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HYPER GENERALIZED WEAKLY SYMMETRIC $(CS)_4$-SPACETIME AND THE RICCI SOLITONS

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Abstract. A hyper generalized weakly symmetric $(CS)_4$-spacetime has been studied. It is found that such a spacetime is a perfect fluid spacetime, space of quasi constant curvature and conformally flat. Also, we point out the sufficient condition for a compact, orientable hyper generalized weakly symmetric $(CS)_4$-spacetime to be conformal to a sphere in 5 dimensinal Euclidean space E_5.

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1. Introduction

In 2003 Shaikh [14] established the notion of Lorentzian concircular structure manifolds (briefly, $(LCS)_n$-manifolds) with an example. Four dimensional Lorentzian concircular structure manifold is termed as $(CS)_4$-spacetime (See [12]).

Definition 1. *A semi-Riemannian manifold* (M^n, g)*,* $n =$*dim* M *is said to be hyper generalized weakly symmetric if its Riemannian curvature tensor* R *admits the relation*

$$\begin{aligned} & (\nabla_X R)(Y,U,V,Z) \\ = \; & \Pi_1(X)R(Y,U,V,Z) + \Psi_1(Y)R(X,U,V,Z) \\ & +\Psi_1(U)R(Y,X,V,Z) + F_1(V)R(Y,U,X,Z) \\ & +F_1(Z)R(Y,U,V,X) + \Pi_2(X)(g\wedge S)(Y,U,V,Z) \\ & +\Psi_2(Y)(g\wedge S)(X,U,V,Z) + \Psi_2(U)\,(g\wedge S)(Y,X,V,Z) \\ & +F_2(V)\,(g\wedge S)(Y,U,X,Z) + F_2(Z)\,(g\wedge S)(Y,U,V,X) \end{aligned} \quad (1)$$

where

$$\begin{aligned} (g\wedge S)(Y,U,V,Z) \;=\; & g(Y,Z)S(U,V) + g(U,V)S(Y,Z) \\ & -g(Y,V)S(U,Z) - g(U,Z)S(Y,V) \end{aligned} \quad (2)$$

and Π_i, Ψ_i *and* $\digamma_i$ *are non-zero* 1*-forms defined as* $\Pi_i(X) = g(X, \epsilon_i)$, $\Psi_i(X) = g(X, \sigma_i)$ *and* $\digamma_i(X) = g(X, \tau_i)$.

The beauty of such manifold is that it has the flavour of,
(i) locally symmetric space [5] (for $\Pi_i = \Psi_i = \digamma_i = 0$,where $i = 1, 2$),
(ii) recurrent space [18] (for $\Pi_1 \neq 0$, $\Pi_2 = \Psi_i = \digamma_i = 0$,where $i = 1, 2$),
(iii) hyper recurrent space [13] (for $\Pi_i \neq 0, \Psi_i = \digamma_i = 0$,where $i = 1, 2$),
(iv) pseudo symmetric space [6] (for $\Pi_1 = \Psi_1 = \digamma_1 = \Pi \neq 0$ and $\Pi_2 = \Psi_2 = \digamma_2 = 0$),
(v) semi-pseudo symmetric space [16] (for $\Psi_1 = \digamma_1$ and $\Pi_1 = \Pi_2 = \Psi_2 = \digamma_2 = 0$),
(vi) hyper semi-pseudo symmetric space (for $\Pi_1 = \Pi_2 = 0, \Psi_1 = \digamma_1 \neq 0$ and $\Psi_2 = \digamma_2 \neq 0$),
(vii) hyper pseudo symmetric space (for $\Pi_i = \Psi_i = \digamma_i \neq 0$ where $i = 1, 2$),
(viii) almost pseudo symmetric space [7] (for $\Pi_1 = \Psi_1 + H_1, H_1 = \Psi_1 = \digamma_1 \neq 0$ $\Pi_2 = \Psi_2 = \digamma_2 = 0$),
(ix) almost hyper pseudo symmetric space (for $\Pi_1 = \Psi_1 + H_1, H_1 = \Psi_1 = \digamma_1 \neq 0$ $\Pi_2 = \Psi_2 + H_2, H_2 = \Psi_2 = \digamma_2 \neq 0$) and
(x) weakly symmetric space [15] (for $\Pi_2 = \Psi_2 = \digamma_2 = 0$).

Definition 2. *A four dimensional Lorentzian manifold is said to be a perfect fluid spacetimes if it satisfies*

$$S(U, V) = \gamma g(U, V) + \nu \delta(U)\delta(V),$$

for any vector fields U *and* V*, where* γ *and* ν *are some scalar functions,* δ *being a non-zero* 1*-form corresponding to an unit timelike vector field* π*, that is,* $g(U, \pi) = \delta(U)$ *and* $g(\pi, \pi) = -1$.

Definition 3. *([8]) A Lorentzian manifold is said to infinitesimally spatially isotropic relative to a unit timelike vector field* ϱ *if the Riemannian curvature tensor* R *satisfies the condition:*

$$R(U, Y)V = \vartheta[g(Y, V)U - g(U, V)Y],$$

for all U*,* Y*,* Z *belongs to* $\varrho^\perp$ *and* $R(U, \varrho)\varrho = \pi U$ *for all* $U \in \varrho^\perp$*, where* ϑ *and* π *are real valued functions.*

First section deals with some basic definations and thereafter we mention known results of $(CS)_4$-spacetimes which are used in sequel. In third section, we show that a hyper generalized weakly symmetric $(CS)_4$-spacetime is a perfect fluid spacetimes, a space of quasi-constant curvature, conformally flat and infinitesimally spatially isotropic relative to the unit timelike vector field ξ. Then we study Ricci solitons and the Poisson equation in that spacetime. Lastly, we obtain the sufficient condition for a compact, orientable hyper generalized weakly symmetric $(CS)_4$-spacetime to be conformal to a sphere in E_5.

2. $(CS)_4$-SPACETIMES

In a $(CS)_4$-spacetime, the following relations hold [[14], [3], [4], [2]]:

$$(\nabla_U \eta)V = \delta\{g(U,V) + \eta(U)\eta(V)\} \qquad (\delta \neq 0), \tag{3}$$

$$\eta(\xi) = -1, \quad \phi \circ \xi = 0, \tag{4}$$

$$\phi U = U + \eta(U)\xi = \frac{1}{\delta}\nabla_U \xi, \tag{5}$$

$$\eta(\phi X) = 0, \quad g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \tag{6}$$

$$\eta(R(X,Y)Z) = (\delta^2 - \epsilon)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)], \tag{7}$$

$$R(X,Y)\xi = (\delta^2 - \epsilon)[\eta(Y)X - \eta(X)Y], \tag{8}$$

$$\begin{aligned} & (\nabla_X R)(Y,Z)\xi \\ = \; & \delta(\delta^2 - \epsilon)[g(X,Z)Y - g(X,Y)Z] \\ & +(2\delta\epsilon - \theta)\eta(X)[\eta(Z)Y - \eta(Y)Z] - \delta R(Y,Z)(X), \end{aligned} \tag{9}$$

$$\begin{aligned} & (\nabla_X R)(Y,Z,V,\xi) \\ = \; & -\delta R(Y,Z,V,X) \\ & -\delta(\delta^2 - \epsilon)[g(X,Z)g(Y,V) - g(X,Y)g(Z,V)] \\ & -(2\delta\epsilon - \theta)\eta(X)[\eta(Z)g(Y,V) - \eta(Y)g(Z,V)], \end{aligned} \tag{10}$$

$$S(X,\xi) = 3(\delta^2 - \epsilon)\eta(X) \tag{11}$$

$$\begin{aligned} & (\nabla_U S)(X,\xi) \\ = \; & 3[\delta(\delta^2 - \epsilon)g(X,U) + (2\delta\epsilon - \theta)\eta(X)\eta(U)] - \delta S(X,U). \end{aligned} \tag{12}$$

for any vector fields X, Y, Z, U, V.

3. Hyper generalized weakly symmetric $(CS)_4$-spacetimes

We first consider a hyper generalized weakly symmetric $(CS)_4$-spacetimes with defining condition (1). Using (2) in (1) and then contracting the resultent, we have

$$
\begin{aligned}
& (\nabla_X S)(Y,Z) \\
= \ & \Pi_1(X)S(Y,Z) + \Psi_1(Y)S(X,Z) + \digamma_1(Z)S(X,Y) \\
& +\Psi_1(R(X,Y)Z) + \digamma_1(R(X,Z)Y) + \Pi_2(X)[2S(Y,Z) + rg(Y,Z)] \\
& +\Psi_2(Y)[2S(X,Z) + rg(X,Z)] + \digamma_2(Z)[2S(Y,X) + rg(Y,X)] \\
& +\Psi_2(LX)g(Y,Z) + \Psi_2(X)S(Y,Z) - \Psi_2(Y)S(X,Z) \\
& -\Psi_2(LY)g(Z,X) + \digamma_2(LX)g(Y,Z) + \digamma_2(X)S(Y,Z) \\
& -\digamma_2(LZ)g(Y,X) - \digamma_2(Z)S(X,Y). \qquad (13)
\end{aligned}
$$

Setting $Z = \xi$ in (13) and then making use of (8), (11) and (12) we get

$$
\begin{aligned}
& 3\{\delta(\delta^2-\epsilon)g(X,Y) + (2\delta\epsilon - \theta)\eta(X)\eta(Y)\} - \delta S(X,Y) \\
= \ & \Pi_1(X)3(\delta^2-\epsilon)\eta(Y) + \Psi_1(Y)3(\delta^2-\epsilon)\eta(X) \\
& +(\delta^2-\epsilon)[\Psi_1(X)\eta(Y) - \Psi_1(Y)\eta(X)] \\
& +(\delta^2-\epsilon)[\eta(Y)\digamma_1(X) - g(X,Y)\digamma_1(\xi)] + S(X,Y)\digamma_1(\xi) \\
& +\Pi_2(X)\eta(Y)[6(\delta^2-\epsilon) + r] + \Psi_2(Y)\eta(X)[6(\delta^2-\epsilon) + r] \\
& +\digamma_2(\xi)[2S(Y,X) + rg(Y,X)] + \Psi_2(LX)\eta(Y) \\
& +\Psi_2(X)3(\delta^2-\epsilon)\eta(Y) - \Psi_2(Y)3(\delta^2-\epsilon)\eta(X) \\
& -\Psi_2(LY)\eta(X) + \digamma_2(LX)\eta(Y) + \digamma_2(X)3(\delta^2-\epsilon)\eta(Y) \\
& -\digamma_2(L\xi)g(Y,X) - \digamma_2(\xi)S(X,Y) \qquad (14)
\end{aligned}
$$

which yields

$$
\begin{aligned}
& 3(2\delta\epsilon - \theta) = r[\Psi_2(\xi) - \Pi_2(\xi) - \digamma_2(\xi)] \\
& -3(\delta^2-\epsilon)[\Pi_1(\xi) + \Psi_1(\xi) + 3\Pi_2(\xi) \\
& +3\Psi_2(\xi) + \digamma_1(\xi) + 2\digamma_2(\xi)] \qquad (15)
\end{aligned}
$$

for $X = Y = \xi$.

Again, setting $Y = \xi$ and $X = \xi$ in succession in (14) and then using the relation

(15), we have respectively

$$\begin{aligned}
& 3\Pi_1(X)(\delta^2-\epsilon)+\Psi_1(X)(\delta^2-\epsilon)+\digamma_1(X)(\delta^2-\epsilon) \\
& +6\Pi_2(X)(\delta^2-\epsilon)+r\Pi_2(X)+\Psi_2(LX)+\digamma_2(LX) \\
& +3\Psi_2(X)(\delta^2-\epsilon)+3\digamma_2(X)(\delta^2-\epsilon) \\
= \ & \eta(X)[3\delta(\delta^2-\epsilon)-3(\delta^2-\epsilon)-\Psi_1(\xi)(\delta^2-\epsilon) \\
& -6\Psi_2(\xi)(\delta^2-\epsilon)+2r\Psi_2(\xi)-\Psi_2(L\xi) \\
& -r\Pi_2(\xi)-3\Pi_1(\xi)(\delta^2-\epsilon)-9\Pi_2(\xi)(\delta^2-\epsilon) \\
& -3\digamma_2(\xi)(\delta^2-\epsilon)-\digamma_1(\xi)(\delta^2-\epsilon)-\digamma_2(L\xi)] \qquad (16)
\end{aligned}$$

and

$$\begin{aligned}
& 2\Psi_1(Y)(\delta^2-\epsilon)+3\Psi_2(Y)(\delta^2-\epsilon) \\
& +r\Psi_2(Y)-\Psi_2(LY) \\
= \ & \eta(Y)[3\delta(\delta^2-\epsilon)-3(\delta^2-\epsilon)-2\Psi_1(\xi)(\delta^2-\epsilon) \\
& -3\Pi_2(\xi)(\delta^2-\epsilon)-6\Psi_2(\xi)(\delta^2-\epsilon) \\
& +r\Psi_2(\xi)+\Psi_2(L\xi)]. \qquad (17)
\end{aligned}$$

Next, in view of (15), (16) and (17), the relationn (14) yields

$$\begin{aligned}
S(X,Y) = \ & \frac{1}{3}[r-3(\delta^2-\epsilon)]g(X,Y) \\
& +\frac{1}{3}[r-12(\delta^2-\epsilon)]\eta(X)\eta(Y). \qquad (18)
\end{aligned}$$

This leads to the followings:

Theorem 1. *Every hyper generalized weakly symmetric $(CS)_4$-spacetime is a perfect fluid spacetime.*

Now with the help of (10), (18) and by the symmetry of the Riemann curvature tensor, one can easily find out

$$\begin{aligned}
R(Y,V,U,Z) = \ & \frac{r-6\left(\delta^2-\epsilon\right)}{6}G(Y,V,U,Z) \\
+ \ & \frac{r-12\left(\delta^2-\epsilon\right)}{6}H(Y,V,U,Z), \qquad (19)
\end{aligned}$$

where $G=g\wedge g$ and $H=g\wedge(\eta\otimes\eta)$. Thus we can state:

Theorem 2. *A hyper generalized weakly symmetric $(CS)_4$-spacetime is a space of quasi constant curvature.*

In an 4-dimensinal semi-Riemannian manifold the Weyl conformal curvature tensor defined as

$$C(X,Y)Z = R(X,Y)Z - \frac{1}{2}[S(Y,Z)X - S(X,Z)Y$$
$$+g(Y,Z)QX - g(X,Z)QY + \frac{r}{3}\{g(Y,Z)X - g(X,Z)Y\}].$$

By virtue of (18) and (19), we can calculate that the Weyl conformal curvature tensor vanishes identically. This infer:

Theorem 3. *Every hyper generalized weakly symmetric $(CS)_4$-spacetime is conformally flat.*

Theorem 4. *([17], Theorem 3.3.) In a hyper generalized weakly symmetric $(CS)_4$-spacetime with constant scalar curvature (mentioned in (15)) the followings are true; i) the characteristic vector field ξ is irrotational, ii) the integral curves of the characteristic vector field ξ are geodesic, iii) the characteristic vector field ξ corresponding to the 1-form η is a unit proper concircular vector field.*

Next, we assume that $\xi^{\perp}$ is an orthonormal 3-dimensional distribution to ξ in hyper generalized weakly symmetric $(CS)_4$-spacetime. Then $g(U,\xi) = 0$, for all $U \in \xi^{\perp}$. Therefore, from (19) we obtain

$$R(U,Y)V = \frac{r - 6\left(\delta^2 - \epsilon\right)}{6}[g(Y,V)U - g(U,V)Y].$$

From the above equation, we have

$$R(U,\xi)\xi = \frac{6\left(\delta^2 - \epsilon\right) - r}{6}U.$$

for all $U \in \xi^{\perp}$. This leads to the followings;

Theorem 5. *A hyper generalized weakly symmetric $(CS)_4$-spacetime is infinitesimally spatially isotropic relative to the unit timelike vector field ξ.*

4. Ricci solitons on hyper generalized weakly symmetric $(CS)_4$-spacetime

Suppose in a $(CS)_4$-spacetime the pair (λ,ξ) defines a Ricci soliton, that is,

$$2S(X,Y) = -(\pounds_{\xi}g)(X,Y) - 2\lambda g(X,Y),$$

for λ a real number. Writting $\pounds_\xi g$ in terms of the Levi-Civita connection ∇, the above equation yields,

$$2S(X,Y) = -g(\nabla_X \xi, Y) - g(X, \nabla_Y \xi) - 2\lambda g(X,Y),$$

for any $X, Y \in \chi(M)$. As a consequence of (5), the above equation becomes

$$S(X,Y) = -(\lambda + \delta) g(X,Y) - \delta \eta(X)\eta(Y). \tag{20}$$

In view of (18) and (20) we obtain

$$\lambda = -\frac{r}{4} - \frac{3}{4}\delta.$$

Therefore;

Theorem 6. *Ricci soliton in a hyper generalized weakly symmetric $(CS)_4$-spacetime is $(-\frac{r}{4} - \frac{3}{4}\delta,\ \xi)$.*

Theorem 7. *If $(\lambda = -(\frac{r}{4} + \frac{3}{4}\delta),\ \xi = grad(f))$ defines a Ricci soliton in a hyper generalized weakly symmetric $(CS)_4$-spacetime, then the Poisson equation satisfied by f is*

$$\Delta(f) = -(4\lambda + r).$$

5. Sufficient condition for a compact, orientable hyper generalized weakly symmetric $(CS)_4$-spacetime to be conformal to a sphere in 5 dimensinal Euclidean space E_5.

Definition 4. *Suppose, (M_1, g_1) and (M_2, g_2) be any two n-dimensional Riemannian manifold. Then (M_1, g_1) is said to be conformal to (M_2, g_2) if, i) there exits a one- one differentiable mapping $\varphi : (M_1, g_1) \to (M_2, g_2)$,ii) the angle between any two vectors at a point p of M_1 is equal to the angle between the corresponding vectors mapped by φ in M_2.*

According to *Watanabe [19]*, if in an n-dimensional Riemannian manifold $\hat{M}$, there exists a non parallel vector field U such that the relation

$$\int_{\hat{M}} S(U,U)dx = \frac{1}{2}\int_{\hat{M}} |dU|^2 dx + \frac{n-1}{n}\int_{\hat{M}} (\partial U)^2 dx \tag{21}$$

satisfies, then $\hat{M}$ is conformal to a sphere in E_{n+1}, where dx is the volume element of $\hat{M}$ and dU and ∂U are the curl and divergence of U respectively. Here, we consider a compact orientable hyper generalized weakly symmetric $(CS)_4$-spactime without boundary.

From (18), we get

$$S(U,\xi) = 3(\delta^2 - \epsilon)\eta(U).$$

Hence,

$$S(\xi,\xi) = 3(\epsilon - \delta^2).$$

In view of this and letting ξ for U, the relation (21) becomes

$$12(\epsilon - \delta^2)\int_{\hat{M}} dx = 2\int_{\hat{M}} |d\xi|^2 dx + 3\int_{\hat{M}} (\partial\xi)^2 dx. \tag{22}$$

Now, assume ξ is a parallel vector field. Then

$$\nabla_U \xi = 0.$$

Hence, from the Ricci identity we have

$$R(U,X)\xi = 0.$$

Which gives after contraction

$$S(V,\xi) = 0.$$

Since $(\delta^2 - \epsilon) \neq 0$ thus from the above, ξ cannot be a parallel vector field. Thus in a compact, orientable hyper generalized weakly symmetric $(CS)_4$-spacetime without boundary the characteristic vector field ξ is not a parallel vector field. Therefore we can state

Theorem 8. *If a compact, orientable hyper generalized weakly symmetric $(CS)_4$-spacetime without boundary admits the relation (22), then it is conformal to a sphere immersed in 5 dimensinal Euclidean space E_5.*

Remark 1. *In [10] authors have proved that 4-dimensinal Lorentzian concircular structure (known as $(CS)_4$)-spacetime coincide with Generalized Robertson-Walker (GRW) spacetimes. Consequently, each of the above mentioned results holds also for hyper generalized weakly symmetric GRW-spacetimes.*

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