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MAPPING PROPERTIES OF AN INTEGRAL OPERATOR ASSOCIATED WITH MITTAG-LEFFLER FUNCTIONS

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Abstract. The main object of this paper is to obtain the mapping properties of an integral operator associated with Mittag - Leffler function of the first kind on a subclass of analytic univalent functions.

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1. Introduction

Let $\mathcal{A}$ stand for the class of functions f of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

which are analytic in the open unit disk $\Delta = \{z \ : \ z \in \mathbb{C} \text{ and } |z| < 1\}$.

The function $f \in \mathcal{A}$ satisfy the normalization condition $f(0) = f'(0) - 1 = 0$. Further, we let S be the subclass of $\mathcal{A}$ consisting of functions $f(z)$ of the form (1), which are univalent in Δ. In 1994, Uralegaddi et al.[18] introduced the class $N(\beta)$ consisting of functions $f(z)$ of the form (1) and satisfy the following analytic criteria

$$\Re\left\{1 + \frac{z\, f''(z)}{f'(z)}\right\} < \beta, \qquad \text{where} \qquad 1 < \beta \leq \frac{3}{2}.$$

It is worthy to note that the class $N(\beta)$ is analogues to the class of convex functions of order β. The class $N(\beta)$ was extensively studied by Uralegaddi et al.[18] for functions $f(z)$ of the form (1) with $a_n \geq 0 \ \ (n \geq 2)$. The class $N(\beta)$ was further generalized by Dixit and Chandra [5], Dixit and Pathak [6] , Dixit *et al.* [7], Porwal and Dixit [12] and Porwal *et al.* [15]etc.

The study of integral operators are interesting topic of research in geometric function theory. In 2008, Breaz [3] studied the integral operator for the class $N(\beta)$. This result was extended by Porwal [10]. Several researchers introduced various integral operators involving Bessel functions and established a co-relation between complex analysis and special functions. In this direction Baricz and Frasin [4], Frasin [8], Magesh *et al.* [9], Porwal and Breaz[11], Porwal and Kumar[13], Porwal *et al.* [14] makes a significant contribution. The integral operators associated with Bessel functions attracts to the researchers to obtain analogues results for other special functions. Recently, Srivastava et al. [17] investigated a new integral operator involving Mittag-Leffler function. Now, we recall the definition of Mittag-Leffler function.

Definition 1. *Let $E_\alpha(z)$ be the function defined by*

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \qquad z \in \mathbb{C}, \;\; \Re(\alpha) > 0.$$

The function $E_\alpha(z)$ was introduced by Mittag-Leffler function in [16]. Wiman [19, 20] generalized the Mittag-Leffler function $E_\alpha(z)$ into a more general function $E_{\alpha,\,\beta}(z)$

$$E_{\alpha,\,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \qquad \alpha,\ \beta \in \mathbb{C}, \;\; \Re(\alpha) > 0. \tag{2}$$

It is easy to verify that Mittag-Leffler function $E_{\alpha,\,\beta}(z)$ is not a member of class $\mathcal{A}$. Therefore, we normalize the Mittag-Leffler function by

$$\begin{aligned} \mathbb{E}_{\alpha,\,\beta}(z) &= \Gamma(\beta) z\, E_{\alpha,\,\beta}(z) \\ \mathbb{E}_{\alpha,\,\beta}(z) &= z + \sum_{n=2}^{\infty} \frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}\, z^n, \end{aligned} \tag{3}$$

where $\alpha, \beta \in C,\ \beta \neq 0, -1, -2, \cdots, \Re(\alpha) > 0$.

In this paper, we consider the case for real-valued α, β and $z \in \Delta$. The function $E_{\alpha,\,\beta}(z)$ is a generalization of very well-known functions. For example,

$$\begin{aligned} \mathbb{E}_{2,\,1}(z) &= z \cosh \sqrt{z} \\ \mathbb{E}_{2,\,2}(z) &= \sqrt{z} \sinh \sqrt{z} \\ \mathbb{E}_{2,\,3}(z) &= 2[\cosh \sqrt{z} - 1] \quad \text{and} \\ \mathbb{E}_{2,\,4}(z) &= \frac{6[\sinh \sqrt{z} - \sqrt{z}]}{\sqrt{z}}. \end{aligned}$$

Bansal and Prajapati[2] (see also [1]) investigated geometric properties of Mittag-Leffler function. In 2017, Srivastava *et.al* [17] introduced a new integral operator associated with Mittag-Leffler function. Now, we introduce a new integral operator associated with Mittag-Leffler function

$$F_{\alpha_j,\,\beta_j,\,\lambda_j}(z)=\int_0^z\prod_{j=1}^{n}\left(\frac{\mathbb{E}_{\alpha_j,\,\beta_j}(t)}{t}\right)^{\frac{1}{\lambda_j}}dt. \tag{4}$$

In the present paper motivated with the above mentioned work we obtain sufficient condition for integral operator $F_{\alpha_j,\,\beta_j,\,\lambda_j}(z)$ to be in the class $N(\beta)$.

To prove our main results we shall require the following lemma.

Lemma 1. *([17])Let $\alpha,\beta\geq 1$. Then*

$$\left|\frac{z\,\mathbb{E}'_{\alpha,\,\beta}(z)}{\mathbb{E}_{\alpha,\,\beta}(z)}-1\right|\leq\frac{2\beta+1}{\beta^2-\beta-1},\quad z\in\Delta.$$

2. Main Results

In our first result, we study the mapping properties for the integral operator defined in (4).

Theorem 2. *Let n be a positive integer and $\alpha_1,\alpha_2,\cdots,\alpha_n\geq 1,\ \beta_1,\beta_2,\cdots,\beta_n\geq\frac{1}{2}(1+\sqrt{5})$ and consider the normalized Mittag-Leffler function $\mathbb{E}_{\alpha_j,\,\beta_j}(z)$ defined by*

$$\mathbb{E}_{\alpha_j,\,\beta_j}(z)=\Gamma(\beta_j)\;z\;E_{\alpha_j,\,\beta_j}(z).$$

Let $\beta=min\{\beta_1,\,\beta_2,\,\cdots,\,\beta_n\}$ and $\lambda_1,\,\lambda_2,\,\cdots,\,\lambda_n$ be non-zero positive real numbers. Moreover, suppose that these numbers satisfy the following inequality

$$0<\frac{2\beta+1}{\beta^2-\beta-1}\;\sum_{j=1}^{n}\frac{1}{\lambda_j}\leq\frac{1}{2}.$$

Then the function $F_{\alpha_j,\,\beta_j,\,\lambda_j}(z)$ defined by (4) is in $N(\delta)$, where $\delta=1+\frac{2\beta+1}{\beta^2-\beta-1}\;\sum_{j=1}^{n}\frac{1}{\lambda_j}$.

Proof. We observe that $\mathbb{E}_{\alpha_j,\,\beta_j}(z)\in\mathcal{A}$, clearly $F_{\alpha_j,\beta_j,\lambda_j}(0)=F'_{\alpha_j,\beta_j,\lambda_j}(0)-1=0$. Differentiating (4), we have

$$F'_{\alpha_j,\beta_j\lambda_j}(z)=\prod_{j=1}^{n}\left(\frac{\mathbb{E}_{\alpha_j,\beta_j}(z)}{z}\right)^{\frac{1}{\lambda_j}}$$

Taking logarithmic differentiation we have

$$\frac{F''_{\alpha_j,\beta_j,\lambda_j}(z)}{F'_{\alpha_j,\beta_j,\lambda_j}(z)} = \sum_{j=1}^{n} \frac{1}{\lambda_j}\left(\frac{\mathbb{E}'_{\alpha_j,\beta_j,\lambda_j}(z)}{\mathbb{E}_{\alpha_j,\beta_j,\lambda_j}(z)} - \frac{1}{z}\right)$$

$$1+\frac{z\,F''_{\alpha_j,\beta_j,\lambda_j}(z)}{F'_{\alpha_j,\beta_j,\lambda_j}(z)} = 1+\sum_{j=1}^{n} \frac{1}{\lambda_j}\left(\frac{z\,\mathbb{E}'_{\alpha_j,\beta_j,\lambda_j}(z)}{\mathbb{E}_{\alpha_j,\beta_j,\lambda_j}(z)} - 1\right)$$

$$\Re\left\{1+\frac{z\,F''_{\alpha_j,\beta_j,\lambda_j}(z)}{F'_{\alpha_j,\beta_j,\lambda_j}(z)}\right\} = \sum_{j=1}^{n} \frac{1}{\lambda_j}\Re\left\{\frac{z\,\mathbb{E}'_{\alpha_j,\beta_j,\lambda_j}(z)}{\mathbb{E}_{\alpha_j,\beta_j,\lambda_j}(z)} - 1\right\}+1$$

$$\leq \sum_{j=1}^{n} \frac{1}{\lambda_j}\left(\frac{2\beta_j+1}{\beta_j^2-\beta_j-1}\right)+1 \qquad \leq 1+\frac{2\beta+1}{\beta^2-\beta-1}\sum_{j=1}^{n}\frac{1}{\lambda_j}.$$

for all $z \in \Delta$ and $(\beta_1, \beta_2, \cdots, \beta_n) \geq \frac{1}{2}(1+\sqrt{5})$. Here, we used that the function $\phi : \left(\frac{1}{2}(1+\sqrt{5}), \infty\right) \to \mathbb{R}$, defined by $\phi(x) = \frac{2x+1}{x^2-x-1}$ is a decreasing function. Therefore, for all $j \in \{1, 2, \cdots, n\}$, we have

$$\frac{2\beta_j+1}{\beta_j^2-\beta_j-1} \leq \frac{2\beta+1}{\beta^2-\beta-1}.$$

Because

$$1 \leq 1+\frac{2\beta+1}{\beta^2-\beta-1}\sum_{j=1}^{n}\frac{1}{\lambda_j} \leq \frac{3}{2}.$$

Further, we obtain $F_{\alpha_j,\beta_j,\lambda_j}(z) \in N(\delta)$, where $\delta = 1+\frac{2\beta+1}{\beta^2-\beta-1}\sum_{j=1}^{n}\frac{1}{\lambda_j}$.

Thus, the proof of Theorem 2 is complete.

Let $n = 1, \alpha_1 = \alpha, \beta_1 = \beta$ and $\lambda_1 = \lambda$ in Theorem 2, we obtain the following result.

Corollary 3. *Let* $\alpha \geq 1, \beta \geq \frac{1}{2}(1+\sqrt{5}),$ *and* $\lambda > 0$. *Moreover, suppose that these numbers satisfy the following inequality* $0 < \frac{2\beta+1}{\lambda(\beta^2-\beta-1)} \leq \frac{1}{2}$. *Then the function* $F_{\alpha,\beta,\lambda}(z)$ *is defined by*

$$F_{\alpha,\beta,\lambda}(z) = \int_0^z \left(\frac{\mathbb{E}_{\alpha,\beta}(t)}{t}\right)^{1/\lambda} dt$$

is in $N(\delta)$, *where* $\delta = 1 + \dfrac{2\beta+1}{\lambda(\beta^2-\beta-1)}$.

Example 1.

1. *If* $0 < \dfrac{5}{\lambda} \leq \dfrac{1}{2}$, *then* $\int\limits_0^z \left(\dfrac{\sinh\sqrt{t}}{\sqrt{t}}\right)^{1/\lambda} dt \in N(\delta),\ \delta = 1 + \dfrac{5}{\lambda}; \lambda \geq 10.$

2. *If* $0 < \dfrac{7}{5\lambda} \leq \dfrac{1}{2}$, *then* $\int\limits_0^z \left(\dfrac{2[\cosh\sqrt{t}-1]}{t}\right)^{1/\lambda} dt \in N(\delta),\ \delta = 1 + \dfrac{7}{5\lambda}; \lambda \geq \dfrac{14}{5}.$

3. *If* $0 < \dfrac{9}{11\lambda} \leq \dfrac{1}{2}$, *then* $\int\limits_0^z \left(\dfrac{6[\sinh\sqrt{t}-\sqrt{t}]}{t^{3/2}}\right)^{1/\lambda} dt \in N(\delta),\ \delta = 1 + \dfrac{9}{11\lambda}; \lambda \geq \dfrac{18}{11}.$

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