CERTAIN CLASSES OF ANALYTIC FUNCTIONS DEFINED BY CONVOLUTION

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Abstract. Making use of a convolution structure, we introduce a new class of analytic functions defined in the open unit disc and investigate coefficient estimates, distortion bounds and the results on integral transform. Further we investigate the neighbourhood results and partial sums.

2000 *Mathematics Subject Classification*: 30C45.

Keywords and Phrases: Univalent, starlike, convex, inclusion relations, generalized hypergeometric functions, $\delta-$neighborhoods, inclusion relations, Hadamard product (convolution), partial sums.

1. Introduction

Let $\mathcal{S}$ be the class of functions f *normalized* by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are *analytic* and *univalent* in the *open* unit disc $\mathcal{U} = \{z : |z| < 1\}$. As usual,we denote by $\mathcal{ST}$ and $\mathcal{CV}$ the subclasses of $\mathcal{S}$ that are respectively, *starlike* and *convex.*

For functions $f \in \mathcal{S}$ given by (1) and $g(z) \in \mathcal{S}$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \tag{2}$$

we recall that the Hadamard product (or convolution) of $f(z)$ and $g(z)$ written symbolically as $f * g$ defined (as usual)by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in U. \tag{3}$$

In terms of the Hadamard product (or convolution), we choose g as a fixed function in $\mathcal{S}$ such that $(f * g)(z) \neq 0$ for any $f \in \mathcal{S}$.For various choices of g we get different linear operators which have been studied in recent past. To illustrate some of these cases which arise from the convolution structure (3), we consider the following examples.

(1) For

$$g(z) = z + \sum_{n=2}^{\infty} \frac{(\alpha_1)_{n-1} \dots (\alpha_l)_{n-1}}{(\beta_1)_{n-1} \dots (\beta_m)_{n-1}} \frac{z^n}{(n-1)!} = z + \sum_{n=2}^{\infty} \Gamma_n z^n \tag{4}$$

where

$$\Gamma_n = \frac{(\alpha_1)_{n-1} \dots (\alpha_l)_{n-1}}{(\beta_1)_{n-1} \dots (\beta_m)_{n-1}} \, \frac{1}{(n-1)!},$$

we get the Dziok–Srivastava operator

$$\Lambda(\alpha_1, \alpha_2, \cdots, \alpha_l; \beta_1, \beta_2, \cdots, \beta_m; z)f(z) \equiv \mathcal{H}_m^l f(z) := (f * g)(z),$$

introduced by Dziok and Srivastava [4]; where $\alpha_1, \alpha_2, \cdots \alpha_l, \beta_1, \beta_2, \cdots \dots, \beta_m$ are positive real numbers, $\beta_j \notin \{0, -1, -2, \cdots\}$ for $j = 1, 2, \cdots, m$, $l \leq m + 1, l, m \in N \cup \{0\}$. Here $(a)_\nu$ denotes the well-known *Pochhammer symbol* (or shifted factorial).

Remark 1. When $l = 1, m = 1; \alpha_1 = a, \alpha_2 = 1; \beta_1 = c$ then the Dizok-Srivastava operator gives the operator due to Carlson-Shaffer operator [1], $\mathcal{L}(a, c)f(z) := (f * g)(z)$. The operator

$$\mathcal{L}(a, c)f(z) \equiv zF(a, 1; c; z) * f(z) := z + \sum_{n=2}^{\infty} \frac{(a)_n}{(c)_n} a_n z^n \quad (c \neq 0, -1, -2, \cdots), \tag{5}$$

where $F(a, b; c; z)$ is the well known *Gaussian hypergeometric* function.

Remark 2. When $l = 1, m = 0; \alpha_1 = \upsilon + 1, \alpha_2 = 1; \beta_1 = 1$ then the Dizok-Srivastava operator yields the Ruscheweyh derivative operator [6] defined by

$$D^\upsilon f(z) := (f * g)(z) = z + \sum_{n=2}^{\infty} \begin{pmatrix} \upsilon + n - 1 \\ n - 1 \end{pmatrix} a_n z^n. \tag{6}$$

(2) Furthermore, if

$$g(z) = z + \sum_{n=2}^{\infty} n \left(\frac{n + \mu}{1 + \mu} \right)^\sigma z^n \quad (\, \mu \geq 0; \; \sigma \in Z) \,, \tag{7}$$

we get the operator, $\mathcal{I}(\mu,\sigma)$ which was studied by Cho and Kim [2] and Cho and Srivastava [3].

(3) Lastly,if

$$g(z) = z + \sum_{n=2}^{\infty} \left(\frac{n+\mu}{1+\mu}\right)^k z^n \quad (\,\lambda \geq 0;\ k \in Z)\,, \tag{8}$$

we get *multiplier transformation* $\mathcal{J}(\mu,k) := (f*g)(z)$ introduced by Cho and Srivastava [3].

Remark 3. When $\mu = 0$, then the operator defined with (8) gives the *Sălăgean operator*

$$g(z) = z + \sum_{n=2}^{\infty} n^k z^n \quad (\,k \geq 0)\,, \tag{9}$$

which was initially studied by Sălăgean [8].

Definition 1. *For $0 \leq \lambda < 1$, $0 \leq \gamma < 1$ we let $\mathcal{ST}(g,\lambda,\gamma)$ be the subclass of S consisting of functions $f(z)$ of the form (1) and satisfying the analytic criteria*

$$Re\left\{\frac{z(f*g)'}{(1-\lambda)(f*g)(z)+\lambda z(f*g)'(z)}\right\} \geq \gamma, \quad z \in \mathcal{U}, \tag{10}$$

where $(f*g)(z)$ as given by (3) and g is fixed function for all $z \in \mathcal{U}$.

We also let

$$\mathcal{T}^*(g,\lambda,\gamma) = \mathcal{ST}(g,\lambda,\gamma) \bigcap \mathcal{T} \tag{11}$$

where $\mathcal{T}$ the subclass of $\mathcal{S}$ consists functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \quad z \in \mathcal{U} \tag{12}$$

was introduced and studied by Silverman [9].

We deem it proper to mention below some of the function classes which emerge from the function class $\mathcal{T}^*(g,\lambda,\gamma)$ defined above. Indeed, we observe that if we specialize the function $g(z)$ by means of (4) to (9), and denote the corresponding reducible classes of functions of $S_n(g;\lambda,b)$, respectively, by $\mathcal{T}_m^l(\lambda,\gamma)$, $\mathcal{T}_c^a(\lambda,\gamma)$, $\mathcal{T}^{\upsilon}(\lambda,\gamma)$, $\mathcal{T}_\mu^\sigma(\lambda,\gamma)$, $\mathcal{T}_\mu^k(\lambda,\gamma)$,and $\mathcal{T}^k(\lambda,\gamma)$ then it follows that viz.

(1) $f(z) \in \mathcal{T}_m^l(\lambda,\gamma)$

$$\Rightarrow \text{Re}\left\{\frac{z(\mathcal{H}_m^l(\alpha_1,\beta_1)f(z))'}{(1-\lambda)\mathcal{H}_m^l(\alpha_1,\beta_1)f(z)+\lambda z(\mathcal{H}_m^l(\alpha_1,\beta_1)f(z))'}\right\} \geq \gamma, \quad z \in \mathcal{U}. \tag{13}$$

(2) $f(z) \in \mathcal{T}_c^a(\lambda,\gamma)$

$$\Rightarrow \text{Re}\left\{\frac{z(\mathcal{L}(a,c)f(z))'}{(1-\lambda)\mathcal{L}(a,c)f(z)+\lambda z(\mathcal{L}(a,c)f(z))'}\right\} \geq \gamma, \quad z \in \mathcal{U}. \tag{14}$$

(3) $f(z) \in \mathcal{T}^v(\lambda,\gamma)$

$$\Rightarrow \text{Re}\left\{\frac{z(D^v f(z))'}{(1-\lambda)D^v f(z)+\lambda z(D^v f(z))'}\right\} \geq \gamma, \quad z \in \mathcal{U}. \tag{15}$$

(4) $f(z) \in \mathcal{T}_\mu^\sigma(\lambda,\gamma)$

$$\Rightarrow \text{Re}\left\{\frac{z(\mathcal{I}(\mu,\sigma)f(z))'}{(1-\lambda)\mathcal{I}(\mu,\sigma)f(z)+\lambda z(\mathcal{I}(\mu,\sigma)f(z))'}\right\} \geq \gamma, \quad z \in \mathcal{U}. \tag{16}$$

(5) $f(z) \in \mathcal{T}_\mu^k(\lambda,\gamma)$

$$\Rightarrow \text{Re}\left\{\frac{z(\mathcal{J}(\mu,k)f(z))'}{(1-\lambda)\mathcal{J}(\mu,k)f(z)+\lambda z(\mathcal{J}(\mu,k)f(z))'}\right\} \geq \gamma, \quad z \in \mathcal{U}. \tag{17}$$

(6) $f(z) \in \mathcal{T}^k(\lambda,\gamma)$

$$\Rightarrow \text{Re}\left\{\frac{z\mathcal{D}^{k+1}f(z)}{(1-\lambda)\mathcal{D}^k f(z)+\lambda z\mathcal{D}^{k+1}f(z)}\right\} \geq \gamma, \quad z \in \mathcal{U}. \tag{18}$$

In the following section we obtain coefficient inequalities and distortion bounds for functions in the class $\mathcal{T}^*(g,\lambda,\gamma)$.

2.Coefficient estimates and Distortion Bounds

First we obtain the coefficient inequalities for $f \in \mathcal{T}^*(g,\lambda,\gamma)$.

Theorem 1. *If $f \in \mathcal{S}$ satisfies*

$$\sum_{n=2}^{\infty}[n-(1+n\lambda-\lambda)\gamma]a_n b_n \leq 1-\gamma \tag{19}$$

then $f \in \mathcal{ST}^*(g,\lambda,\gamma)$.

Proof. Assume that the inequality(19)holds and let $|z|=1$. Then,

$$\left|\frac{z(f*g)'}{(1-\lambda)(f*g)(z)+\lambda z(f*g)'(z)}-1\right| = \left|\frac{z+\sum\limits_{n=2}^{\infty}na_nb_nz^n}{z+\sum\limits_{n=2}^{\infty}(1+n\lambda-\lambda)a_nb_nz^n}-1\right|$$

$$\leq \frac{\sum\limits_{n=2}^{\infty}na_nb_nz^{n-1}-\sum\limits_{n=2}^{\infty}(1+n\lambda-\lambda)a_nb_nz^{n-1}}{1-\sum\limits_{n=2}^{\infty}(1+n\lambda-\lambda)a_nb_nz^{n-1}}.$$

Letting $z \longrightarrow 1$ we have

$$\frac{\sum\limits_{n=2}^{\infty} n a_n b_n - \sum\limits_{n=2}^{\infty}(1+n\lambda-\lambda)a_n b_n}{1-\sum\limits_{n=2}^{\infty}(1+n\lambda-\lambda)a_n b_n}.$$

The above expression is bounded above by $1-\gamma$ that is ,

$$\frac{\sum\limits_{n=2}^{\infty} n a_n b_n - \sum\limits_{n=2}^{\infty}(1+n\lambda-\lambda)a_n b_n}{1-\sum\limits_{n=2}^{\infty}(1+n\lambda-\lambda)a_n b_n} \le 1-\gamma.$$

This shows that $\frac{z(f*g)'}{(1-\lambda)(f*g)(z)+\lambda z(f*g)'(z)} - 1$ lies in a circle centered at $w = 1$ whose radius is $1-\gamma$.Which completes the proof.

Theorem 2. *Let the function f be defined by (12). Then $f \in \mathcal{T}^*(g, \lambda, \gamma)$ if and only if*

$$\sum_{n=2}^{\infty}[n-(1+n\lambda-\lambda)\gamma]a_n b_n \le (1-\gamma) \tag{20}$$

The result is sharp for the function

$$f(z) = z - \frac{(1-\gamma)}{[n-(1+n\lambda-\lambda)\gamma]b_n} z^n, \quad n \ge 2. \tag{21}$$

Proof. In view of Theorem 1, we need only to prove the necessity. If $f(z) \in \mathcal{T}^*(g, \lambda, \gamma)$ and z is real then assume that the inequality(19)holds and let $|z| = 1$. Then,

$$\mathrm{Re}\left\{\frac{z(f*g)'}{(1-\lambda)(f*g)(z)+\lambda z(f*g)'(z)}\right\} = \mathrm{Re}\left\{\frac{z-\sum\limits_{n=2}^{\infty} n a_n b_n z^n}{z-\sum\limits_{n=2}^{\infty}(1+n\lambda-\lambda)a_n b_n z^n} - 1\right\} \ge \gamma,$$

$z \in \mathcal{U}$. Choose the values of z on the real axis so that $\frac{z(f*g)'}{(1-\lambda)(f*g)(z)+\lambda z(f*g)'(z)}$ is real and upon clearing the denominator in the above inequality and letting $z = 1^-$, through the real values we get

$$1-\sum_{n=2}^{\infty} n a_n b_n z^{n-1} \ge \gamma\left\{1-\sum_{n=2}^{\infty}(1+n\lambda-\lambda)a_n b_n z^{n-1}\right\}$$

which yields (20). The result is sharp for the function given by(21).

Corollary 1. *If $f(z)$ of the form (12) is in $\mathcal{T}^*(g,\lambda,\gamma)$, then*

$$a_n \leq \frac{(1-\gamma)}{[n-(1+n\lambda-\lambda)\gamma]b_n}, \quad n \geq 2, \tag{22}$$

with equality only for functions of the form (21).

Now we state coefficient inequalities for other subclasses defined in this paper,as a corollary.

Corollary 2. *A function $f \in \mathcal{T}_m^l(\lambda,\gamma)$ if and only if*

$$\sum_{n=2}^{\infty}[n-(1+n\lambda-\lambda)\gamma]a_n\Gamma_n \leq 1-\gamma,$$

where $\Gamma_n = \frac{(\alpha_1)_{n-1}(\alpha_2)_{n-1},\dots,(\alpha_l)_{n-1}}{(\beta_1)_{n-1}(\beta_2)_{n-1},\dots,(\beta_m)_{n-1}(n-1)!}$.

Remark 4. For specific choices of parameters l, m, α_1, β_1 (as mentioned in the Remarks 1 and 2), Corollary 2, would yield the coefficient bound inequalities for the subclasses of functions $\mathcal{T}_c^a(\lambda,\gamma), \mathcal{T}^v(\lambda,\gamma)$ of functions respectively.

Corollary 3. *A function $f \in \mathcal{T}_\mu^\sigma(\lambda,\gamma)$ if and only if*

$$\sum_{n=2}^{\infty} n\left(\frac{n+\mu}{1+\mu}\right)^{\sigma}[n-(1+n\lambda-\lambda)\gamma]a_n \leq 1-\gamma.$$

Corollary 4. *A function $f \in \mathcal{T}_\mu^k(\lambda,\gamma)$ if and only if*

$$\sum_{n=2}^{\infty}\left(\frac{n+\mu}{1+\mu}\right)^{k}[n-(1+n\lambda-\lambda)\gamma]a_n \leq 1-\gamma. \tag{23}$$

Remark 5. When $\mu = 0$, (23) would give the coefficient bound inequalities for the subclass $\mathcal{T}^k(\lambda,\gamma)$ of functions.

Theorem 3.(Distortion Bounds) *Let the function $f(z)$ defined by (12) belong to $\mathcal{T}^*(g,\lambda,\gamma)$. Then for $|z| \leq r$,*

$$r - \frac{(1-\gamma)}{[2-(1+\lambda)\gamma]b_2}r^2 \leq |f(z)| \leq r + \frac{(1-\gamma)}{[2-(1+\lambda)\gamma]b_2}r^2 \tag{24}$$

and

$$1-\frac{2(1-\gamma)}{[2-(1+\lambda)\gamma]b_2}r \le |f'(z)| \le 1+\frac{2(1-\gamma)}{[2-(1+\lambda)\gamma]b_2}r \tag{25}$$

for $z \in \mathcal{U}$.

Proof. In the view of (20), we have

$$\begin{aligned}[2-(1+\lambda)\gamma]b_2\sum_{n=2}^{\infty}a_n = & \sum_{n=2}^{\infty}[n-(1+n\lambda-\lambda)\gamma]a_nb_n \\ \le & (1-\gamma)\end{aligned}$$

which is equivalent to

$$\sum_{n=2}^{\infty}a_n \le \tfrac{(1-\gamma)}{[2-(1+\lambda)\gamma]b_2}. \tag{26}$$

Using (12) and (26) we obtain

$$\begin{aligned}|f(z)| &\ge |z|-|z|^2\sum_{n=2}^{\infty}a_n \\ &\ge |z|-|z|^2\frac{(1-\gamma)}{[2-(1+\lambda)\gamma]b_2} \\ &\ge |z|\left\{1-\frac{(1-\gamma)}{[2-(1+\lambda)\gamma]b_2}|z|\right\} \end{aligned}\tag{27}$$

and

$$|f(z)| \le |z|\left\{1+\frac{(1-\gamma)}{[2-(1+\lambda)\gamma]b_2}|z|\right\} \tag{28}$$

From (27) and (28) with $|z| \le r$, we have (24). Again using (12) and (26) we have,

$$\begin{aligned}|f'(z)| &\ge 1-2|z|\sum_{n=2}^{\infty}a_n \\ &\ge 1-\frac{2(1-\gamma)}{[2-(1+\lambda)\gamma]b_2}|z| \end{aligned}\tag{29}$$

and

$$|f'(z)| \le 1+\frac{2(1-\gamma)}{[2-(1+\lambda)\gamma]b_2}|z| \tag{30}$$

From (29) and (30) with $|z| \le r$, we have (25).

3. Extreme Points and Closure Theorems

In this section we discuss the closure properties.

Theorem 4.(Extreme Points) *Let*

$$\begin{aligned} f_1(z) &= z \quad \text{and} \\ f_n(z) &= z - \tfrac{(1-\gamma)}{[n-(1+n\lambda-\lambda)]b_n} z^n, \ \ n \geq 2, \end{aligned} \tag{31}$$

for $0 \leq \gamma < 1$, $\lambda \geq 0$. *Then* $f(z)$ *is in the class* $\mathcal{T}^*(g, \lambda, \gamma)$ *if and only if it can be expressed in the form*

$$f(z) = \sum_{n=1}^{\infty} \mu_n f_n(z), \tag{32}$$

where $\mu_n \geq 0$ *and* $\sum\limits_{n=1}^{\infty} \mu_n = 1$.

Proof. Suppose $f(z)$ can be written as in (32). Then

$$f(z) \ = \ z - \sum_{n=2}^{\infty} \mu_n \frac{(1-\gamma)}{[n-(1+n\lambda-\lambda)]b_n} z^n.$$

Now,

$$\sum_{n=2}^{\infty} \frac{[n-(1+n\lambda-\lambda)]b_n}{(1-\gamma)} \mu_n \frac{(1-\gamma)}{[n-(1+n\lambda-\lambda)]b_n} \ = \ \sum_{n=2}^{\infty} \mu_n = 1 - \mu_1 \leq 1.$$

Thus $f \in \mathcal{T}^*(g, \lambda, \gamma)$. Conversely, let us have $f \in \mathcal{T}^*(g, \lambda, \gamma)$. Then by using (32), we set

$$\mu_n = \frac{[n-(1+n\lambda-\lambda)]b_n}{(1-\gamma)} a_n, \quad n \geq 2$$

and $\mu_1 = 1 - \sum_{n=2}^{\infty} \mu_n$. Then we have $f(z) = \sum_{n=1}^{\infty} \mu_n f_n(z)$ and hence this completes the proof of Theorem 4.

Theorem 5.(Closure Theorem) *Let the functions* $f_j(z)$ *be defined for* $j = 1, 2, \ldots, m$ *defined by*

$$f_j(z) = z - \sum_{n=2}^{\infty} a_{n,j} z^n, \quad a_{n,j} \geq 0, \ \ z \in \mathcal{U} \tag{33}$$

be in the class $\mathcal{T}^*(g, \lambda, \gamma_j)$ $(j = 1, 2, \ldots m)$ *respectively. Then the function* $h(z)$ *defined by*

$$h(z) = z - \frac{1}{m} \sum_{n=2}^{\infty} \left(\sum_{j=1}^{m} a_{n,j} \right) z^n$$

is in the class $\mathcal{T}^*(g,\lambda,\gamma)$, *where* $\gamma = \min_{1\le j\le m}\{\gamma_j\}$ *where* $0 \le \gamma_j < 1$.

Proof. Since $f_j(z) \in \mathcal{T}^*(g,\lambda,\gamma_j)$ $(j = 1,2,3,\dots m)$ by applying Theorem 2, to (33) we observe that

$$\begin{aligned}&\sum_{n=2}^{\infty}[n-(1+n\lambda-\lambda)\gamma]b_n\left(\frac{1}{m}\sum_{j=1}^{m}a_{n,j}\right)\\ &\quad=\frac{1}{m}\sum_{j=1}^{m}\left(\sum_{n=2}^{\infty}[n-(1+n\lambda-\lambda)\gamma]b_n a_{n,j}\right)\\ &\quad\le\frac{1}{m}\sum_{j=1}^{m}(1-\gamma_j)\le 1-\gamma\end{aligned}$$

which in view of Theorem 2, implies that $h(z) \in \mathcal{T}^*(g,\lambda,\gamma)$ and so the proof is complete.

Theorem 6. *The class* $\mathcal{T}^*(g,\lambda,\gamma)$ *is a convex set.*

Proof. Let the functions defined by (33) be in the class $\mathcal{T}^*(g,\lambda,\gamma)$. It sufficient to show that the function $h(z)$ defined by

$$h(z) = \mu f_1(z) + (1-\mu) f_2(z), \quad 0 \le \mu \le 1,$$

is in the class $\mathcal{T}^*(g,\lambda,\gamma)$. Since

$$h(z) = z - \sum_{n=2}^{\infty}[\mu a_{n,1} + (1-\mu)a_{n,2}]z^n,$$

an easy computation with the aid of Theorem 2 gives,

$$\begin{aligned}&\sum_{n=2}^{\infty}[n-(1+n\lambda-\lambda)\gamma]\mu b_n a_{n,1}+\sum_{n=2}^{\infty}[n-(1+n\lambda-\lambda)\gamma](1-\mu)b_n a_{n,2}\\ &\le \mu(1-\gamma)+(1-\mu)(1-\gamma)\\ &\le 1-\gamma,\end{aligned}$$

which implies that $h \in \mathcal{T}^*(g,\lambda,\gamma)$. Hence $\mathcal{T}^*(g,\lambda,\gamma)$ is convex.

4. Radii of Starlikeness and Convexity

In this section we obtain the radii of close-to-convexity, starlikeness and convexity for the class $\mathcal{T}^*(g,\lambda,\gamma)$.

Theorem 7. *Let the function $f(z)$ defined by (12) belong to the class $\mathcal{T}^*(g,\lambda,\gamma)$. Then $f(z)$ is close-to-convex of order ρ $(0 \le \rho < 1)$ in the disc $|z| < r_1$, where*

$$r_1 := \left[\frac{(1-\rho)}{n}\frac{[n-(1+n\lambda-\lambda)\gamma]b_n}{(1-\gamma)}\right]^{\frac{1}{n-1}} \quad (n \geq 2). \tag{34}$$

The result is sharp, with extremal function $f(z)$ given by (31).

Proof. Given $f \in \mathcal{T}$, and f is close-to-convex of order ρ, we have

$$|f'(z) - 1| < 1 - \rho. \tag{35}$$

For the left hand side of (35) we have

$$|f'(z) - 1| \leq \sum_{n=2}^{\infty} n a_n |z|^{n-1}.$$

The last expression is less than $1-\rho$ if

$$\sum_{n=2}^{\infty} \frac{n}{1-\rho} a_n |z|^{n-1} < 1.$$

Using the fact, that $f \in \mathcal{T}^*(g,\lambda,\gamma)$ if and only if

$$\sum_{n=2}^{\infty} \frac{[n-(1+n\lambda-\lambda)\gamma]b_n}{(1-\gamma)} a_n \leq 1,$$

We can say (35) is true if

$$\frac{n}{1-\rho}|z|^{n-1} \leq \frac{[n-(1+n\lambda-\lambda)\gamma]b_n}{(1-\gamma)}.$$

Or, equivalently,

$$|z|^{n-1} = \left[\left(\frac{1-\rho}{n}\right)\frac{[n-(1+n\lambda-\lambda)\gamma]b_n}{(1-\gamma)}\right]$$

which completes the proof.

Theorem 8. *Let $f \in \mathcal{T}^*(g,\lambda,\gamma)$. Then*

(i) f is starlike of order $\rho (0 \le \rho < 1)$ in the disc $|z| < r_2$; that is, $Re \left\{\frac{zf'(z)}{f(z)}\right\} > \rho$, $(|z| < r_2 \;;\; 0 \le \rho < 1)$, where

$$r_2 = \inf_{n \geq 2}\left[\left(\frac{1-\rho}{n-\rho}\right)\frac{[n-(1+n\lambda-\lambda)\gamma]b_n}{(1-\gamma)}\right]^{\frac{1}{n-1}} \quad . \tag{36}$$

(ii) f *is convex of order* ρ $(0 \leq \rho < 1)$ *in the unit disc* $|z| < r_3$, *that is* $Re\ \left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \rho$, $(|z| < r_3; 0 \leq \rho < 1)$, *where*

$$r_3 = \inf_{n \geq 2} \left[\left(\frac{1-\sigma}{n(n-\rho)} \right) \frac{[n-(1+n\lambda-\lambda)\gamma]b_n}{(1-\gamma)} \right]^{\frac{1}{n-1}} . \tag{37}$$

Each of these results are sharp for the extremal function $f(z)$ *given by (31).*
Proof. (i) Given $f \in \mathcal{T}$, and f is starlike of order ρ, we have

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 - \rho. \tag{38}$$

For the left hand side of (38) we have

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \frac{\sum\limits_{n=2}^{\infty} (n-1)a_n\ |z|^{n-1}}{1 - \sum\limits_{n=2}^{\infty} a_n\ |z|^{n-1}}.$$

The last expression is less than $1 - \sigma$ if

$$\sum_{n=2}^{\infty} \frac{n-\rho}{1-\rho} a_n\ |z|^{n-1} < 1.$$

Using the fact, that $f \in \mathcal{T}^*(g, \lambda, \gamma)$ if and only if

$$\sum_{n=2}^{\infty} \frac{[n-(1+n\lambda-\lambda)\gamma]}{(1-\gamma)} a_n b_n \leq 1.$$

We can say (38) is true if

$$\frac{n-\rho}{1-\rho}|z|^{n-1} < \frac{[n-(1+n\lambda-\lambda)\gamma]b_n}{(1-\gamma)}.$$

Or, equivalently,

$$|z|^{n-1} = \left[\left(\frac{1-\rho}{n-\rho} \right) \frac{[n-(1+n\lambda-\lambda)\gamma]b_n}{(1-\gamma)} \right]$$

which yields the starlikeness of the family.

(ii) Given $f \in \mathcal{T}$, and f is convex of order ρ, we have

$$\left| \frac{zf''(z)}{f'(z)} \right| < 1 - \rho. \tag{39}$$

For the left hand side of (39) we have

$$\left|\frac{zf''(z)}{f'(z)}\right| \leq \frac{\sum\limits_{n=2}^{\infty} n(n-1)a_n\ |z|^{n-1}}{1-\sum\limits_{n=2}^{\infty} na_n\ |z|^{n-1}}.$$

The last expression is less than $1-\rho$ if

$$\sum_{n=2}^{\infty} \frac{n(n-\rho)}{1-\rho} a_n\ |z|^{n-1} < 1.$$

Using the fact, that $f \in \mathcal{T}^*(g,\lambda,\gamma)$ if and only if

$$\sum_{n=2}^{\infty} \frac{[n-(1+n\lambda-\lambda)\gamma]}{(1-\gamma)} a_n b_n \leq 1.$$

We can say (39) is true if

$$\frac{n(n-\rho)}{1-\rho}|z|^{n-1} < \frac{[n-(1+n\lambda-\lambda)\gamma]b_n}{(1-\gamma)}.$$

Or, equivalently,

$$|z|^{n-1} = \left[\left(\frac{1-\rho}{n(n-\rho)}\right)\frac{[n-(1+n\lambda-\lambda)\gamma]b_n}{(1-\gamma)}\right]$$

which yields the convexity of the family.

5. Neighbourhood Results

In this section we discuss neighbourhood results of the class $\mathcal{T}^*(g,\lambda,\gamma)$. Following [5,7], we define the $\delta-$ neighborhood of function $f(z) \in \mathcal{T}$ by

$$N_\delta(f) := \left\{ h \in T : h(z) = z - \sum_{n=2}^{\infty} d_n z^n \ \text{ and } \ \sum_{n=2}^{\infty} n|a_n - d_n| \leq \delta \right\}. \tag{40}$$

Particulary for the identity function $e(z) = z$, we have

$$N_\delta(e) := \left\{ h \in T : g(z) = z - \sum_{n=2}^{\infty} d_n z^n \ \text{ and } \ \sum_{n=2}^{\infty} n|d_n| \leq \delta \right\}. \tag{41}$$

Theorem 9. *If*

$$\delta := \frac{2(1-\gamma)}{[2-(1+\lambda)\gamma]b_2} \tag{42}$$

then $\mathcal{T}^*(g,\lambda,\gamma) \subset N_\delta(e)$.

Proof. For $f \in \mathcal{T}^*(g,\lambda,\gamma)$, Theorem 2, immediately yields

$$[2-(1+\lambda)\gamma]b_2 \sum_{n=2}^{\infty} a_n \leq 1-\gamma,$$

so that

$$\sum_{n=2}^{\infty} a_n \leq \frac{(1-\gamma)}{[2-(1+\lambda)\gamma]b_2}. \tag{43}$$

On the other hand, from (20) and (43) that

$$\begin{aligned} b_2\sum_{n=2}^{\infty} na_n &\leq (1-\gamma)+(1+\lambda)\gamma b_2\sum_{n=2}^{\infty} a_n \\ &\leq (1-\gamma)+(1+\lambda)\gamma b_2\frac{(1-\gamma)}{[2-(1+\lambda)\gamma]b_2} \\ &\leq \frac{2(1-\gamma)}{[2-(1+\lambda)\gamma]}, \end{aligned}$$

that is

$$\sum_{n=2}^{\infty} na_n \leq \frac{2(1-\gamma)}{[2-(1+\lambda)\gamma]b_2} := \delta \tag{44}$$

which, in view of the definition (41) proves Theorem 9.

Now we determine the neighborhood for the class $\mathcal{T}^{*(\rho)}(g,\lambda,\gamma)$ which we define as follows. A function $f \in \mathcal{T}$ is said to be in the class $\mathcal{T}^{*(\rho)}(g,\lambda,\gamma)$ if there exists a function $h \in \mathcal{T}^{*(\rho)}(g,\lambda,\gamma)$ such that

$$\left|\frac{f(z)}{h(z)}-1\right| < 1-\rho, \quad (z \in \mathcal{U}, \quad 0 \leq \rho < 1). \tag{45}$$

Theorem 10. *If* $h \in \mathcal{T}^{*(\rho)}(g,\lambda,\gamma)$ *and*

$$\rho = 1 - \frac{\delta[2-(1+\lambda)\gamma]b_2}{2([2-(1+\lambda)\gamma]b_2-(1-\gamma))} \tag{46}$$

then

$$N_\delta(h) \subset \mathcal{T}^{*(\rho)}(g,\lambda,\gamma). \tag{47}$$

Proof. Suppose that $f \in N_\delta(h)$ we then find from (40) that

$$\sum_{n=2}^{\infty} n|a_n - d_n| \leq \delta$$

which implies that the coefficient inequality

$$\sum_{n=2}^{\infty} |a_n - d_n| \leq \frac{\delta}{2}.$$

Next, since $h \in \mathcal{T}^*(g, \lambda, \gamma)$, we have

$$\sum_{n=2}^{\infty} d_n = \frac{(1-\gamma)}{[2-(1+\lambda)\gamma]b_2}$$

so that

$$\begin{aligned}
\left|\frac{f(z)}{h(z)} - 1\right| &< \frac{\sum\limits_{n=2}^{\infty} |a_n - d_n|}{1 - \sum\limits_{n=2}^{\infty} d_n} \\
&\leq \frac{\delta}{2} \times \frac{[2-(1+\lambda)\gamma]b_2}{[2-(1+\lambda)\gamma]b_2 - (1-\gamma)} \\
&\leq \frac{\delta[2-(1+\lambda)\gamma]b_2}{2([2-(1+\lambda)\gamma]b_2 - (1-\gamma))} \\
&= 1 - \rho.
\end{aligned}$$

provided that ρ is given precisely by (47). Thus by definition, $f \in \mathcal{T}^{*(\rho)}(g, \lambda, \gamma)$ for ρ given by (47), which completes the proof.

6. Partial Sums

Following the earlier works by Silverman [10] and Silvia [11] on partial sums of analytic functions. We consider in this section partial sums of functions in the class $\mathcal{ST}^*(g, \lambda, \gamma)$ and obtain sharp lower bounds for the ratios of real part of $f(z)$ to $f_k(z)$ and $f'(z)$ to $f_k'(z)$.

Theorem 11. *Let $f(z) \in \mathcal{ST}^*(g, \lambda, \gamma)$. Define the partial sums $f_1(z)$ and $f_k(z)$, by*

$$f_1(z) = z; \text{ and } f_k(z) = z + \sum_{n=2}^{k} a_n z^n, \ (k \in N/1) \tag{48}$$

Suppose also that

$$\sum_{n=2}^{\infty} c_n|a_n| \leq 1,$$

where

$$c_n := \frac{[n-(1+n\lambda-\lambda)\gamma]b_n}{(1-\gamma)}. \tag{49}$$

Then $f \in \mathcal{ST}^*(g,\lambda,\gamma)$. *Furthermore,*

$$Re\left\{\frac{f(z)}{f_k(z)}\right\} > 1 - \frac{1}{c_{k+1}} \quad z \in \mathcal{U}, k \in N \tag{50}$$

and

$$Re\left\{\frac{f_k(z)}{f(z)}\right\} > \frac{c_{k+1}}{1+c_{k+1}}. \tag{51}$$

Proof. For the coefficients c_n given by (49) it is not difficult to verify that

$$c_{n+1} > c_n > 1. \tag{52}$$

Therefore we have

$$\sum_{n=2}^{k} |a_n| + c_{k+1}\sum_{n=k+1}^{\infty} |a_n| \leq \sum_{n=2}^{\infty} c_n|a_n| \leq 1 \tag{53}$$

by using the hypothesis (49). By setting

$$\begin{aligned} g_1(z) &= c_{k+1}\left\{\frac{f(z)}{f_k(z)} - \left(1-\frac{1}{c_{k+1}}\right)\right\} \\ &= 1 + \frac{c_{k+1}\sum\limits_{n=k+1}^{\infty} a_n z^{n-1}}{1+\sum\limits_{n=2}^{k} a_n z^{n-1}} \end{aligned} \tag{54}$$

and applying (53), we find that

$$\begin{aligned} \left|\frac{g_1(z)-1}{g_1(z)+1}\right| &\leq \frac{c_{k+1}\sum\limits_{n=k+1}^{\infty} |a_n|}{2-2\sum\limits_{n=2}^{n} |a_n| - c_{k+1}\sum\limits_{n=k+1}^{\infty} |a_n|} \\ &\leq 1, \ \ z \in U, \end{aligned} \tag{55}$$

which readily yields the assertion (50) of Theorem 11. In order to see that

$$f(z) = z + \frac{z^{k+1}}{c_{k+1}} \tag{56}$$

gives sharp result, we observe that for $z = re^{i\pi/k}$ that $\frac{f(z)}{f_k(z)} = 1 + \frac{z^k}{c_{k+1}} \to 1 - \frac{1}{c_{k+1}}$ as $z \to 1^-$. Similarly, if we take

$$\begin{aligned} g_2(z) &= (1 + c_{k+1}) \left\{ \frac{f_k(z)}{f(z)} - \frac{c_{k+1}}{1 + c_{k+1}} \right\} \\ &= 1 - \frac{(1 + c_{n+1}) \sum\limits_{n=k+1}^{\infty} a_n z^{n-1}}{1 + \sum\limits_{n=2}^{\infty} a_n z^{n-1}} \end{aligned} \tag{57}$$

and making use of (53), we can deduce that

$$\left| \frac{g_2(z) - 1}{g_2(z) + 1} \right| \leq \frac{(1 + c_{k+1}) \sum\limits_{n=k+1}^{\infty} |a_n|}{2 - 2 \sum\limits_{n=2}^{k} |a_n| - (1 - c_{k+1}) \sum\limits_{n=k+1}^{\infty} |a_n|} \tag{58}$$

which leads us immediately to the assertion (51) of Theorem 11.

The bound in (51) is sharp for each $k \in N$ with the extremal function $f(z)$ given by (56). The proof of the Theorem 11, is thus complete.

Theorem 12. *If $f(z)$ of the form (1) satisfies the condition (22). Then*

$$Re \left\{ \frac{f'(z)}{f_k'(z)} \right\} \geq 1 - \frac{k+1}{c_{k+1}}. \tag{59}$$

Proof. By setting

$$\begin{aligned} g(z) &= c_{k+1} \left\{ \frac{f'(z)}{f_k'(z)} - \left(1 - \frac{k+1}{c_{k+1}} \right) \right\} \\ &= \frac{1 + \frac{c_{k+1}}{k+1} \sum\limits_{n=k+1}^{\infty} n a_n z^{n-1} + \sum\limits_{n=2}^{\infty} n a_n z^{n-1}}{1 + \sum\limits_{n=2}^{k} n a_n z^{n-1}} \end{aligned}$$

$$= 1 + \frac{\frac{c_{k+1}}{k+1}\sum\limits_{n=k+1}^{\infty} n a_n z^{n-1}}{1+\sum\limits_{n=2}^{k} n a_n z^{n-1}}.$$

$$\left|\frac{g(z)-1}{g(z)+1}\right| \leq \frac{\frac{c_{k+1}}{k+1}\sum\limits_{n=k+1}^{\infty} n|a_n|}{2-2\sum\limits_{n=2}^{k} n|a_n| - \frac{c_{k+1}}{k+1}\sum\limits_{n=k+1}^{\infty} n|a_n|}. \tag{60}$$

Now

$$\left|\frac{g(z)-1}{g(z)+1}\right| \leq 1$$

if

$$\sum_{n=2}^{k} n|a_n| + \frac{c_{k+1}}{k+1}\sum_{n=k+1}^{\infty} n|a_n| \leq 1 \tag{61}$$

since the left hand side of (61) is bounded above by $\sum\limits_{n=2}^{k} c_n|a_n|$ if

$$\sum_{n=2}^{k}(c_n - n)|a_n| + \sum_{n=k+1}^{\infty} c_n - \frac{c_{k+1}}{k+1} n|a_n| \geq 0, \tag{62}$$

and the proof is complete. The result is sharp for the extremal function $f(z) = z + \frac{z^{k+1}}{c_{k+1}}$.

Theorem 13. *If $f(z)$ of the form (1.1) satisfies the condition (22) then*

$$Re\left\{\frac{f_k'(z)}{f'(z)}\right\} \geq \frac{c_{k+1}}{k+1+c_{k+1}}. \tag{63}$$

Proof. By setting

$$g(z) = [(k+1)+c_{k+1}]\left\{\frac{f_k'(z)}{f'(z)} - \frac{c_{k+1}}{k+1+c_{k+1}}\right\}$$

$$= 1 - \frac{\left(1+\frac{c_{k+1}}{k+1}\right)\sum\limits_{n=k+1}^{\infty} n a_n z^{n-1}}{1+\sum\limits_{n=2}^{k} n a_n z^{n-1}}$$

and making use of (62), we deduce that

$$\left|\frac{g(z)-1}{g(z)+1}\right| \leq \frac{\left(1+\frac{c_{k+1}}{k+1}\right)\sum\limits_{n=k+1}^{\infty} n|a_n|}{2-2\sum\limits_{n=2}^{k} n|a_n| - \left(1+\frac{c_{k+1}}{k+1}\right)\sum\limits_{n=k+1}^{\infty} n|a_n|} \leq 1,$$

which leads us immediately to the assertion of the Theorem 13.

Concluding Remarks. We observe that if we specialize the function $g(z)$ by means of (4) to (9) we would arrive at the analogous results for the classes $\mathcal{T}_m^l(\lambda,\gamma)$, $\mathcal{T}_c^a(\lambda,\gamma)$, $\mathcal{T}^v(\lambda,\gamma)$, $\mathcal{T}_\mu^\sigma(\lambda,\gamma)$, $\mathcal{T}_\mu^k(\lambda,\gamma)$ and $\mathcal{T}^k(\lambda,\gamma)$ (defined above in Section 1). These obvious consequences of our results being straightforward, further details are hence omitted here.

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