|T\|_1$ for all $T \in L^1(\psi_0)$.

Let $p_1, p_2, p \in [1, \infty]$ such that $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$. If $T \in L^{p_1}(\psi_0)$ and $S \in L^{p_2}(\psi_0)$, then the operator TS is densely defined and preclosed, its closure $[TS]$ belongs to $L^p(\psi_0)$, and

$$\|[TS]\|_p \leq \|T\|_{p_1} \|S\|_{p_2}.$$

In particular, if $T \in L^p(\psi_0)$ and $S \in L^q(\psi_0)$, where $\frac{1}{p} + \frac{1}{q} = 1$, then $[TS] \in L^1(\psi_0)$ and $\|[TS]\|_1 \leq \|T\|_p \|S\|_q$ (Hölder's inequality); furthermore, $\int [TS] d\psi_0 = \int [ST] d\psi_0$.

If $p \in [1, \infty)$ and $\frac{1}{p} + \frac{1}{q} = 1$, then we identify $L^q(\psi_0)$ with the dual space of $L^p(\psi_0)$ by means of the form $(T, S) \mapsto \int [TS] d\psi_0$, $T \in L^p(\psi_0)$. $S \in L^q(\psi_0)$. In particular, $L^1(\psi_0)$ is the predual of $M = L^\infty(\psi_0)$. The space $L^2(\psi_0)$ is a Hilbert space with the inner product $(T|S)_{L^2(\psi_0)} = \int [S * T] d\psi_0$.

Remark 2.8. Suppose that G is unimodular. Then the $\alpha-$*homogeneous* operators for any α are simply the operators affiliated with M and the canonical weight

φ_0 on M is a trace. We claim that $\int T d\psi_0 = \varphi_0(T)$ for all positive self-adjoint operators T affiliated with M, where we have written $\varphi_0(T)$ for the value of $\varphi = \varphi_0(T.)$ at 1 (with $\varphi_0(T.)$ defined as in [17, §4]). To see this, recall that $\frac{d\varphi_0}{d\psi_0} = \Delta = 1$, so that using [2, Theorem 9, (2)], we have

$$T^{it} = (D\varphi : D\varphi_0)_t = \left(\frac{d\varphi}{d\psi_0}\right)^{it} \left(\frac{d\varphi_0}{d\psi_0}\right)^{-it} = \left(\frac{d\varphi}{d\psi_0}\right)^{it}$$

for all $t \in \mathbb{R}$. Thus $T = \frac{d\varphi}{d\psi_0}$, and $\int T d\psi_0 = \varphi(1) = \varphi_0(T)$. (When proving $T = \frac{d\varphi}{d\psi_0}$, we implicitly assumed that T is injective so that $\varphi = \varphi_0(T.)$ is faithful. In the general case, denote by $Q \in M$ the projection onto $N(T)$, note that $T+Q$ is positive self-adjoint, affiliated with M, and injective, and verify that

$$T + Q = \frac{d\varphi_0((T+Q).)}{d\psi_0} = \frac{d\varphi_0(T.)}{d\psi_0} + \frac{d\varphi_0(Q.)}{d\psi_0}.$$

Since the supports of $\frac{d\varphi_0(T.)}{d\psi_0}$ and $\frac{d\varphi_0(Q.)}{d\psi_0}$ are $1-Q$ and Q, respectively, we conclude that $T = \frac{d\varphi_0(T.)}{d\psi_0}$ as desired.) It follows that in this case the spaces $L^p(\psi_0)$ reduce the ordinary $L^p(M, \varphi_0)$ (discussed in the introduction).

Returning to the general case, we now proceed to a more detailed study of the spaces $L^p(\psi_0)$. For this, we shall need the following slightly generalized version of [12, II, Proposition 2].

Lemma 2.9. *Suppose that T is a positive self-adjoint operator on $L^2(G)$ and $\alpha-$homogeneous for some $\alpha \in \mathbb{R}$. Let $\xi \in \mathfrak{A}_l$. Then for each $n \in \mathbb{N}$ there exists $\xi_n \in \mathfrak{A}_l \cap \left(\cap_{\beta \in \mathbb{R}_+} D(T^\beta)\right)$ such that*
(i) $\forall n \in \mathbb{N} : \|\lambda(\xi_n)\| \leq \|\lambda(\xi)\|$,
(ii) $\xi_n \to \xi$ as $n \to \infty$,
(iii) $T^\beta \xi_n \to T^\beta \xi$ as $n \to \infty$ whenever ξ and $\beta \in \mathbb{R}_+$ satisfy $\xi \in D(T^\beta)$.

Proof. For each $n \in N$, define $f_n : [0, \infty) \to \mathbb{C}$ by

$$f_n(x) = \begin{cases} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} x^{\frac{it}{\sqrt{n}}} dt & \text{if } x > 0 \\ 1 & \text{if } x = 0. \end{cases}$$

Since for all $x \in [0, \infty)$ we have $|f_n(x)| \leq \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} dt = 1$, the operators $f_n(T)$ are bounded. For each $\eta \in \mathbb{N}$, put $\xi_n = f_n(T)\xi$.

To prove that the ξ_n belong to $\mathfrak{A}_l$ and satisfy (i), denote by P the projection onto $N(T)^\perp$ and observe that for all $\eta \in \mathcal{K}(G)$ we have

$$\begin{aligned} f_n(T)P\xi * \eta &= \lambda'(\eta) f_n(T)P\xi \\ &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} \lambda'(\eta) T^{\frac{it}{\sqrt{n}}} \xi dt \\ &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} T^{\frac{it}{\sqrt{n}}} \lambda'(\Delta^{\frac{i\alpha t}{\sqrt{n}}} \eta) \xi dt \\ &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} T^{\frac{it}{\sqrt{n}}} (\xi * \Delta^{\frac{i\alpha t}{\sqrt{n}}} \eta) dt, \end{aligned}$$

where we have used Lemma 2.4. It follows that

$$\|f_n(T)P\xi * \eta\|_2 \leq \frac{1}{\sqrt{\pi}} \int e^{-t^2} \|\lambda(\xi)\| \, \|\Delta^{\frac{i\alpha t}{\sqrt{n}}}\eta\|_2 dt \leq \|\lambda(\xi)\| \, \|\eta\|_2.$$

On the other hand,

$$\|(1-P)\xi * \eta\|_2 \leq \|\lambda((1-P)\xi)\| \, \|\eta\|_2 \leq \|\lambda(\xi)\| \, \|\eta\|_2,$$

since $P \in M$.

In all, $f_n(T)\xi = f_n(T)P\xi + (1-P)\xi$ belongs to $\mathfrak{A}_l$ and $\|\lambda(f_n(T)\xi)\| \leq \|\lambda(\xi)\|$.

Now, to see that $\xi_n \in D(T^\beta)$ for all $\beta \in \mathbb{R}_+$, note that

$$\begin{aligned} f_n(x) &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} e^{\frac{it}{\sqrt{n}} \log x} \, dt \\ &= e^{-\frac{1}{4n}(\log x)^2} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(t - \frac{i}{2\sqrt{n}} \log x)^2} dt \\ &= e^{-\frac{1}{4n}(\log x)^2} \end{aligned}$$

for all $x > 0$. Then $x \mapsto x^\beta f_n(x) = e^{(\beta \log x - \frac{1}{4n}(\log x)^2)}$ is bounded, so that $T^\beta f_n(T)$ is a bounded operator, and thus $f_n(T)\xi \in D(T^\beta)$.

Since f_n is bounded and $f_n(x) \to 1$ as $n \to \infty$ for all $x \in [0, \infty)$, we have

$$f_n(T)\zeta \to \zeta \quad \text{as} \quad n \to \infty$$

for all ζ. From this, we immediately get (ii) and (iii). Indeed, $\xi_n = f_n(T)\xi \to \xi$, and if $\xi \in D(T^\beta)$, then

$$T^\beta \xi_n = T^\beta f_n(T)\xi = f_n(T)T^\beta \xi \to T^\beta \xi.$$

$\square$

Proposition 2.10. *Let T be a closed densely defined $(-1)-$homogeneous operator on $L^2(G)$. Then the following conditions are equivalent:*
(i) $T \in L^1(\psi_0)$,
(ii) there exists a constant $C \geq 0$ such that

$$\forall \xi \in \mathfrak{A}_l \cap D(T) \quad \forall \eta \in \mathfrak{A}_l : |(T\xi|\eta)| \leq C\|\lambda(\xi)\| \, \|\lambda(\eta)\|,$$

(iii) there exists a constant $C \geq 0$ such that

$$\forall \xi \in \mathfrak{A}_l \cap D(|T|^{\frac{1}{2}}) : \| \, |T|^{\frac{1}{2}}\xi\|^2 \leq C\|\lambda(\xi)\|^2,$$

(iv) there exists an approximate identity $(\xi_i)_{i \in I}$ in $K(G)_+$ such that all $\xi_i \in D(|T|^{\frac{1}{2}})$ and

$$\liminf_{i \in I} \| \, |T|^{\frac{1}{2}}\xi_i\| < \infty.$$

If $T \in L^1(\psi_0)$, then $\mathfrak{A}_l \subseteq D(|T|^{\frac{1}{2}})$, and for any approximate identity $(\xi_i)_{i \in I}$ in $\mathcal{K}(G)_+$ we have

$$\|T\|_1 = \lim_{i \in I} \||T|^{\frac{1}{2}}\xi_i\|^2.$$

Furthermore, $\|T\|_1$ is the smallest C satisfying (ii) and the smallest C satisfying (iii).

Proof. Let $T = U|T|$ be the polar decomposition of T.

First, suppose that $T \in L^1(\psi_0)$. Then $|T| \in L^1(\psi_0)_+$, and therefore $|T| = \frac{d\varphi}{d\psi_0}$ for some positive functional φ on M. Recall that $\mathfrak{A}_l \subseteq D(|T|^{\frac{1}{2}})$. Thus for all $\xi \in \mathfrak{A}_l \cap D(T)$ and $\eta \in \mathfrak{A}_l$ we have

$$\begin{aligned} |(T\xi|\eta)| &= \left|(|T|^{\frac{1}{2}}\xi \mid |T|^{\frac{1}{2}}U^*\eta)\right| \\ &= |\varphi(\lambda(\xi)\lambda(U^*\eta))| \\ &\leq \|\varphi\| \, \|\lambda(\xi)\| \, \|\lambda(U^*\eta)\| \\ &\leq \|T\|_1 \|\lambda(\xi)\| \, \|\lambda(\eta)\|, \end{aligned}$$

i.e. (ii) holds.

Next, suppose that T satisfies (ii). Then for all $\xi \in \mathfrak{A}_l \cap D(|T|)$ we have

$$\begin{aligned} \| \, |T|^{\frac{1}{2}}\xi\|^2 &= |(T\xi|U\xi)| \\ &\leq C\|\lambda(\xi)\| \, \|\lambda(U\xi)\| \\ &\leq C\|\lambda(\xi)\|^2. \end{aligned}$$

Now if $\xi \in \mathfrak{A}_l \cap D(|T|^{\frac{1}{2}})$, there exist (by Lemma 2.9) $\xi_n \in \mathfrak{A}_l \cap D(|T|)$ such that $|T|^{\frac{1}{2}}\xi_n \to |T|^{\frac{1}{2}}\xi$ and $\|\lambda(\xi_n)\| \leq \|\lambda(\xi)\|$. Since

$$\| \, |T|^{\frac{1}{2}}\xi_n\|^2 \leq C\|\lambda(\xi_n)\|^2 \leq C\|\lambda(\xi)\|^2,$$

we conclude that $\| \, |T|^{\frac{1}{2}}\xi\|^2 \leq C\|\lambda(\xi)\|^2$. Thus (iii) is proved.

Now suppose that T satisfies (iii). First we show that this implies $\mathfrak{A}_l \subseteq D(|T|^{\frac{1}{2}})$. Let $\xi \in \mathfrak{A}_l$. Then by Lemma 2.9 there exist $\xi_n \in \mathfrak{A}_l \cap D(|T|^{\frac{1}{2}})$ such that $\xi_n \to \xi$ and $\|\lambda(\xi_n)\| \leq \|\lambda(\xi)\|$. Then for all $\eta \in D(|T|^{\frac{1}{2}})$ we have

$$\begin{aligned} \left|(|T|^{\frac{1}{2}}\xi_n|\eta)\right| &\leq \| \, |T|^{\frac{1}{2}}\xi_n\| \, \|\eta\| \\ &\leq C^{1/2}\|\lambda(\xi_n)\| \, \|\eta\| \\ &\leq C^{1/2}\|\lambda(\xi)\| \, \|\eta\| \end{aligned}$$

and

$$(|T|^{\frac{1}{2}}\xi_n|\eta) = (\xi_n| \, |T|^{\frac{1}{2}}\eta) \to (\xi \mid |T|^{\frac{1}{2}}\eta).$$

We conclude that

$$\forall \eta \in D(|T|^{\frac{1}{2}}) : \left|(\xi \mid |T|^{\frac{1}{2}}\eta)\right| \leq C^{1/2}\|\lambda(\xi)\| \, \|\eta\|.$$

Thus $\xi \in D(|T|^{\frac{1}{2}})$ as wanted.

Now, still assuming (iii), let us prove (iv). Let $(\xi_i)_{i\in I}$ be any approximate identity in $\mathcal{K}(G)_+$. Then automatically all $\xi_i \in \mathcal{K}(G) \subseteq \mathfrak{A}_l \subseteq D(|T|^{\frac{1}{2}})$, and $\|\lambda(\xi_i)\| \leq \|\xi_i\|_1 = 1$ so that

$$\| \, |T|^{\frac{1}{2}}\xi_i\|^2 \leq C\|\lambda(\xi_i)\|^2 \leq C,$$

whence $\liminf_{i\in I} \| \, |T|^{\frac{1}{2}}\xi_i\| \leq C^{\frac{1}{2}} < \infty$.

Finally, suppose that T satisfies (iv) for some $(\xi_i)_{i\in I}$. Note that since $\int(\xi_i * \xi_i^*)(x)dx = 1$, $(\xi_i * \xi_i^*)_{i\in I}$ is again an approximate identity in $\mathcal{K}(G)_+$. Therefore,

$\lambda(\xi_i)\lambda(\xi_i)^* = \lambda(\xi_i * \xi_i^*)$ convergence strongly, and hence weakly, to 1 in M. Since all $\|\lambda(\xi_i)\lambda(\xi_i)^*\| \leq 1$, this convergence is also σ-weak, and by the σ-weak lower semicontinuity of φ, this implies

$$\begin{aligned}\varphi(1) &\leq \liminf_{i\in I} \varphi(\lambda(\xi_i)\lambda(\xi_i)^*)\\ &= \liminf_{i\in I} \|\, |T|^{\frac{1}{2}}\xi_i\|^2\\ &\leq C \liminf_{i\in I} \|\lambda(\xi_i)\|^2\\ &\leq C < \infty\,.\end{aligned}$$

Since $\varphi(1) = \int |T| d\psi_0 < \infty$, we have $T \in L^1(\psi_0)$, i.e. (i) holds.

Note that once $\varphi(1) < \infty$ is established, φ is known to be σ-weakly lower continuous and thus

$$\varphi(1) = \lim_{i\in I} \varphi(\lambda(\xi_i)\lambda(\xi_i)^*) = \lim_{i\in I} \|\, |T|^{\frac{1}{2}}\xi_i\|^2$$

for any approximate identity $(\xi_i)_{i\in I}$, i.e.

$$\|T\|_1 = \lim_{i\in I} \|\, |T|^{\frac{1}{2}}\xi_i\|^2.$$

In the course of the proof we observed that $\|T\|_1$ may be used as the constant C in (ii), that every constant C satisfying (ii) also satisfies (iii), and that any C satisfying (iii) is bigger than $\lim_{i\in I} \|\, |T|^{\frac{1}{2}}\xi_i\|^2$, i.e. bigger than $\|T\|_1$. This proves the remarks that end Proposition 2.10 $\square$

As an immediate corollary, we have:

Proposition 2.11. *Let T be a closed densely defined $(-\frac{1}{2})$-homogeneous operator on $L^2(G)$. Then the following conditions are equivalent:*

(i) $T \in L^2(\psi_0)$,

(ii) *there exists a constant $C \geq 0$ such that*

$$\forall \xi \in \mathfrak{A}_l \cap D(T) : \|T\xi\| \leq C\|\lambda(\xi)\|,$$

(iii) *there exists an approximate identity $(\xi_i)_{i\in I}$ in $\mathcal{K}(G)_+$ such that all $\xi_i \in D(T)$ and*

$$\liminf_{i\in I} \|T\xi_i\| < \infty.$$

If $T \in L^2(\psi_0)$, then $\mathfrak{A}_l \subseteq D(T)$, and for any approximate identity $(\xi_i)_{i\in I}$ in $\mathcal{K}(G)_+$ we have

$$\|T\|_2 = \lim_{i\in I} \|T\xi_i\|;$$

furthermore, $\|T\|_2$ is the smallest constant C satisfying (ii).

We now come to the case of a general $p \in [1, \infty)$. Suppose that $T \in L^P(\psi_0)$ and $S \in L^q(\psi_0)$, where $\frac{1}{p} + \frac{1}{q} = 1$. Then by [12, II, Proposition 5, 1], we have

$$(T\xi|S\eta) = \langle [S * T], \lambda(\xi)\lambda(\eta)^*\rangle$$

for all $\xi \in \mathfrak{A}_l \cap D(T)$ and $\eta \in \mathfrak{A}_l \cap D(S)$. (Here, $\langle ., . \rangle$ denotes the form giving the duality of $L^1(\psi_0)$ and M.) Using Hölder's inequality, we get

$$|(T\xi|S\eta)| \leq \|[S^*T]\|_1 \|\lambda(\xi)\lambda(\eta)^*\| \leq \|T\|_p \|S\|_q \|\lambda(\xi)\|\, \|\lambda(\eta)\|$$

for all such ξ and η. This kind of inequality in fact characterizes $L^p(\psi_0)$-operators among all $(-\frac{1}{p})-$homogeneous operators:

Proposition 2.12. *Let $p \in [1, \infty]$ and define q by $\frac{1}{p} + \frac{1}{q} = 1$. Let T be a closed densely defined $(-\frac{1}{p})-$homogeneous operator on $L^2(G)$. Then the following conditions are equivalent:*

(i) $T \in L^p(\psi_0)$,

(ii) *there exists a constant $C \geq 0$ such that*

$$\forall S \in L^q(\psi_0)\ \forall \xi \in \mathfrak{A}_l \cap D(T)\ \forall \eta \in \mathfrak{A}_l \cap D(S) : |(T\xi|S\eta)| \leq C\|S\|_q \|\lambda(\xi)\|\,\|\lambda(\eta)\|.$$

If $T \in L^p(\psi_0)$, then $\|T\|_p$ is the smallest C satisfying (ii).

Proof. In view of the remarks preceding this proposition, we just have to show that if T satisfies (ii) for some constant C, then $T \in L^p(\psi_0)$, and $\|T\|_p \leq C$.

Therefore suppose that T with polar decomposition $T = U|T|$ satisfies (ii). Then also

$$|(|T|\xi|S\eta)| = |(T\xi|U^*S\eta)| \leq C\|[U^*S]\|_q \|\lambda(\xi)\|\,\|\lambda(\eta)\| \leq C\|S\|_q\|\lambda(\xi)\|\,\|\lambda(\eta)\|$$

for all S, ξ, and η chosen as in (ii). Then we may assume that T is positive self-adjoint.

Let $S \in L^q(\psi_0)$ and $\eta \in \mathfrak{A}_l \cap D(T^{\frac{1}{2}}S)$. We claim that for all $\xi \in \mathfrak{A}_l \cap D(T^{\frac{1}{2}})$ we have

$$|(T^{\frac{1}{2}}\xi|(T^{\frac{1}{2}}S\eta)| \leq C\|S\|_q\|\lambda(\xi)\|\,\|\lambda(\eta)\|. \tag{2.3}$$

If $\xi \in \mathfrak{A}_l \cap D(T)$, this follows directly from the hypothesis. In case of a general $\xi \in \mathfrak{A}_l \cap D(T^{\frac{1}{2}})$, choose (by Lemma 2.9) $\xi_n \in \mathfrak{A}_l \cap D(T)$ such that $T^{\frac{1}{2}}\xi_n \to T^{\frac{1}{2}}\xi$ and $\|\lambda(\xi_n)\| \leq \|\lambda(\xi)\|$. Then (2.3) follows by passing to the limit.

Now since T is $-\frac{1}{p}$-homogeneous, there exist $T_i \in L^p(\psi_0)_+$ satisfying $T_i^p \leq T^p$ and $\int T^p d\psi_0 = \sup T_i^P d\psi_0$. (To see this, recall that $T^p = \frac{d\varphi}{d\psi_0}$ for some normal semi-finite weight φ on M; put $T_i = (\frac{d\varphi_i}{d\psi_0})^{1/p}$ where φ_i are positive normal functionals such that $\varphi_i \nearrow \varphi$; then $\frac{d\varphi_i}{d\psi_0} \leq \frac{d\varphi}{d\psi_0}$ by [2, Proposition 8], and $\int T^p d\psi_0 = \varphi(1) = \sup \varphi_i(1) = \sup \int T_i^p d\psi_0$.)

Since the function $t \to t^{1/p}$ is operator monotone on $[0, \infty)$ (by [16, Proposition 1.3.8]), we have $T_i \leq T$, i.e. $D(T_i^{\frac{1}{2}}) \supseteq D(T^{\frac{1}{2}})$ and

$$\forall \xi \in D(T^{\frac{1}{2}}) : \|T_i^{\frac{1}{2}}\xi\| \leq \|T^{\frac{1}{2}}\xi\|,$$

for each $i \in I$ (cf. also the remark following this proof).

For each i, let B_i be the bounded operator characterized by $B_iT^{\frac{1}{2}}\xi = T_i^{\frac{1}{2}}\xi$ for all $\xi \in D(T^{\frac{1}{2}})$ and $B_i\xi = 0$ for all $\xi \in R(T^{\frac{1}{2}})^{\perp}$. Then $\|B_i\| \leq 1$. Since $B_iT^{\frac{1}{2}} \subseteq T_i^{\frac{1}{2}}$, and since $T^{\frac{1}{2}}$ and $T_i^{\frac{1}{2}}$ are $(-\frac{1}{p})$-homogeneous, B_i is 0-homogeneous, i.e. $B_i \in M$. Put $A_i = B_i^*$. Then $A_i \in M$, $\|A_i\| \leq 1$, and

$$T_i^{\frac{1}{2}} \subseteq T^{\frac{1}{2}}A_i.$$

Using this, the fact that

$$T_i^{p-1} = T_i^{\frac{p}{q}} \in L^q(\psi_0) \quad \text{with} \quad \|T_i^{p-1}\|_q = \|T_i\|_p^{p-1},$$

and (2.3), we find that for all $\xi \in \mathfrak{A}_l \cap \left(\cap_{\beta\in\mathbb{R}_+} D(T_i^\beta)\right)$, we have

$$\begin{aligned}
\|T_i^{\frac{p}{2}}\xi\|^2 &= (T_i^{\frac{1}{2}}\xi|T_i^{\frac{1}{2}}T_i^{p-1}\xi) \\
&= (T^{\frac{1}{2}}A_i\xi|T^{\frac{1}{2}}A_iT_i^{p-1}\xi) \\
&\leq C\|[A_iT_i^{p-1}]\|_q\|\lambda(A_i\xi)\|\,\|\lambda(\xi)\| \\
&\leq C\|A_i\|\,\|T_i^{p-1}\|_q\|A_i\|\,\|\lambda(\xi)\|^2 \\
&= C\|T_i\|_p^{p-1}\|\lambda(\xi)\|^2.
\end{aligned}$$

By means of Lemma 2.9, we conclude that the estimate

$$\|T_i^{\frac{p}{2}}\|^2 \leq C\|T_i\|_p^{p-1}\|\lambda(\xi)\|^2$$

holds for all $\xi \in \mathfrak{A}_l \cap D(T_i^{p/2})$. Thus by Proposition 2.10,

$$\|T_i\|_p^p = \|T_i^p\|_1 \leq C\|T_i\|_p^{p-1},$$

i.e.

$$\|T_i\|_p \leq C.$$

Since this holds for all i, we have

$$\int T^p d\psi_0 = \sup \int T_i^p d\psi_0 \leq C^p < \infty;$$

thus $T \in L^p(\psi_0)$ and $\|T_i\|_p \leq C$. □

Remark 2.13. we have used the fact that if a continuous function f on $[0,\infty)$ is operator monotone in the sense that $R \leq S$ implies $f(R) \leq f(S)$ for all positive bounded operators R and S, then the same is true for all - possibly unbounded - positive self-adjoint R and S. To see this, suppose that $R \leq S$. Then for all $\varepsilon \in \mathbb{R}_+$, we have $R(1+\varepsilon R)^{-1} \leq S(1+\varepsilon S)^{-1}$ by [17, §4], and hence $f(R(1+\varepsilon R)^{-1}) \leq f(S(1+\varepsilon S)^{-1})$. Now if $\xi \in D(f(S)^{\frac{1}{2}})$, we have by the spectral theory

$$\begin{aligned}
(f(R(1+\varepsilon R)^{-1})\xi|\xi) &\leq (f(S+(1+\varepsilon S)^{-1})\xi|\xi) \\
&\qquad \to \|f(S)^{\frac{1}{2}}\xi\|^2 \quad \text{as} \quad \varepsilon \to 0.
\end{aligned}$$

Again by the spectral theory, we conclude that $\xi \in D(f(R)^{\frac{1}{2}})$ and that

$$\|f(R)^{\frac{1}{2}}\xi\|^2 = \lim_{\varepsilon\to 0}(f(R(1+\varepsilon R)^{-1})\xi|\xi) \leq \|f(S)^{\frac{1}{2}}\xi\|^2.$$

In all, we have proved that $f(R) \leq f(S)$.

Recall from [12, §1, Théorème 4, 1)], that if T_1 and T_2 belong to some $L^p(\psi_0)$, $1 \leq p < \infty$, and if $T_2 \subseteq T_1$, then $T_1 = T_2$. Actually, a stronger result holds:

Lemma 2.14. *Let $p \in [1, \infty]$. Let $T_1 \in L^p(\psi_0)$ and let T_2 be a closed densely defined $(-\frac{1}{p})$-homogeneous operator on $L^2(G)$. If $T_2 \subseteq T_1$ or $T_1 \subseteq T_2$, then $T_1 = T_2$.*

Proof. First suppose that $T_2 \subseteq T_1$. If $p = \infty$, the result is well-known (a closed densely defined operator having a bounded and everywhere defined extension is equal to that extension). If $p \in [1, \infty)$, we conclude by Proposition 2.12 that also $T_2 \in L^p(\psi_0)$, and thus by [12, §1, Théorème 4, 1)], $T_1 = T_2$. (Alternatively, this can be proved directly, i.e. without using Proposition 2.12, by the methods of the proof of [12, §1, Théorème 4, 1)].)

If $T_1 \subseteq T_2$, apply the first part of the proof to $T_2^* \subseteq T_1^*$. □

A specific form of this lemma will be crucial to much of the following:

Proposition 2.15. *Let $p \in [1, \infty]$.*

1) *Let T and S be closed densely defined $(-\frac{1}{p})$-homogeneous operators on $L^2(G)$ with $\mathcal{K}(G) \subseteq D(T)$ and $\mathcal{K}(G) \subseteq D(S)$. Suppose that $T\xi = S\xi$ for all $\xi \in \mathcal{K}(G)$. If one of the operators, say T, belongs to $L^p(\psi_0)$, we may conclude that $T = S$.*
2) *If $T \in L^p(\psi_0)$ and $\mathcal{K}(G) \subseteq D(T)$, then $T = [T|_{\mathcal{K}(G)}]$.*

Proof. (of both parts). Suppose that $T \in L^p(\psi_0)$. Then $T|_{\mathcal{K}(G)}$, being a restriction of a $(-\frac{1}{p})$-homogeneous operator to a right invariant subspace, is itself $(-\frac{1}{p})$-homogeneous. Therefore also $[T|_{\mathcal{K}(G)}]$ is $(-\frac{1}{p})$-homogeneous. Since $[T|_{\mathcal{K}(G)}] \subseteq T$, we conclude by the above lemma that $T = [T|_{\mathcal{K}(G)}]$. This proves 2). As for 1), note that $S \supseteq S|_{\mathcal{K}(G)} = T|_{\mathcal{K}(G)}$, and thus $S \supseteq [T|_{\mathcal{K}(G)}] = T$. Again we conclude $S = T$. □

Finally, for later reference, we summarize in a lemma some remarks of Hilsum [12]:

Lemma 2.16. *Let $q \in [2, \infty)$. Let $T \in L^q(\psi_0)$. Then $\mathfrak{A}_l \subseteq D(T)$, and for all $\xi \in \mathfrak{A}_l$ we have*

$$\|T\xi\| \le \|T\|_q \|\lambda(\xi)\|^{2/q} \|\xi\|^{1-2/q}.$$

Proof. Since $|T|^{\frac{q}{2}} \in L^2(\psi_0)$, we have $\mathfrak{A}_l \subseteq D(|T|^{\frac{q}{2}})$. Now let $\xi \in \mathfrak{A}_l$. Then by the spectral theory $\xi \in D(|T|)$ and

$$\begin{aligned} \|\,|T|\xi\|^2 &\le (\|\,|T|^{\frac{q}{2}}\xi\|^2)^{2/q} . (\|\xi\|^2)^{1-2/q} \\ &\le (\|\,|T|^q\|_1 \|\lambda(\xi)\|^2)^{2/q} . \|\xi\|^{2(1-2/q)} \\ &= (\|T\|_q \|\lambda(\xi)\|^{2/q} \|\xi\|^{1-2/q})^2. \end{aligned}$$

□

3. The Plancherel transformation

Given any functions $f \in L^2(G)$ and $\xi \in L^2(G)$, the convolution product $f * \Delta^{\frac{1}{2}}\xi$ exists and belongs to $L^\infty(G)$. Thus the following definition makes sense:

Definition 3.1. Let $f \in L^2(G)$. The Plancherel transform $\mathcal{P}(f)$ of f is the operator on $L^2(G)$ given by

$$\mathcal{P}(f)\xi = f * \Delta^{\frac{1}{2}}\xi, \quad \xi \in D(\mathcal{P}(f)),$$

where

$$D(\mathcal{P}(f)) = \{\xi \in L^2(G) | f * \Delta^{\frac{1}{2}}\xi \in L^2(G)\}.$$

Theorem 3.2. *(Plancherel).*

(1) *Let $f \in L^2(G)$. Then $\mathcal{P}(f)$ belongs to $L^2(\psi_0)$, and*

$$\|\mathcal{P}(f)\|_2 = \|f\|_2.$$

(2) *The Plancherel transformation $\mathcal{P} : L^2(G) \to L^2(\psi_0)$ is a unitary transformation of $L^2(G)$ onto $L^2(\psi_0)$.*

Proof. (1) First note that $\mathcal{P}(f)$ is $(-\frac{1}{2})$-homogeneous: for all $x, y \in G$ and $\xi \in D(\mathcal{P}(f))$, we have

$$\begin{aligned}
\rho(x)(\mathcal{P}(f)\xi)(y) &= \Delta^{\frac{1}{2}}(x)(f * \Delta^{\frac{1}{2}}\xi)(yx) \\
&= \Delta^{\frac{1}{2}}(x) \int f(z)\Delta^{\frac{1}{2}}(z^{-1}yx)\xi(z^{-1}yx)dz \\
&= \Delta^{\frac{1}{2}}(x) \int f(z)\Delta^{\frac{1}{2}}(z^{-1}y)(\rho(x)\xi)(z^{-1}y)dz \\
&= \Delta^{\frac{1}{2}}(x)(f * \Delta^{\frac{1}{2}}\rho(x)\xi)(y),
\end{aligned}$$

i.e. $\rho(x)\mathcal{P}(f) \subseteq \Delta^{\frac{1}{2}}\mathcal{P}(f)\rho(x)$.

We next show that $\mathcal{P}(f)$ is closed. Suppose that $\xi_n \to \xi$ in $L^2(G)$ and $\mathcal{P}(f)\xi_n \to \eta$ in $L^2(G)$, where all the $\xi_n \in D(\mathcal{P}(f))$. Then $f * \Delta^{\frac{1}{2}}\xi_n \to f * \Delta^{\frac{1}{2}}\xi$ uniformly (by a simple case of Lemma 1.1). Since $f * \Delta^{\frac{1}{2}}\xi_n \to \eta$ in $L^2(G)$, we conclude that $\eta = f * \Delta^{\frac{1}{2}}\xi$. Thus $\xi \in D(\mathcal{P}(f))$ and $\mathcal{P}(f)\xi = \eta$, so that $\mathcal{P}(f)$ is closed. Obviously, $\mathcal{K}(G) \subseteq D(\mathcal{P}(f))$. In all, we have shown that $\mathcal{P}(f)$ is closed, densely defined, and $(-\frac{1}{2})$-homogeneous, so that we are now in a position to apply Proposition 2.11.

Let $(\xi_i)_{i \in I}$ be an approximate identity in $\mathcal{K}(G)_+$. Then

$$\mathcal{P}(f)\xi_i = f * \Delta^{\frac{1}{2}}\xi_i \to f \quad \text{in} \quad L^2(G).$$

Thus $\|\mathcal{P}(f)\xi_i\| \to \|f\|_2$. By Proposition 2.11 we conclude that $\mathcal{P}(f) \in L^2(\psi_0)$ and that

$$\|\mathcal{P}(f)\|_2 = \|f\|_2.$$

(2) the map $\mathcal{P}$ is linear: if $f_1, f_2 \in L^2(G)$, then $[\mathcal{P}(f_1) + \mathcal{P}(f_2)]$ and $\mathcal{P}(f_1 + f_2)$ obviously agree on $\mathcal{K}(G)$ and therefore by Proposition 2.15, we have

$$\mathcal{P}(f_1 + f_2) = [\mathcal{P}(f_1) + \mathcal{P}(f_2)].$$

Now, to prove that $\mathcal{P}$ is surjective, let $T \in L^2(\psi_0)$. We shall show that there exists a function $f \in L^2(G)$ such that $T = \mathcal{P}(f)$. Let $(\varepsilon_i)_{i \in T}$ be an approximation identity in $\mathcal{K}(G)_+$. Then for all $\eta, \zeta \in \mathcal{K}(G)$ we have

$$\begin{aligned}(\eta * \Delta^{-\frac{1}{2}}\tilde{\zeta}|T\xi_i) &= (\eta|(T\xi_i) * \Delta^{\frac{1}{2}}\zeta) \\ &= (\eta|T(\xi_i * \zeta)) \\ &= (T * \eta|\xi_i * \zeta) \\ &\quad \to (T^*\eta|\zeta) = (\eta|T\zeta)\end{aligned}$$

where we have used the $(-\dfrac{1}{2})$-homogeneity of T and the fact that $\mathcal{K}(G) \subseteq D(T^*)$ since $T^* \in L^2(\psi_0)$. Thus we can define a linear functional F on the dense subspace $\mathcal{K}(G) * \mathcal{K}(G)$ of $L^2(G)$ by

$$F(\xi) = \lim_i (\xi|T\xi_i).$$

Since

$$|(\xi|T\xi_i)| \leq \|\xi\|_2 \|T\xi_i\|_2 \leq \|\xi\|_2 \|T\|_2 \|\lambda(\xi_i)\| \leq \|T\|_2 \|\xi\|_2,$$

this functional is bounded and therefore is given by some $f \in L^2(G)$:

$$\forall\, \xi \in \mathcal{K}(G) * \mathcal{K}(G) : F(\xi) = (\xi|f).$$

In particular, we have

$$(\eta|T\zeta) = F(\eta * \Delta^{-\frac{1}{2}}\tilde{\zeta}) = (\eta * \Delta^{-\frac{1}{2}}\tilde{\zeta}|f)$$

for all $\eta, \zeta \in \mathcal{K}(G)$. Since

$$(\eta * \Delta^{-\frac{1}{2}}\tilde{\zeta}|f) = (\eta|f * \Delta^{\frac{1}{2}}\zeta) = (\eta|\mathcal{P}(f)\zeta),$$

this implies

$$\forall\, \zeta \in \mathcal{K}(G) : T\zeta = \mathcal{P}(f)\zeta,$$

and we conclude, by Proposition 2.15, that $T = \mathcal{P}(f)$. □

Proposition 3.3. 1) *For all $T \in M$ and all $f \in L^2(G)$, we have*

$$\mathcal{P}(Tf) = [T\mathcal{P}(f)].$$

2) *For all $f \in L^2(G)$, we have*

$$\mathcal{P}(Jf) = \mathcal{P}(f)^*.$$

Proof. 1) Let $f \in L^2(G)$ and $T \in M$. Then $[T\mathcal{P}(f)]$ and $\mathcal{P}(Tf)$ both belong to $L^2(\psi_0)$, and for all $\xi \in \mathcal{K}(G)$ we have

$$\mathcal{P}(Tf)\xi = (Tf) * \Delta^{\frac{1}{2}}\xi = T(f * \Delta^{\frac{1}{2}}\xi) = [T\mathcal{P}(f)]\xi,$$

since T commutes with right convolution. By Proposition 2.15 we conclude that $\mathcal{P}(Tf) = [T\mathcal{P}(f)]$.

2) Let $f \in L^2(G)$. Then for all $\xi, \eta \in \mathcal{K}(G)$ we have

$$\begin{aligned}(\mathcal{P}(Jf)\xi|\eta) &= (Jf * \Delta^{\frac{1}{2}}\xi|\eta)\\ &= (Jf|\eta * \Delta^{-\frac{1}{2}}\tilde{\xi})\\ &= (J(\eta * \Delta^{-\frac{1}{2}}\tilde{\zeta})|f)\\ &= (\xi * \Delta^{-\frac{1}{2}}\tilde{\eta}|f)\\ &= (\xi|f * \Delta^{\frac{1}{2}}\eta)\\ &= (\xi|\mathcal{P}(f)\eta),\end{aligned}$$

so that $\mathcal{P}(Jf)|_{\mathcal{K}(G)} \subseteq (\mathcal{P}(f)|_{\mathcal{K}(G)})^* = [\mathcal{P}(f)|_{\mathcal{K}(G)}]^* = \mathcal{P}(f)^*$ (since $\mathcal{P}(f) = [\mathcal{P}(f)|_{\mathcal{K}(G)}]$). We conclude by Proposition 2.15 that $\mathcal{P}(Jf) = \mathcal{P}(f)^*$. $\square$

Proposition 3.4. *Let $f \in L^2(G)$. Then $\mathcal{P}(f) \geq 0$ if and only if*

$$\int f(x)(\xi * J\xi)(x)dx \geq 0$$

for all $\xi \in \mathcal{K}(G)$.

Proof. . For all $\xi \in \mathcal{K}(G)$ we have

$$\int f(x)(\xi * J\xi)(x)dx = (f|\overline{\xi} * \Delta^{-\frac{1}{2}}\check{\xi}) = (f * \overline{\xi}|\overline{\xi}) = (\mathcal{P}(f)\overline{\xi}|\overline{\xi}).$$

Since $\mathcal{P}(f) = [\mathcal{P}(f)|_{\mathcal{K}(G)}]$, we have $\mathcal{P}(f) \geq 0$ if and only if $(\mathcal{P}(f)\eta|\eta) \geq 0$ for all $\eta \in \mathcal{K}(G)$, and the result follows. $\square$

By [10, Theorem 1.21 (3)] (or, to be precise, its spatial analogue obtained by the methods of [12, §1] connecting abstract [10] and spatial [12] L^p spaces), $L^2(\psi_0)_+$ is a selfdual cone in $L^2(\psi_0)$. By Proposition 3.4 and the unitarity of $\mathcal{P}$ we conclude that

$$P_0 = \{f \in L^2(G) \mid \forall \xi \in \mathcal{K}(G) : \int f(x)(\xi * J\xi)(x) \geq 0\}$$

is a selfdual cone in $L^2(G)$. Denote by P the ordinary selfdual cone in $L^2(G)$ associated with the achieved left Hilbert algebra $\mathfrak{A}_l \cap \mathfrak{A}_l^*$, i.e. let P be the closure in $L^2(G)$ of the set $\{\lambda(\xi)(J\xi)|\xi \in \mathfrak{A}_l \cap \mathfrak{A}_l^*\}$ (see [8, §1]). Since P is selfdual, we have

$$P = \{f \in L^2(G) \mid \forall \xi \in \mathfrak{A}_l \cap \mathfrak{A}_l^* : (f|\lambda(\xi)(J\xi)) \geq 0\}.$$

Thus $P \subseteq P_0$. Since P and P_0 are both selfdual, this implies that $P = P_0$. We have proved

Corollary 3.5. *A function $f \in L^2(G)$ belongs to the positive selfdual cone of $L^2(G)$ if and only if*

$$\forall \xi \in \mathcal{K}(G) : \int f(x)(\xi * J\xi)(x)dx \geq 0.$$

Remark 3.6. This result is similar to the characterization of the cone $P^\flat$ given in [18, p. 392] and proved in general in [9, Corollary 8]. The methods of [9] would also apply for our result. Our proof is based on the fact that $\mathcal{P}(f) = [\mathcal{P}(f)|_{\mathcal{K}(G)}]$.

Note 3.7. We have proved that $\mathcal{P} : L^2(G) \to L^2(\psi_0)$ carries the left regular representation on $L^2(G)$ into left multiplication on $L^2(\psi_0)$, takes J into $*$, and maps the positive selfdual cone of $L^2(G)$ onto $L^2(\psi_0)_+$. That a unitary transformation $L^2(G) \to L^2(\psi_0)$ having these properties exists (and is unique) also follows from [8, Theorem 2.3], since both representations of M are standard (by the spatial analogue of [10, Theorem 1.21, (3)]). In our approach, we have given a simple and direct definition of $\mathcal{P}$.

We can give an explicit description of the inverse of $\mathcal{P}$:

Proposition 3.8. *Let $T \in L^2(\psi_0)$, and let $(\xi_i)_{i\in I}$ be an approximate identity in $\mathcal{K}(G)_+$. Then*
$$\mathcal{P}^{-1}(T) = \lim_{i\in I} T\xi_i.$$

Proof. Let $f \in \mathcal{P}^{-1}(T)$. Then
$$T\xi_i = \mathcal{P}(f)\xi_i = f * \Delta^{\frac{1}{2}}\xi_i \to f$$
in $L^2(G)$. □

Remark 3.9. From Proposition 2.11 we already knew that for any approximate identity $(\xi_i)_{i\in I}$, the $\|T\xi_i\|$ tend to a limit and that this limit is independent of the choice of $(\xi_i)_{i\in I}$. Now, using that $L^2(\psi_0) = \mathcal{P}(L^2(G))$, we have proved that the same holds for the $T\xi_i$ themselves.

As a corollary, we have the following characterization of the inner product in $L^2(\psi_0)$, generalizing the formula for $\|T\|_2$ given in Proposition 2.11:

Corollary 3.10. *Let $T, S \in L^2(\psi_0)$. Then*
$$(T|S)_{L^2(\psi_0)} = \lim_{i\in I}(T\xi_i|S\xi_i)$$
for any approximate identity $(\xi_i)_{i\in I}$ in $\mathcal{K}(G)_+$.

Proof. Since $\mathcal{P}$ is unitary, we have
$$(T|S)_{L^2(\psi_0)} = (\mathcal{P}^{-1}(T)|\mathcal{P}^{-1}(S))_{L^2(G)} = \lim_{i\in I}(T\xi_i|S\xi_i)_{L^2(G)}.$$
□

4. The L^p Fourier transformations

Let $p \in [1, 2]$ and define $q \in [2, \infty]$ by $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

Definition 4.1. Let $f \in L^p(G)$. The L^p Fourier transform of f is the operator $\mathcal{F}_p(f)$ on $L^2(G)$ given by
$$\mathcal{F}_p(f)\xi = f * \Delta^{\frac{1}{q}}\xi,\ \ \xi \in D(\mathcal{F}_p(f)),$$
where $D(\mathcal{F}_p(f)) = \{\xi \in L^2(G) \mid f * \Delta^{\frac{1}{q}}\xi \in L^2(G)\}$.

Note that by Lemma 1.1 the convolution product $f * \Delta^{\frac{1}{q}}\xi$ exists and belongs to $L^r(G)$, where $r \in [2, \infty]$ is given by $\dfrac{1}{p} + \dfrac{1}{2} - \dfrac{1}{r} = 1$, whenever $f \in L^p(G)$ and $\xi \in L^2(G)$, so that the definition of $D(\mathcal{F}_p(f))$ makes sense.

Remark 4.2. For $p=1$, we write $\mathcal{F}_1=\mathcal{F}$; we have $\mathcal{F}(f)\xi=f*\xi$ and $D(\mathcal{F}(f))=L^2(G)$, so that $\mathcal{F}(f)$ is simply $\lambda(f)$. For $p=2$, we have $\mathcal{F}_2(f)=\mathcal{P}(f)$.

Now again let $p\in[1,2]$. Let $f\in L^p(G)$. Then the operator $\mathcal{F}_p(f)$ is closed. To see this, suppose that $\xi_i\in D(\mathcal{F}_p(f))$ converges in $L^2(G)$ to some $\xi\in L^2(G)$ and $\mathcal{F}_p(f)\xi_i$ converges in $L^2(G)$ to some $\eta\in L^2(G)$. Now by Lemma 1.1 we have $\mathcal{F}_p(f)\xi_i=f*\Delta^{\frac{1}{q}}\xi_i\to f*\Delta^{\frac{1}{q}}\xi$ in $L^r(G)$ (where $\frac{1}{p}+\frac{1}{2}-\frac{1}{r}=1$). Therefore $f*\Delta^{\frac{1}{q}}\xi=\eta$, so that $f*\Delta^{\frac{1}{q}}\xi\in L^2(G)$, i.e. $\xi\in D(\mathcal{F}_p(f))$ and $\mathcal{F}_p(f)\xi=\eta$ as wanted.

Next we show that $\mathcal{F}_p(f)$ is $(-\frac{1}{q})$-homogeneous. For all $\xi\in D(\mathcal{F}_p(f))$ and all $x,y\in G$ we have

$$\begin{aligned}
\rho(x)(\mathcal{F}_p(f)\xi)(y)&=\Delta^{\frac{1}{2}}(x)(f*\Delta^{\frac{1}{q}}\xi)(yx)\\
&=\Delta^{\frac{1}{2}}(x)\int f(z)\Delta^{\frac{1}{q}}(z^{-1}yx)\xi(z^{-1}yx)dz\\
&=\Delta^{\frac{1}{q}}(x)\int f(z)\Delta^{\frac{1}{q}}(z^{-1}y)\Delta^{\frac{1}{2}}(x)\xi(z^{-1}yx)dz\\
&=\Delta^{\frac{1}{q}}(x)\int f(z)\Delta^{\frac{1}{q}}(z^{-1}y)(\rho(x)\xi)(z^{-1}y)dz\\
&=\Delta^{\frac{1}{q}}(x)(f*\Delta^{\frac{1}{q}}\rho(x)\xi)(y)\\
&=\Delta^{\frac{1}{q}}(x)(\mathcal{F}_p(f)\rho(x)\xi)(y),
\end{aligned}$$

i.e.

$$\rho(x)\mathcal{F}_p(f)\subseteq\Delta^{\frac{1}{q}}(x)\mathcal{F}_p(f)\rho(x)$$

for all $x\in G$ as wanted.

Finally, note that if $\xi\in L^2(G)\cap L^s(G)$ where $s\in[1,2]$ is given by $\frac{1}{p}+\frac{1}{s}-\frac{1}{2}=1$, then $\xi\in D(\mathcal{F}_p(f))$ by Lemma 1.1. In particular, $\mathcal{K}(G)\subseteq D(\mathcal{F}_p(f))$. In all, we have proved that for all $f\in L^p(G)$, $\mathcal{F}_p(f)$ is closed, densely defined, and $(-\frac{1}{q})$-homogeneous. We shall see, using the criterion from Proposition 2.12, that actually $\mathcal{F}_p(f)\in L^q(\psi_0)$. The proof is based on interpolation from the special cases

$$\mathcal{F}:L^1(G)\to L^\infty(\psi_0)$$

and

$$\mathcal{P}:L^2(G)\to L^2(\psi_0).$$

First we restrict our attention to $f\in\mathcal{K}(G)$

Lemma 4.3. *Let $p\in[1,2]$. Denote by A the closed strip $\{\alpha\in\mathbb{C}\mid\frac{1}{2}\leq\mathrm{Re}(\alpha)\leq 1\}$. Let $f\in\mathcal{K}(G)$ and $\xi\in\mathfrak{A}_l$. Then:*

(*i*) *for each $\alpha\in A$, the convolution product*

$$\xi_\alpha=\mathrm{sg}(f)|f|^{p\alpha}*\Delta^{1-\alpha}\xi$$

exists, and $\xi_\alpha\in L^2(G)$;

(*ii*) *the function*

$$\alpha \mapsto \xi_\alpha,\ \ \alpha \in A,$$

with values in $L^2(G)$ *is bounded;*

(*iii*) *for each* $\eta \in L^2(G)$*, the scalar function*

$$\alpha \mapsto (\xi_\alpha|\eta),\ \ \alpha \in A,$$

is continuous on A *and analytic in the interior of* A*.*

Proof. Write $g = \Delta^{-1/p}\check{f}$. Then

$$\forall \alpha \in A : \mathrm{sg}(f)|f|^{p\alpha} = \Delta^{-\alpha}(\mathrm{sg}(g)|g|^{p\alpha})^\vee.$$

Note that g as well as all $\mathrm{sg}(g)|g|^{p\alpha}, \alpha \in A$, belong to $\mathcal{K}(G)$.

For each $\eta \in \mathcal{K}(G)$, we define

$$H_\eta(\alpha) = \int \xi(x)(\mathrm{sg}(g)|g|^{p\alpha} * \Delta^{1-\alpha}\eta)(x)dx, \quad \alpha \in A, \tag{4.1}$$

i.e.

$$H_\eta(\alpha) = \int\int \xi(x)(\mathrm{sg}(g)|g|^{p\alpha})(y)\Delta^{1-\alpha}(y^{-1}x)\eta(y^{-1}x)dydx \tag{4.2}$$

(later we shall recognize $H_\eta(\alpha)$ as simply $(\xi_\alpha|\overline{\eta})$).

Note that

$$\begin{aligned}\forall \alpha \in A :&\| \, |\mathrm{sg}(g)|g|^{p\alpha}| * |\Delta^{1-\alpha}\eta|\, \|_2 \\ &\le \| \, |g|^{p\mathrm{Re}(\alpha)}\|_1 \|\Delta^{1-\mathrm{Re}(\alpha)}|\eta|\, \|_2 \\ &\le K < \infty,\end{aligned} \tag{4.3}$$

where K is a constant independent of $\alpha \in A$. In particular, this allows us to apply Fubini's theorem to the double integral (4.2). We find

$$\begin{aligned}H_\eta(\alpha) &= \int\int \xi(x)(\mathrm{sg}(g)|g|^{p\alpha})(y^{-1})\Delta^{1-\alpha}(yx)\eta(yx)\Delta^{-1}(y)dydx \\ &= \int\int \xi(y^{-1}x)(\mathrm{sg}(g)|g|^{p\alpha})(y^{-1})\Delta^{1-\alpha}(x)\eta(x)\Delta^{-1}(y)dxdy \\ &= \int\int (\mathrm{sg}(f)|f|^{p\alpha})(y)\Delta^{1-\alpha}(y^{-1}x)\xi(y^{-1}x)\eta(x)dydx;\end{aligned}$$

it also follows that the convolution integral

$$\xi_\alpha(x) = \int (\mathrm{sg}(f)|f|^{p\alpha})(y)\Delta^{1-\alpha}(y^{-1}x)\xi(y^{-1}x)dy$$

exists, so that we can write

$$H_\eta(\alpha) = \int \xi_\alpha(x)\eta(x)dx.$$

Now we shall prove that there exists a constant $C \ge 0$ independent of α such that

$$\forall \eta \in \mathcal{K}(G) : \left| \int \xi_\alpha(x)\eta(x)dx \right| \le C\|\eta\|_2. \tag{4.4}$$

This will imply that each $\xi_\alpha,\ \alpha \in A$, is in $L^2(G)$ with $\|\xi_\alpha\|_2 \le C$, i.e. (*i*) and (*ii*) will be proved.

Let us prove (4.4). Without loss of generality, we may assume that $\|f\|_p = 1$. We want to show then that

$$\forall \eta \in \mathcal{K}(G) : |H_\eta(\alpha)| \leq (\|\lambda(\xi)\| + \|\xi\|_2)\|\eta\|_2. \tag{4.5}$$

To do this, we shall apply the Phragmen–Lindelöf principle [24, p.93].

Fix $\eta \in \mathcal{K}(G)$. By (4.2), H_η is continuous on A and analytic in the interior of A (the integrand in (4.2) can be majorized by an integrable function that is independent of α). Furthermore, H_η is bounded (use (4.3) and (4.1)). Finally, we shall estimate H_η on the boundaries of A.

Let $t \in \mathbb{R}$. Then $\Delta^{-it}\xi \in \mathfrak{A}_l$ and $\|\lambda(\Delta^{-it}\xi)\| \leq \|\lambda(\xi)\|$.

Now

$$\begin{aligned}&\mathcal{P}(\mathrm{sg}(f)|f|^{p(\frac{1}{2}+it)})(\Delta^{-it}\xi)\\ &\qquad = \mathrm{sg}(f)|f|^{p(\frac{1}{2}+it)} * \Delta^{1-(\frac{1}{2}+it)}\xi = \xi_{\frac{1}{2}+it},\end{aligned}$$

so that $\xi_{\frac{1}{2}+it} \in L^2(G)$ with

$$\begin{aligned}\|\xi_{\frac{1}{2}+it}\|_2 &\leq \|\mathcal{P}(\mathrm{sg}(f)|f|^{p(\frac{1}{2}+it)})\|_2 \|\lambda(\Delta^{-it}\xi)\|\\ &\leq \|\mathrm{sg}(f)|f|^{p(\frac{1}{2}+it)}\|_2 \|\lambda(\xi)\|\\ &= \|\,|f|^{\frac{p}{2}}\|_2 \|\lambda(\xi)\|\\ &= \|\lambda(\xi)\|\end{aligned}$$

(where we have used Proposition 2.11, the fact that $\mathcal{P}$ is unitary, and the hypothesis $\|f\|_p = 1$). Similarly,

$$\mathcal{F}(\mathrm{sg}(f)|f|^{p(1+it)})(\Delta^{-it}\xi) = \mathrm{sg}(f)|f|^{p(1+it)} * \Delta^{1-(1+it)}\xi = \xi_{1+it},$$

so that $\xi_{1+it} \in L^2(G)$ with

$$\begin{aligned}\|\xi_{1+it}\|_2 &\leq \|\mathcal{F}(\mathrm{sg}(f)|f|^{p(1+it)})\|_\infty \|\Delta^{-it}\xi\|_2\\ &\leq \|\mathrm{sg}(f)|f|^{p(1+it)}\|_1 \|\xi\|_2\\ &= \|\,|f|^p\|_1 \|\xi\|_2\\ &= \|\xi\|_2\end{aligned}$$

(where we have used that $\mathcal{F} : L^1(G) \to L^\infty(\psi_0)$ is norm-decreasing).

It follows that

$$\begin{aligned}\forall t \in \mathbb{R} : |H_\eta(\frac{1}{2} + it)| &= |\int \xi_{\frac{1}{2}+it}(x)\eta(x)dx|\\ &\leq \|\xi_{\frac{1}{2}+it}\|_2 \|\eta\|_2 \leq \|\lambda(\xi)\|\,\|\eta\|_2\end{aligned}$$

and

$$\begin{aligned}\forall t \in \mathbb{R} : |H_\eta(1 + it)| &= |\int \xi_{1+it}(x)\eta(x)dx|\\ &\leq \|\xi_{1+it}\|_2 \|\eta\|_2 \leq \|\xi\|_2 \|\eta\|_2.\end{aligned}$$

Then by the Phragmen–Lindelöf principle, we have established (4.5) and thus (i) and (ii).

Finally, (iii) is easy. Indeed, since $\alpha \mapsto \xi_\alpha$ as is bounded, each $\alpha \mapsto (\xi_\alpha|\eta)$, where $\eta \in L^2(G)$, can be uniformly approximated by functions $\alpha \mapsto (\xi_\alpha|\zeta)$ with $\zeta \in \mathcal{K}(G)$, so we just have to prove (iii) in the case of $\eta \in \mathcal{K}(G)$. This is already done since $(\xi_\alpha|\eta) = H_{\overline{\eta}}(\alpha)$. □

Lemma 4.4. *Let $p \in [1,2]$. Let $f \in \mathcal{K}(G)$ and $S \in L^p(\psi_0)$. Then for all $\xi \in \mathfrak{A}_l$ and $\eta \in \mathfrak{A}_l \cap D(S)$ we have*

$$|(\mathcal{F}_p(f)\xi|S\eta)| \leq \|f\|_p \|S\|_p \|\lambda(\xi)\| \, \|\lambda(\eta)\|.$$

Note that $\xi \in D(\mathcal{F}_p(f))$ by Lemma 4.3.

Proof. We may assume that $\|f\|_p = 1$ and $\|S\|_p = 1$. Furthermore, by Lemma 2.9, we need only consider $\eta \in \mathfrak{A}_l \cap D(|S|^p)$.

Let $\xi \in \mathfrak{A}_l$ and $\eta \in \mathfrak{A}_l \cap D(|S|^p)$. For each α in the closed strip $A = \{\alpha \in \mathbb{C} | \frac{1}{2} \leq \mathrm{Re}(\alpha) \leq 1\}$, put $\xi_\alpha = \mathrm{sg}(f)|f|^{p\alpha} * \Delta^{1-\alpha}\xi$ as in Lemma 4.3. Note that for all $\alpha \in A$ we have (by the spectral theory) $\eta \in D(U|S|^{p\alpha})$ and

$$\|U|S|^{p\alpha}\eta\|_2^2 \leq \| \, |S|^{\frac{p}{2}}\eta\|_2^2 + \| \, |S|^p\eta\|_2^2,$$

where $S = U|S|$ is the polar decomposition of S. For each $\alpha \in A$, put

$$\eta_\alpha = U|S|^{p\alpha}\eta.$$

Then the function $\alpha \mapsto \eta_\alpha$ with values in $L^2(G)$ is bounded on A. Furthermore, by [22, 9, 15], it is continuous on A and analytic in the interior of A.

Now for each $\alpha \in A$, let

$$H(\alpha) = (\xi_\alpha|\eta_{\overline{\alpha}}).$$

Then obviously H is bounded on A (by Lemma 4.3 (ii), $\alpha \mapsto \xi_\alpha$ is bounded). Furthermore, H is continuous on A. To see this, note that

$$\forall \alpha, \alpha_0 \in A : (\xi_\alpha|\eta_{\overline{\alpha}}) - (\xi_{\alpha_0}|\eta_{\overline{\alpha}_0}) = (\xi_\alpha|\eta_{\overline{\alpha}} - \eta_{\overline{\alpha}_0}) + (\xi_\alpha - \xi_{\alpha_0}|\eta_{\overline{\alpha}_0}),$$

the continuity follows since $\alpha \mapsto \xi_\alpha$ is bounded and weakly continuous (Lemma 4.3 (iii)). Finally, we claim that H is analytic in the interior of A. First note that for each $\zeta \in L^2(G)$ the function $\alpha \mapsto (\zeta|\eta_{\overline{\alpha}})$, being equal to $\alpha \mapsto \overline{(\eta_{\overline{\alpha}}|\zeta)}$, is analytic. Next, recall that $\alpha \mapsto \xi_\alpha$ is actually analytic as a function with values is $L^2(G)$ (by Lemma 4.3 (iii) and [19, Theorem 3.31]). Then, writing

$$\frac{(\xi_\alpha|\eta_{\overline{\alpha}}) - (\xi_{\alpha_0}|\eta_{\overline{\alpha}_0})}{\alpha - \alpha_0} = (\frac{1}{\alpha - \alpha_0}(\xi_\alpha - \xi_{\alpha_0})|\eta_{\overline{\alpha}}) + \frac{(\xi_{\alpha_0}|\eta_{\overline{\alpha}}) - (\xi_{\alpha_0}|\eta_{\overline{\alpha}_0})}{\alpha - \alpha_0},$$

we find that H has a derivative at each point α_0 in the interior of A.

Now suppose that

$$\forall t \in \mathbb{R} : |H(\frac{1}{2} + it)| \leq \|\lambda(\xi)\| \, \|\lambda(\eta)\| \tag{4.6}$$

and

$$\forall t \in \mathbb{R} : |H(1 + it)| \leq \|\lambda(\xi)\| \, \|\lambda(\eta)\|. \tag{4.7}$$

Then by the Phragmen–Lindelöf principle [24, p. 93] we infer that

$$\forall \alpha \in A : |H(\alpha)| \leq \|\lambda(\xi)\| \, \|\lambda(\eta)\|,$$

in particular,

$$|(\mathcal{F}_p(f)\xi|S\eta)| \le \|\lambda(\xi)\|\,\|\lambda(\eta)\|$$

as desired, since

$$H(\frac{1}{p}) = (f * \Delta^{1-\frac{1}{p}}\xi|U|S|\eta) = (\mathcal{F}_p(f)|S\eta).$$

So we just have to prove (4.6) and (4.7).

Since $S \in L^p(\psi_0)$ with $\|S\|_p = 1$ we have

$$U|S|^{\frac{p}{2}} \in L^2(\psi_0) \quad \text{with} \quad \|U|S|^{\frac{p}{2}}\|_2 = 1 \tag{4.8}$$

and

$$U|S|^p \in L^1(\psi_0) \quad \text{with} \quad \|U|S|^p\|_1 = 1. \tag{4.9}$$

Now let $t \in \mathbb{R}$. Then by Lemma 2.5, we have

$$|S|^{-pit}\eta \in \mathfrak{A}_l \quad \text{with} \quad \|\lambda(|S|^{-pit}\eta)\| \le \|\lambda(\eta)\|. \tag{4.10}$$

Using this, Proposition 2.11, the estimate $\|\xi_{\frac{1}{2}+it}\|_2 \le \|\lambda(\xi)\|$ given in the proof of Lemma 4.3, and (4.8), we get

$$\begin{aligned}
|H(\tfrac{1}{2}+it)| &= |(\xi_{\frac{1}{2}+it}|U|S|^{\frac{p}{2}}|S|^{-pit}\eta)| \\
&\le \|\xi_{\frac{1}{2}+it}\|_2 \|U|S|^{\frac{p}{2}}|S|^{-pit}\eta\|_2 \\
&\le \|\lambda(\xi)\|\,\|U|S|^{\frac{p}{2}}\|_2 \|\lambda(|S|^{-pit}\eta)\|_2 \\
&\le \|\lambda(\xi)\|\,\|\lambda(\eta)\|,
\end{aligned}$$

i.e. (4.6) is proved. To prove (4.7), note that

$$\begin{aligned}
\xi_{1+it} &= \operatorname{sg}(f)|f|^{p(1+it)} * \Delta^{1-(1+it)}\xi \\
&= \lambda(\operatorname{sg}(f)|f|^{p(1+it)})\Delta^{-it}\xi \in \mathfrak{A}_l
\end{aligned}$$

and

$$\begin{aligned}
\|\lambda(\xi_{1+it})\| &\le \|\lambda(\operatorname{sg}(f)|f|^{p(1+it)})\|\,\|\lambda(\Delta^{-it}\xi)\| \\
&\le \|\operatorname{sg}(f)|f|^{p(1+it)}\|_1 \|\lambda(\xi)\| \\
&\le \|\lambda(\xi)\|,
\end{aligned}$$

since $\|\operatorname{sg}(f)|f|^{p(1+it)}\|_1 = \|\,|f|^p\|_1 = 1$. Using this together with (4.10), Proposition 2.10, and (4.9), we find

$$\begin{aligned}
|H(1+it)| &= |(\xi_{1+it}|U|S|^p|S|^{-pit}\eta)| \\
&\le \|\lambda(\xi_{1+it})\|\,\|U|S|^p\|_1 \|\lambda(|S|^{-pit}\eta)\| \\
&\le \|\lambda(\xi)\|\,\|\lambda(\eta)\|,
\end{aligned}$$

so that (4.7) is proved. □

In the formulation of the following theorem we include the case $p = 2$. Note however that the proof is based on the results for this special case (they were used for the preceding lemmas).

Theorem 4.5 (Hausdorff–Young). *Let $p \in]1,2]$ and $\frac{1}{p}+\frac{1}{q}=1$.*

1) *Let $f \in L^p(G)$. Then $\mathcal{F}_p(f) \in L^q(\psi_0)$ and*
$$\|\mathcal{F}_p(f)\|_q \leq \|f\|_p.$$
2) *The mapping*
$$\mathcal{F}_p : L^p(G) \to L^q(\psi_0)$$
is linear, norm-decreasing, injective, and has dense range.
3) *For all $h \in L^1(G)$ and $f \in L^p(G)$, we have*
$$\mathcal{F}_p(h * f) = [\lambda(h)\mathcal{F}_p(f)].$$
4) *For all $f \in L^p(G)$, we have*
$$\mathcal{F}_p(J_p f) = \mathcal{F}_p(f)^*.$$

Proof. 1) First suppose that $f \in \mathcal{K}(G)$. Then, using Proposition 2.12, we conclude from Lemma 4.4 that $\mathcal{F}_p(f) \in L^q(\psi_0)$ with $\|\mathcal{F}_p(f)\|_q \leq \|f\|_p$. Thus we have defined a norm-decreasing mapping
$$\mathcal{F}_p|_{\mathcal{K}(G)} : L^p(G) \to L^q(\psi_0).$$
Furthermore $\mathcal{F}_p|_{\mathcal{K}(G)}$ is linear: for all $f_1, f_2 \in \mathcal{K}(G)$ and all $\xi \in \mathcal{K}(G)$ we have
$$(f_1+f_2) * \Delta^{\frac{1}{q}}\xi = f_1 * \Delta^{\frac{1}{q}}\xi + f_2 * \Delta^{\frac{1}{q}}\xi$$
so that $\mathcal{F}_p(f_1+f_2) = [\mathcal{F}_p(f_1)+\mathcal{F}_p(f_2)]$ by Proposition 2.15. Now $\mathcal{F}_p|_{\mathcal{K}(G)}$ extends by continuity to a norm-decreasing linear mapping
$$\mathcal{F}'_p : L^p(G) \to L^q(\psi_0).$$
We claim that for all $f \in L^p(G)$, we have
$$\mathcal{F}'_p(f) = \mathcal{F}_p(f).$$
this will prove 1).

Let $f \in L^p(G)$. Then $\mathcal{F}'_p(f) \in L^q(\psi_0)$ and $\mathcal{K}(G) \subseteq D(\mathcal{F}'_p(f))$ by Lemma 2.16. On the other hand, by the remarks at the beginning of this section, $\mathcal{F}_p(f)$ is closed, densely defined, and $(-\frac{1}{q})$-homogeneous, an $\mathcal{K}(G) \subseteq D(\mathcal{F}_p(f))$. Thus ny Lemma 2.7, to conclude that $\mathcal{F}'_p(f) = \mathcal{F}_p(f)$ we just have to show that
$$\forall \xi \in \mathcal{K}(G) : \ \mathcal{F}'_p(f)\xi = \mathcal{F}_p(f)\xi.$$
Now, take $f_n \in \mathcal{K}(G)$ such that $f_n \to f$ in $L^p(G)$. Then for all $\xi \in \mathcal{K}(G)$, we have
$$\begin{aligned}\mathcal{F}_p(f_n)\xi &= f_n * \Delta^{\frac{1}{q}}\xi \\ &\to f * \Delta^{\frac{1}{q}}\xi = \mathcal{F}_p(f)\xi \text{ in } L^p(G).\end{aligned}$$
On the other hand, since $\mathcal{F}'_p$ is continuous,
$$\mathcal{F}_p(f_n)\xi = \mathcal{F}'_p(f_n)\xi \to \mathcal{F}'_p(f)\xi \text{ in } L^2(G)$$
by Lemma 2.16. We conclude that $\mathcal{F}_p(f)\xi = \mathcal{F}'_p(f)\xi$ as desired. Thus 1) is proved.

2) By the proof of 1), we just have to show that $\mathcal{F}_p$ is injective and has dense range. The injectivity is evident: if $\mathcal{F}_p(f) = 0$ for some $f \in L^p(G)$, then $f * \Delta^{\frac{1}{q}}\xi = 0$ for all $\xi \in \mathcal{K}(G)$, and thus $f = 0$. That $\mathcal{F}_p(L^p(G))$ is dense will be proved later.

3) For all $h \in L^1(G)$, $f \in L^p(G)$, and $\xi \in \mathcal{K}(G)$ we have

$$h * (f * \Delta^{\frac{1}{q}}\xi) = (h * f) * \Delta^{\frac{1}{q}}\xi$$

(in $L^p(G)$). Thus by Proposition 2.15,

$$\lambda(h)\mathcal{F}_p(f)] = \mathcal{F}_p(h * f).$$

4)Let $f \in \mathcal{K}(G)$. Then for $\xi, \eta \in \mathcal{K}(G)$ we have

$$\begin{aligned}(\mathcal{F}_p(J_p f)\xi|\eta) &= (J_p f * \Delta^{\frac{1}{q}}\xi|\eta)\\ &= (\Delta^{\frac{1}{q}}\xi|\Delta^{-1}(J_p f)\breve{\ } * \eta)\\ &= (\xi|\Delta^{\frac{1}{q}}(\Delta^{-1}\Delta^{\frac{1}{p}} f * \eta))\\ &= (\xi|f * \Delta^{\frac{1}{q}}\eta)\\ &= (\xi|\mathcal{F}_p(f)\eta),\end{aligned}$$

so that $\mathcal{F}_p(J_p f)|_{\mathcal{K}(G)} \subseteq (\mathcal{F}_p(f)|_{\mathcal{K}(G)})^*$. By Proposition 2.15, we conclude that

$$\mathcal{F}_p(J_p f) = \mathcal{F}_p(f)^*.$$

By the continuity of J_p, $\mathcal{F}_p$, and $*$, this holds for all $f \in L^p(G)$.

Finally, let us show that $\mathcal{F}_p(L^p(G))$ is dense in $L^q(\psi_0)$. By the duality between $L^q(\psi_0)$ and $L^p(\psi_0)$, this is equivalent to proving that if $T \in L^p(\psi_0)$ satisfies $\int[\mathcal{F}_p(f)T]d\psi_0 = 0$ for all $f \in L^p(G)$ for all $f \in L^p(G)$, then $T = 0$.

Suppose that $T \in L^p(\psi_0)$ is such that

$$\forall f \in L^p(G) : \int[\mathcal{F}_p(f)T]d\psi_0 = 0.$$

Let $f \in L^p(G)$. Then for all $h \in L^1(G)$ we have

$$\int[\mathcal{F}_p(h * f)T]d\psi_0 = 0.$$

Alternatively stated, since $[\mathcal{F}_p(h * f)T] = [[\lambda(h)\mathcal{F}_p(f)]T] = [\lambda(h)[\mathcal{F}_p(f)T]]$, we have

$$\forall h \in L^1(G) : \int[\lambda(h)[\mathcal{F}_p(f)T]]d\psi_0 = 0.$$

We conclude that the normal functional on M defined by $[\mathcal{F}_p(f)T] \in L^1(\psi_0)$ is 0, so that

$$[\mathcal{F}_p(f)T] = 0.$$

Changing f into $J_p f$ and using 4) this gives

$$\forall f \in L^p(G) : \ [\mathcal{F}_p(f) * T] = 0.$$

Now let $\xi \in D(T)$. Then using [12, II, Proposition 5],[1] we find that

$$\begin{aligned}\forall f, \eta \in \mathcal{K}(G) : (T\xi | f * \Delta^{\frac{1}{q}}\eta) &= (T\xi | \mathcal{F}_p(f)\eta) \\ &= \langle [\mathcal{F}_p(f) * T], \lambda(\xi)\lambda(\eta)^* \rangle = 0.\end{aligned}$$

Thus $T\xi = 0$. This proves that $T = 0$ as wanted. □

Proposition 4.6. *Let $p \in [1, 2]$. Let $f \in L^p(G)$ and $\mathcal{F}_p(f) \geq 0$ if and only if*

$$\forall \xi \in \mathcal{K}(G) : \int f(x)(\xi * J_p\xi)(x)dx \geq 0.$$

Proof. We have

$$(\mathcal{F}_p(f)\xi|\xi) = \int (f * \Delta^{\frac{1}{p}}\xi)(x)\overline{\xi(x)}dx = \int f(x)(\overline{\xi} * \Delta^{-\frac{1}{p}}\check{\xi})(x)dx$$

for all $\xi \in \mathcal{K}(G)$. The result follows by changing ξ into $\overline{\xi}$ and recalling that $\mathcal{F}_p(f) = [\mathcal{F}_p(f)|_{\mathcal{K}(G)}]$. □

The L^p Fourier transformations are well-behaved with respect to convolution as the following proposition shows. The result generalizes 3) of theorem.

Proposition 4.7. *Let $p_1, p_2, p \in [1, 2]$ such that $\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p} = 1$. Define $q_1 \in [2, \infty]$ by $\frac{1}{p_1} + \frac{1}{q_1} = 1$. Let $f_1 \in L^{p_1}(G)$ and $f_2 \in L^{p_2}(G)$. Then*

$$\mathcal{F}_p(f_1 * \Delta^{\frac{1}{q_1}} f_2) = [\mathcal{F}_{p_1}(f_1)\mathcal{F}_{p_2}(f_2)].$$

Proof. By Lemma 1.1, we have $f_1 * \Delta^{\frac{1}{q_1}} f_2 \in L^p(G)$, and $(f_1, f_2) \mapsto \mathcal{F}_p(f_1 * \Delta^{\frac{1}{q_1}} f_2)$ maps $L^{p_1}(G) \times L^{p_2}(G)$ continuously into $L^q(\psi_0)$ (where $\frac{1}{p} + \frac{1}{q} = 1$). Also $[\mathcal{F}_{p_1}(f_1)\mathcal{F}_{p_2}(f_2)]$ is continuous as a function of $(f_1, f_2) \in L^{p_1}(G) \times L^{p_2}(G)$ with values in $L^q(\psi_0)$. Thus we need only prove the statement for $f_1, f_2 \in \mathcal{K}(G)$. Since

$$(f_1 * \Delta^{\frac{1}{q_1}} f_2) * \Delta^{\frac{1}{q_1}}\xi = f_1 * \Delta^{\frac{1}{q_1}}(f_2 * \Delta^{\frac{1}{q_2}}\xi)$$

(where $\frac{1}{p_2} + \frac{1}{q_2} = 1$) for all $f_1, f_2, \xi \in \mathcal{K}(G)$, the result follows by Proposition 2.15 as usual. □

We conclude this section by the following characterization of the image of $L^p(G)$ under $\mathcal{F}_p$:

Proposition 4.8. *Let $p \in]1, 2]$ and $\frac{1}{p} + \frac{1}{q} = 1$. Let $T \in L^q(\psi_0)$.*

1) *If $T = \mathcal{F}_p(f)$ for some $f \in L^p(G)$, then for any approximate identity $(\xi_i)_{i\in I}$ in $\mathcal{K}(G)_+$ we have*
$$T\xi_i \to f \text{ in } L^p(G).$$
In particular, $\lim_{i\in I} \|T\xi_i\|_p = \|f\|_p < \infty$.
2) *Conversely, suppose that for some approximate identity $(\xi_i)_{i\in I}$ in $\mathcal{K}(G)_+$ we have $T\xi_i \in L^p(G)$ for all $i \in I$ and*
$$\liminf_{i\in I} \|T\xi_i\|_p < \infty.$$
Then $T \in \mathcal{F}_p(L^p(G))$.

Proof. The first part is obvious since $T\xi_i = f * \Delta^{\frac{1}{q}}\xi_i \to f$ in $L^p(G)$ and therefore $\|T\xi_i\|_p \to \|f\|_p$. Now suppose that the hypothesis of 2) holds for some $(\xi_i)_{i\in I}$. We the proceed as in the proof of the surjectivity of $\mathcal{P}$ (Theorem 3.2). for all $\eta, \zeta \in \mathcal{K}(G)$ we have

$$\begin{aligned}(\eta * \Delta^{-\frac{1}{q}}\tilde{\zeta}|T\xi_i) =&(\eta|(T\xi_i) * \Delta^{\frac{1}{q}}\zeta)\\ =&(\eta|T(\xi_i * \zeta))\\ =&(T * \eta|\xi_i * \zeta)\\ &\to (T * \eta|\zeta) = (\eta|T\zeta).\end{aligned}$$

Thus we can define a linear functional F on $\mathcal{K}(G) * \mathcal{K}(G)$ by

$$F(\xi) = \lim_{i\in I} \int \xi(x)\overline{(T\xi_i)(x)}dx.$$

Since

$$\left|\int \xi(x)\overline{(T\xi_i)(x)}dx\right| \le \|\xi\|_q\|T\xi_i\|_p$$

we have

$$|F(\xi)| \le (\liminf_{i\in I} \|T\xi_i\|_p) \cdot \|\xi\|_q.$$

Now since $\mathcal{K}(G) * \mathcal{K}(G)$ is dense in $L^q(G)$, F extends to a bounded functional on $L^q(G)$ and therefore is given by some $\overline{f} \in L^p(G)$:

$$F(\xi) = \int \xi(x)\overline{f(x)}dx.$$

In particular,

$$(\eta|T\zeta) = F(\eta * \Delta^{-\frac{1}{q}}\tilde{\zeta}) = \int (\eta * \Delta^{-\frac{1}{q}}\tilde{\zeta})(x)\overline{f(x)}dx$$

for all $\eta, \zeta \in \mathcal{K}(G)$. Since

$$\int (\eta * \Delta^{-\frac{1}{q}}\tilde{\zeta})(x)\overline{f(x)}dx = \int \eta(x)\overline{(f * \Delta^{\frac{1}{q}}\zeta)(x)}dx = (\eta|\mathcal{F}_p(f)\zeta),$$

this implies that

$$\forall \zeta \in \mathcal{K}(G) : T\zeta = \mathcal{F}_p(f)\zeta,$$

and we conclude by Proposition 2.15 that $T = \mathcal{F}_p(f)$. □

Remark 4.9. For $p = 1$, part 2) of the above proposition fails. (for counter-example, take $T = \lambda(x)$, $x \in G$.)

5. The L^p Fourier contransformation

Definition 5.1. Let $p \in [1, 2]$ and $\frac{1}{p} + \frac{1}{q} = 1$. For each $T \in L^p(\psi_0)$, denote by $\overline{\mathcal{F}}_p(T)$ the unique function in $L^q(G)$ such that

$$\int h(x)\overline{\mathcal{F}}_p(T)(x)dx = \int [\mathcal{F}_p(h)T]d\psi_0$$

for all $h \in L^p(G)$ (or just $h \in \mathcal{K}(G)$, or $h \in \mathcal{K}(G)$, or $h \in \mathcal{K}(G) * \mathcal{K}(G)$). The mapping

$$\overline{\mathcal{F}}_p : L^p(\psi_0) \to L^q(G)$$

thus define will be called the L^p fourier transformation. For $p = 1$, we write $\overline{\mathcal{F}} = \overline{\mathcal{F}}_1$.

Note that if $1 < p \leq 2$, then $\overline{\mathcal{F}}_p$ is simply the transpose of $\mathcal{F}_p : L^p(G) \to L^q(\psi_0)$ when we identify the dual spaces of $L^p(G)$ and $L^q(\psi_0)$ with $L^q(G)$ and $L^p(\psi_0)$, respectively.

The mapping $\overline{\mathcal{F}}$ takes an element $T \in L^1(\psi_0)$ into the unique function $\varphi \in A(G)$ that defines the same element of M_* as T does; in particular,

$$\overline{\mathcal{F}}\left(\frac{d\varphi}{d\psi_0}\right) = \varphi$$

for all $\varphi \in (M_*)^+ \simeq A(G)_+$.

In view of these remarks, we obviously have

Theorem 5.2. 1) *Let $p \in]1, 2]$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\overline{\mathcal{F}}_p : L^p(\psi_0) \to L^q(G)$$

is linear, norm-decreasing, injective, and has dense range.

2) *The mapping*

$$\overline{\mathcal{F}} : L^1(\psi_0) \to A(G)$$

is an isometry of $L^1(\psi_0)$ onto $A(G)$.

Remark 5.3. With our definition of the contransformations, $\overline{\mathcal{F}}_2$ is not exactly the inverse of $\mathcal{P}$; they are related by the formula

$$\forall T \in L^2(\psi_0) : \overline{\mathcal{F}}_2(T) = \overline{\mathcal{P}^{-1}(T^*)}$$

(since for all $h \in L^2(G)$ we have

$$\begin{aligned}\int h(x)\overline{\mathcal{F}}_2(T)(x)dx &= \int [\mathcal{F}_2(h)T]d\psi_0 = (\mathcal{F}_2(h)|T^*)_{L^2(\psi_0)} \\ &= \left(h|\mathcal{P}^{-1}(T^*)\right)_{L^2(G)} = \int h(x)\overline{\mathcal{P}^{-1}(T^*)(x)}dx.\end{aligned}$$

It follows that $\overline{\mathcal{F}}_2 : L^2(\psi_0) \to L^2(G)$ is unitary.

The classical Hausdorff–Young theorem [24, p.101] has a second part, stating that with each $c \in l_p(\mathbb{Z})$, $1 \leq p \leq 2$, we can associate a function $f \in L^q(\mathbb{T})$ with $\|f\|_q \leq \|c\|_p$, such that c is the sequence of Fourier coefficients of f. Theorem 5.2 is a generalization of this result. Indeed, let $T \in L^p(\psi_0)$ and put $g = \Delta^{-\frac{1}{q}}\overline{\mathcal{F}}_p(T)\check{}$. Then $g \in L^q(G)$ and $\|g\|_q = \|\overline{\mathcal{F}}_p(T)\|_q \leq \|T\|_p$, and we shall see that T is close to being the "L^q Fourier transform" of g in the sense that $T\xi = g * \Delta^{\frac{1}{p}}\xi$ for certain ξ (note that we do not in general define L^q Fourier transforms for $q \geq 2$).

Proposition 5.4. *Let $p \in [1, 2]$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then for all $T \in L^p(\psi_0)$, we have*

$$\overline{\mathcal{F}}_p(T^*) = J_q(\overline{\mathcal{F}}_p(T)).$$

Proof. For all $h \in L^p(G)$ we have

$$\begin{aligned}\int h(x)\overline{\mathcal{F}}_p(T^*)(x)dx &= \int [\mathcal{F}_p(h)T^*]d\psi_0 \\ &= \overline{\int [T\mathcal{F}_p(h)^*]d\psi_0} = \overline{\int [T\mathcal{F}_p(J_p h)]d\psi_0} \\ &= \overline{\int \mathcal{F}_p(T)(x)\Delta^{-\frac{1}{q}}(x)\overline{h(x^{-1})}dx} \\ &= \int \Delta^{-\frac{1}{q}}(x)\overline{\mathcal{F}_p(T)(x^{-1})}h(x)dx.\end{aligned}$$

□

Lemma 5.5. *Let $h, k \in \mathcal{K}(G)$ and put $\varphi = h * \tilde{k}$. Then $[\lambda(\varphi)\Delta] \in L^1(\psi_0)$ and*

$$\int [\lambda(\varphi)\Delta]\psi_0 = \varphi(e).$$

Proof. Since

$$\begin{aligned}\lambda(\varphi)\Delta &= \lambda(h)\lambda(\tilde{k})\Delta^{\frac{1}{2}}\Delta^{\frac{1}{2}} \\ &\subseteq \lambda(h)\Delta^{\frac{1}{2}}\lambda(\Delta^{-\frac{1}{2}}\tilde{k})\Delta^{\frac{1}{2}} \subseteq \mathcal{P}(h)\mathcal{P}(k)^*,\end{aligned}$$

the closure $[\lambda(\varphi)\Delta]$ exists and $[\lambda(\varphi)\Delta] \subseteq [\mathcal{P}(h)\mathcal{P}(k)^*]$. One easily checks that for all $x \in G$ we have $\rho(x)\lambda(\varphi)\Delta \subseteq \Delta(x)\lambda(\varphi)\Delta\rho(x)$, i.e. that $\lambda(\varphi\Delta)$ is (-1)-homogeneous. Then also $[\lambda(\varphi)\Delta]$ is (-1)-homogeneous, and we conclude by Proposition 2.15 that $[\lambda(\varphi)\Delta] = [\mathcal{P}(h)\mathcal{P}(k)^*]$, so that $[\lambda(\varphi)\Delta] \in L^1(\psi_0)$ and

$$\begin{aligned}\int [\lambda(\varphi)\Delta]d\psi_0 &= (\mathcal{P}(h)|\mathcal{P}(k))_{L^2(\psi_0)} \\ &= \int h(x)\overline{k(x)}dx = (h * \tilde{k})(e) = \varphi(e).\end{aligned}$$

□

Suppose that $f_1 \in L^{p_1}(G)$ and $f_2 \in L^{p_2}(G)$, where $p_1, p_2 \in [1, 2]$. In Proposition 4.7, a formula relating $f_1 * \Delta^{\frac{1}{q_1}} f_2$ and $[\mathcal{F}_{p_1}(f_1)\mathcal{F}_{p_2}(f_2)]$ was given in the case where $\frac{1}{p_1} + \frac{1}{p_2} \geq \frac{3}{2}$ (under this assumption, $p \in [1, 2]$ satisfying $\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p} = 1$ exists). The following proposition takes care of the case where $\frac{1}{p_1} + \frac{1}{p_2} \leq \frac{3}{2}$.

Proposition 5.6. *Let $p_1, p_2 \in [1, 2]$ and $q \in [2, \infty]$ such that $\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{q} = 1$. Let $f_1 \in L^{p_1}(G)$ and $f_2 \in L^{p_2}(G)$. Then*

$$\overline{\mathcal{F}}_p([\mathcal{F}_{p_1}(f_1)\mathcal{F}_{p_2}(f_2)]) = \Delta^{-\frac{1}{q}}(f_1 * \Delta^{\frac{1}{q_1}} f_2)\check{},$$

where $\frac{1}{p} + \frac{1}{q}$ and $\frac{1}{p_1} + \frac{1}{q_1} = 1$.

Proof. Both expressions exist, belong to $L^q(G)$, and are continuous as functions of $(f_1, f_2) \in L^{p_1}(G) \times L^{p_2}(G)$. Thus we need only prove the formula for $f_1, f_2 \in \mathcal{K}(G)$. In this case, for all $h \in \mathcal{K}(G)$ and $\xi \in \mathcal{K}(G)$ we have

$$h * \Delta^{\frac{1}{q}}(f_1 * \Delta^{\frac{1}{q_1}}(f_2 * \Delta^{\frac{1}{q_2}}\xi)) = h * \Delta^{\frac{1}{q}}(f_1 * \Delta^{\frac{1}{q_1}} f_2) * \Delta\xi,$$

where $\frac{1}{p_2} + \frac{1}{q_2} = 1$. We conclude by Proposition 2.15 that

$$\forall h \in \mathcal{K}(G) : [\mathcal{F}_p(h)[\mathcal{F}_{p_1}(f_1)\mathcal{F}_{p_2}(f_2)]] = [\lambda(h * \Delta^{\frac{1}{q}} f)\Delta],$$

where we have written $f = f_1 * \Delta^{\frac{1}{q_1}} f_2$. Using this and Lemma 5.5, we find

$$\begin{aligned}\forall h \in \mathcal{K}(G) : &\int [\mathcal{F}_p(h)[\mathcal{F}_{p_1}(f_1)\mathcal{F}_{p_2}(f_2)]]d\psi_0 \\ &= \int [\lambda(h * \Delta^{\frac{1}{q}} f)\Delta] d\psi_0 \\ &= (h * \Delta^{\frac{1}{q}} f)(e) \\ &= \int h(x)\Delta^{\frac{1}{q}}(x^{-1})f(x^{-1})dx.\end{aligned}$$

We conclude that

$$\overline{\mathcal{F}}_p([\mathcal{F}_{p_1}(f_1)\mathcal{F}_{p_2}(f_2)]) = \Delta^{-\frac{1}{q}} \check{f}$$

as desired. □

Corollary 5.7. *Let $f, g \in L^2(G)$. Then*

$$f * \tilde{g} = \overline{\mathcal{F}}([\mathcal{P}(\overline{g}\mathcal{P}(\overline{f})^*]).$$

Proof. Letting $p_1 = p_2 = 2$ and $q = \infty$ in Proposition 5.6, we obtain

$$\mathcal{F}([\mathcal{P}(\overline{g})\mathcal{P}(\overline{f})^*]) = \overline{\mathcal{F}}([\mathcal{F}_2(\overline{g})\mathcal{F}_2(J\overline{f})]) = (\overline{g} * \Delta^{\frac{1}{2}} J\overline{f})\check{} = f * \tilde{g}.$$

□

Remark 5.8. Since $A(G) = \overline{\mathcal{F}}(L^1(\psi_0))$ and since every $T \in L^1(\psi_0)$ can be written $T = [RS^*]$ where $R, S \in L^2(\psi_0) = \mathcal{P}(L^2(G))$ (just put $R = U|T|^{\frac{1}{2}}$ and $S^* = |T|^{\frac{1}{2}}$, where $T = U|T|$ is the polar decomposition of T), we have reproved the fact [6] that $A(G) = \{f * \tilde{g} | f, g \in L^2(G)\}$. It also follows that $\|\varphi\|_{A(G)} \leq \|f\|_2\|g\|_2$ whenever $\varphi = f * \tilde{g}$, $f, g \in L^2(G)$ (since $\|[\mathcal{P}(\overline{g})\mathcal{P}(\overline{f})^*]\|_1 \leq \|\mathcal{P}(\overline{g})\|_2\|\mathcal{P}(\overline{f})\|_2$), and that, given $\varphi \in A(G)$, there exist $f, g \in L^2(G)$ with $\varphi = f * \tilde{g}$ such that $\|\varphi\|_{A(G)} = \|f\|_2\|g\|_2$ (use that $\|T\|_1 = \|U|T|^{\frac{1}{2}}\|_2\||T|^{\frac{1}{2}}\|_2$ for $T \in L^1(\psi_0)$).

Proposition 5.9. *Let $p \in [1, 2]$ and $q_1, q_2 \in [2, \infty]$ such that $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{p}$. Let $T \in L^{q_1}(\psi_0)$ and $S \in L^{q_2}(\psi_0)$. Then*

$$\langle T\xi | S\eta \rangle = \int \overline{\mathcal{F}}_p([S * T])(x)(\xi * J_p\eta)(x)dx$$

for all $\xi, \eta \in \mathcal{K}(G)$.

Proof. By Lemma 2.16, the left hand side of the equation to be proved is a continuous function of T and S. The same is true of the right hand side. Therefore it is enough to prove the statement for T and S belonging to the (dense) sets $\mathcal{F}_{p_1}(\mathcal{K}(G))$ and $\mathcal{F}_{p_2}(\mathcal{K}(G))$ (where, as usual, $\frac{1}{p_1} + \frac{1}{q_1} = 1$, $\frac{1}{p_2} + \frac{1}{q_2} = 1$).

Now suppose that $T = \mathcal{F}_{p_1}(h)$ and $S = \mathcal{F}_{p_2}(k)$ where $h, k \in \mathcal{K}(G)$. Then

$$\begin{aligned}(T\xi|S\eta) &= (h * \Delta^{\frac{1}{q_1}}\xi | k * \Delta^{\frac{1}{q_2}}\eta) \\ &= (\Delta^{\frac{1}{q_1}}\xi * \Delta^{-\frac{1}{q_2}}\tilde{\eta} | \Delta^{-1}\tilde{h} * k) \\ &= (\xi * \Delta^{-\frac{1}{q_1}-\frac{1}{q_2}}\tilde{\eta} | \Delta^{-\frac{1}{p_1}}\tilde{h} * \Delta^{-\frac{1}{q_1}}k) \\ &= \int (\xi * J_p\eta)(x)(\Delta^{-\frac{1}{p_1}}\check{h} * \Delta^{-\frac{1}{q_1}}\overline{k})(x)dx.\end{aligned}$$

Since

$$\begin{aligned}\overline{\mathcal{F}}_p([S * T]) &= \overline{\mathcal{F}_p}([\mathcal{F}_{p_2}(J_{p_2}k)\mathcal{F}_{p_1}(h)]) \\ &= \Delta^{-\frac{1}{q}}(J_{p_2}k * \Delta^{\frac{1}{q_2}}h)\check{} \\ &= \Delta^{-1+\frac{1}{p}}\Delta^{-\frac{1}{q_2}}\check{h} * \Delta^{-1+\frac{1}{p}}\Delta^{\frac{1}{p_2}}\overline{k} \\ &= \Delta^{-\frac{1}{p_1}}\check{h} * \Delta^{-\frac{1}{q_1}}\overline{k}\end{aligned}$$

we have proved the formula. □

Proposition 5.10. *Let $p \in [1,2]$ and $\frac{1}{p} + \frac{1}{q} = 1$. Let $T \in L^p(\psi_0)$ with polar decomposition $T = U|T|$. Put $g = \Delta^{-\frac{1}{q}}\overline{\mathcal{F}}_p(T)\check{}$.*

Then

$$(|T|^{\frac{1}{2}}\xi \,|\, |T|^{\frac{1}{2}}U^*\eta) = \int (g * \Delta^{\frac{1}{p}}\xi)(x)\overline{\eta(x)}dx$$

for all $\xi, \eta \in \mathcal{K}(G)$.

Proof. Put $q_1 = q_2 = 2p$. then $|T|^{\frac{1}{2}} \in L^{q_1}(\psi_0)$ and $|T|^{\frac{1}{2}}U^* \in L^{q_2}(\psi_0)$, and by Proposition 5.9 we get

$$\begin{aligned}(|T|^{\frac{1}{2}}\xi| \, |T|^{\frac{1}{2}}U^*\eta) &= \int \overline{\mathcal{F}}_p(T)(x)(\xi * J_p\eta)(x)dx \\ &= \int \overline{\mathcal{F}}(T)(x^{-1}(\Delta^{\frac{1}{p}}\overline{\eta} * \check{\xi})(x^{-1})\Delta^{-1}(x)dx \\ &= \int g(x)(\overline{\eta} * \Delta^{-\frac{1}{p}}\check{\xi})(x)dx \\ &= \int (g * \Delta^{\frac{1}{p}}\xi)(x)\overline{\eta(x)}dx.\end{aligned}$$

□

Proposition 5.11. *Let $p \in [1,2]$ and $T \in L^p(\psi_0)$. Put $g = \Delta^{-\frac{1}{q}}\overline{\mathcal{F}}_p(T)\check{}$. Let $\xi \in \mathcal{K}(G)$. Then $\xi \in D(T)$ if and only if $g * \Delta^{\frac{1}{p}}\xi \in L^2(G)$, and if this is the case, we have*

$$T\xi = g * \Delta^{\frac{1}{p}}\xi.$$

Proof. First suppose that $\xi \in D(T)$. Then for all $\eta \in \mathcal{K}(G)$ we have

$$\int (T\xi)(x)\overline{\eta(x)}dx = (T\xi|\eta) = (|T|^{\frac{1}{2}}\xi \,|\, |T|^{\frac{1}{2}}U^*\eta) = \int (g * \Delta^{\frac{1}{p}}\xi)(x)\overline{\eta(x)}dx.$$

Hence $g * \Delta^{\frac{1}{p}}\xi = T\xi$ and thus $g * \Delta^{\frac{1}{p}}\xi \in L^2(G)$.

Conversely, if $g * \Delta^{\frac{1}{p}}\xi \in L^2(G)$, then

$$\begin{aligned}(|T|^{\frac{1}{2}}\xi \,|\, |T|^{\frac{1}{2}}U^*\eta|) &= \left|\int (g * \Delta^{\frac{1}{p}}\xi)(x)\overline{\eta(x)}dx\right| \\ &\le \|g * \Delta^{\frac{1}{p}}\xi\|_2\|\eta\|_2\end{aligned}$$

for all $\eta \in \mathcal{K}(G)$.

We conclude that $|T|^{\frac{1}{2}}\xi \in D([|T|^{\frac{1}{2}}U^*|_{\mathcal{K}(G)}]^*)$. Now $[|T|^{\frac{1}{2}}U^*|_{\mathcal{K}(G)}]^* = [|T|^{\frac{1}{2}}U^*]^* = U|T|^{\frac{1}{2}}$, so that $|T|^{\frac{1}{2}}\xi \in D(U|T|^{\frac{1}{2}})$, whence $\xi \in D(T)$. □

Theorem 5.12. *Let $p \in [1, 2]$ and $T \in L^p(\psi_0)$. Put $g = \Delta^{-\frac{1}{q}}\overline{\mathcal{F}}_p(T)\check{}$. Suppose that $g \in L^2(G)$. Then T is the closure of the operator*

$$\xi \mapsto g * \Delta^{\frac{1}{p}}\xi, \qquad \xi \in \mathcal{K}(G).$$

Proof. When $g \in L^2(G)$, we have $g * \Delta^{\frac{1}{p}} \in L^2(G)$ for all $\xi \in \mathcal{K}(G)$. Thus, by Proposition 5.11, $\mathcal{K}(G) \subseteq D(T)$, and $T\xi = g * \Delta^{\frac{1}{p}}\xi$ for all $\xi \in \mathcal{K}(G)$. Since $T = [T|_{\mathcal{K}(G)}]$ by Proposition 2.15, the theorem is proved. □

As a corollary, we have

Theorem 5.13 (Fourier inversion)**.** *Let $p \in [1, 2]$ and $\frac{1}{p} + \frac{1}{q} = 1$.*

1) *Let $T \in L^p(\psi_0)$. Put $g = \Delta^{-\frac{1}{q}}\overline{\mathcal{F}}_p(T)\check{}$. If $g \in L^r(G)$ for some $r \in [1, 2]$, then $\mathcal{F}_r(q)\Delta^{\frac{1}{r}-\frac{1}{q}}$ is closable, and*
$$T = \left[\mathcal{F}_r(q)\Delta^{\frac{1}{r}-\frac{1}{q}}\right].$$
2) *Let $f \in L^p(G)$. If for some $r \in [1, 2]$, the closure $S = \left[\mathcal{F}_p(f)\Delta^{\frac{1}{r}-\frac{1}{q}}\right]$ exists and belongs to $L^r(\psi_0)$, then*
$$f = \Delta^{-\frac{1}{s}}\overline{\mathcal{F}}_r(S)\check{},$$
where $\frac{1}{r} + \frac{1}{s} = 1$.

Proof. 1) Since $g \in L^r(G) \cap L^q(G)$, we also have $g \in L^2(G)$. Then by Theorem 5.12 we have

$$T\xi = g * \Delta^{\frac{1}{p}}\xi = g * \Delta^{\frac{1}{s}}\Delta^{-1} = \frac{1}{r} + \frac{1}{p}\xi = \mathcal{F}_r(g)\Delta^{\frac{1}{r}-\frac{1}{q}}\xi$$

for all $\xi \in \mathcal{K}(G)$. Thus $T|_{\mathcal{K}(G)} \subseteq \mathcal{F}_r(g)\Delta^{\frac{1}{r}-\frac{1}{q}}$. As is easily seen $\mathcal{F}_r(g)\Delta^{\frac{1}{r}-\frac{1}{q}}$ is $(-\frac{1}{p})$-homogeneous. It is also closable, since its adjoint is densely defined (indeed, $(\mathcal{F}_r(g)\Delta^{\frac{1}{r}-\frac{1}{q}})^* \subseteq (T|_{\mathcal{K}(G)})^* = T^*$ so that $(\mathcal{F}_r(g)\Delta^{\frac{1}{r}-\frac{1}{q}})^* = T^*$). We conclude that $T = [\mathcal{F}_r(g)\Delta^{\frac{1}{r}-\frac{1}{q}}]$ (since $T \subseteq [\mathcal{F}_r(g)\Delta^{\frac{1}{r}-\frac{1}{q}}]$).

2) For all $\xi \in \mathcal{K}(G)$, we have $\xi \in D(S)$ and by Proposition 5.11,

$$f * \Delta^{\frac{1}{r}}\xi = \mathcal{F}_p(f)\Delta^{\frac{1}{r}-\frac{1}{q}}\xi = S\xi = \Delta^{-\frac{1}{s}}\overline{\mathcal{F}}_r(S)\check{} * \xi.$$

The result follows. □

Putting $p = r = 1$ in the first part of Theorem 5.12 and recalling that $\overline{\mathcal{F}}\left(\frac{d\varphi}{d\psi_0}\right) = \varphi$ for $\varphi \in A(G)_+$ we obtain

Corollary 5.14. *Let $\varphi \in A(G)_+$. If $\check{\varphi} \in L^1(G)$, then*

$$\frac{d\varphi}{d\psi_0} = [\lambda(\check{\varphi})\Delta].$$

Finally we shall give some results on positive operators $T \in L^p(\psi_0)$ valid without any restriction on $\mathcal{F}_p(T)$.

Note that for all $f \in L^q(G)$ and $\xi, \eta \in \mathcal{K}(G)$ we have

$$\begin{aligned}\int f(x)(\xi * J_p\eta)(x)dx &= \int\int f(x)\xi(y)\Delta^{-\frac{1}{p}}(y^{-1}x)\tilde{\eta}(y^{-1}x)dydx \\ &= \int\int f(yx)\xi(y)\Delta^{-\frac{1}{p}}(x)\tilde{\eta}(x)dxdy \\ &= \int\int f(yx^{-1})\xi(y)\Delta^{\frac{1}{q}}(x)\overline{\eta(x)}dxdy.\end{aligned}$$

Proposition 5.15. *Let $p \in [1, 2]$ and $\frac{1}{p} + \frac{1}{q} = 1$. Let $T \in L^p(\psi_0)_+$. Put $f = \overline{\mathcal{F}}_p(T)$. Let*

$$q(\xi) = \int f(x)(\xi * J_p\xi)(x)dx = \int\int f(yx^{-1})\Delta^{\frac{1}{q}}(x)\xi(y)\overline{\xi(x)}dydx$$

for all $\xi \in \mathcal{K}(G)$. Then q is a closable positive quadratic form, and the positive self-adjoint operator associated with its closure is T.

Proof. By (the proof of) Proposition 5.10, we have

$$(T^{\frac{1}{2}}\xi|T^{\frac{1}{2}}\xi) = \int f(x)(\xi * J_p\xi)(x)dx = q(\xi)$$

for all $\xi \in \mathcal{K}(G)$, and $T^{\frac{1}{2}} = [T^{\frac{1}{2}}|_{\mathcal{K}(G)}]$. Thus q is a closable positive quadratic form with closure corresponding to T. □

Corollary 5.16. *Let $\varphi \in A(G)_+$. Then $\frac{d\varphi}{d\psi_0}$ is the positive self-adjoint operator associated with the closure of the positive quadratic form q given by*

$$q(\xi) = \int \varphi(x)(\xi * \xi^*)(x)dx = \int\int \varphi(yx^{-1})\xi(y)\overline{\xi(x)}dydx$$

for all $\xi \in \mathcal{K}(G)$.

Remark 5.17. This result also follows directly from the definition of $\frac{d\varphi}{d\psi_0}$. Indeed,

$$\left\|\left(\frac{d\varphi}{d\psi_0}\right)^{\frac{1}{2}}\xi\right\|^2 = \varphi(\lambda(\xi)\lambda(\xi)^*) = \int \varphi(x)(\xi * \xi^*)(x)dx$$

for all $\xi \in \mathcal{K}(G)$, and we have $\left(\frac{d\varphi}{d\psi_0}\right)^{\frac{1}{2}} = \left[\left(\frac{d\varphi}{d\psi_0}\right)^{\frac{1}{2}}\Big|_{\mathcal{K}(G)}\right]$ by Proposition 2.15 (or, alternatively, by an application of [9, Theorem] together with the fact that $\left(\frac{d\varphi}{d\psi_0}\right)^{\frac{1}{2}} = \left[\left(\frac{d\varphi}{d\psi_0}\right)^{\frac{1}{2}}\Big|_{\mathfrak{A}_l}\right]$).

Actually, the property of defining closable quadratic forms on $\mathcal{K}(G)$ characterizes $A(G)_+$-functions among all positive definite continuous functions. The precise statement is as follows:

Theorem 5.18. *Let φ be a positive definite continuous function. define q on $\mathcal{K}(G)$ by*

$$q(\xi) = \int \varphi(x)(\xi * \xi^*)(x)dx = \int\int \varphi(yx^{-1})\xi(y)\overline{\xi(x)}dydx, \qquad \xi \in \mathcal{K}(G).$$

Then q is a positive quadratic form on $\mathcal{K}(G)$, and q is closable if and only if $\varphi \in A(G)$.

Proof. That q is a quadratic form is obvious, and since φ is positive definite, q is positive.

Now suppose that q is closable. Denote by T the positive self-adjoint operator associated with its closure; Then T is characterized by the properties $\mathcal{K}(G) \subseteq D(T^{\frac{1}{2}})$, $T^{\frac{1}{2}} = [T^{\frac{1}{2}}|_{\mathcal{K}(G)}]$, and

$$\forall \xi \in \mathcal{K}(G): \ \|T^{\frac{1}{2}}\xi\|^2 = q(\xi).$$

Let us show that T is (-1)-homogeneous. Suppose that $x \in G$. Then $T_x = \Delta^{-1}(x)\rho(x)T\rho(x^{-1})$ is positive self-adjoint and $T_x^{\frac{1}{2}} = \Delta^{-\frac{1}{2}}(x)\rho(x)T^{\frac{1}{2}}\rho(x^{-1})$. Therefore $\mathcal{K}(G) \subseteq D(T_x^{\frac{1}{2}})$ and $T_x^{\frac{1}{2}} = [T_x^{\frac{1}{2}}|_{\mathcal{K}(G)}]$. Furthermore, for all $\xi \in \mathcal{K}(G)$ we have

$$\begin{aligned}
\|T_x^{\frac{1}{2}}\xi\|^2 &= \|\Delta^{-\frac{1}{2}}(x)\rho(x)T^{\frac{1}{2}}\rho(x^{-1})\xi\| \\
&= \Delta^{-1}(x)\|T^{\frac{1}{2}}\rho(x^{-1})\xi\|^2 \\
&= \Delta^{-1}(x)q(\rho(x^{-1})\xi) \\
&= \Delta^{-1}(x)\int\int \varphi(yz^{-1})(\rho(x^{-1})\xi)(y)\overline{(\rho(x^{-1})\xi)(z)}dydz \\
&= \int\int \Delta^{-1}(x)\varphi(yz^{-1})\Delta^{\frac{1}{2}}(x^{-1})\xi(yx^{-1})\Delta^{\frac{1}{2}}(x^{-1})\overline{\xi(zx^{-1})}dydz \\
&= \Delta^{-1}(x)\int\int \varphi(yxz^{-1})\xi(y)\overline{\xi(zx^{-1})}dydz \\
&= \int\int \varphi(yz^{-1})\xi(y)\overline{\xi(z)}dzdy \\
&= q(\xi).
\end{aligned}$$

We conclude from the characterization of T that $T_x = T$, so that

$$\forall x \in G: \ \Delta^{-1}(x)\rho(x)T\rho(x^{-1}) = T,$$

i.e. T is (-1)-homogeneous.

Now let $(\xi_i)_{i\in I}$ be an approximate identity in $\mathcal{K}(G)_+$. Then

$$\begin{aligned}\|T^{\frac{1}{2}}\xi_i\|^2 &= q(\xi_i)\\ &= \int \varphi(x)(\xi_i * \xi_i^*)(x)dx\\ &\leq \sup\{|\varphi(x)|\ |x \in \operatorname{supp}(\xi_i * \xi_i^*)\} \cdot \|\xi_i * \xi_i^*\|_1\\ &\leq \sup\{|\varphi(x)|\ |x \in \operatorname{supp}(\xi_i * \xi_i^*)\}.\end{aligned}$$

Since φ is continuous and the support of the $\xi_i * \xi_i^*$ tend to $\{e\}$, we get

$$\liminf_{i\in I} \|T^{\frac{1}{2}}\xi_i\|^2 \leq \varphi(e).$$

By Proposition 2.10, this shows that $T \in L^1(\psi_0)$.

Put $\varphi_1 = \overline{\mathcal{F}}(T) \in A(G)$. Then

$$\int \varphi_1(x)(\xi * \xi^*)(x)dx = \|T^{\frac{1}{2}}\xi\|^2 = q(\xi) = \int \varphi(x)(\xi * \xi^*)(x)dx$$

for all $\xi \in \mathcal{K}(G)$. We conclude that $\varphi = \varphi_1$ and thus $\varphi \in A(G)$. □

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