A RELATED FIXED POINT THEOREM IN N COMPLETE METRIC SPACES

Fayçal Merghadi, Abdelkrim Aliouche, Brian Fisher

Abstract. We prove a related fixed point theorem for n mappings in n complete metric spaces via implicit relations. Our result generalizes Theorem 1.1 of [1].

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1. Introduction

Recently, A. Aliouche and B. Fisher [1] proved the following related fixed point theorem in two complete metric spaces.

Theorem 1.1 *Let (X,d) and (Y,ρ) be complete metric spaces and let S,T be mappings of Y into X and of X into Y respectively, satisfying the inequalities*

$$f\left(\rho\left(Tx,TSy\right),d\left(x,Sy\right),\rho\left(y,Tx\right),\rho\left(y,TSy\right)\right)\leq 0\ ,$$

$$g\left(d\left(Sy,STx\right),\rho\left(y,Tx\right),d\left(x,Sy\right),d\left(x,STx\right)\right)\leq 0$$

for all x in X and y in Y, where $f,g\in F$. Then ST has a unique fixed point in X and TS has a unique fixed point v in Y. Further, $Tu=v$ and $Sv=u$.

We denote by $\mathbb{R}_+$ the set of non-negative reals and by Φ the set of all functions $\phi:\mathbb{R}_+^4\to\mathbb{R}$ such that
(i) ϕ is upper semi continuous in each coordinate variable,
(ii) if either $\phi\left(u,v,0,u\right)\leq 0$ or $\phi\left(u,v,u,0\right)\leq 0$ for all $u,v\in\mathbb{R}_+$, then there exists a real constant c, with $0\leq c<1$, such that $u\leq cv$.

Example 1.2. $\phi\left(t_1,t_2,t_3,t_4\right)=t_1-c\max\left\{t_2,t_3,t_4\right\},\ 0\leq c<1.$

Example 1.3. $\phi\left(t_1,t_2,t_3,t_4\right)=t_1^2-c_1\max\left\{t_2^2,t_3^2,t_4^2\right\}-c_2\max\left\{t_1t_3,t_2t_4\right\}-c_3t_3t_4$, where $c_1,c_2,c_3\in\mathbb{R}_+$ and $c_1+c_2<1$.

2. Main results

Theorem 2.1. *Let (X_i, d_i) be n complete metric spaces and let $\{A_i\}_{i=1}^{i=n}$ be n mappings such that $A_i : X_i \to X_{i+1}$ for $i = 1, \ldots, n-1$ and $A_n : X_n \to X_1$, satisfying the inequalities*

$$\phi_1 \begin{pmatrix} d_1(A_nA_{n-1}...A_2x_2, A_nA_{n-1}...A_1x_1), \\ d_2(x_2, A_1A_nA_{n-1}...A_2x_2), \\ d_1(x_1, A_nA_{n-1}...A_2x_2), d_1(x_1, A_nA_{n-1}...A_1x_1) \end{pmatrix} \leq 0 \qquad (2.1)$$

for all $x_1 \in X_1$ and $x_2 \in X_2$, in general

$$\phi_i \begin{pmatrix} d_i(A_{i-1}A_{i-2}...A_1A_nA_{n-1}...A_{i+1}x_{i+1}, A_{i-1}A_{i-2}...A_1A_nA_{n-1}...A_ix_i), \\ d_{i+1}(x_{i+1}, A_iA_{i-1}..A_1A_nA_{n-1}...A_{i+1}x_{i+1}), \\ d_i(x_i, A_{i-1}A_{i-2}..A_1A_nA_{n-1}...A_{i+1}x_{i+1}), \\ d_i(x_i, A_{i-1}A_{i-2}...A_1A_nA_{n-1}...A_ix_i) \end{pmatrix} \leq 0 \qquad (2.i)$$

for all $x_i \in X_i$, $x_{i+1} \in X_{i+1}$ and $i = 2, \ldots, n-1$, and

$$\phi_n \begin{pmatrix} d_n(A_{n-1}A_{n-2}...A_1x_1, A_{n-1}A_{n-2}...A_1A_nx_n), \\ d_1(x_1, A_nA_{n-1}A_{n-2}...A_1x_1), \\ d_n(x_n, A_{n-1}A_{n-2}...A_1x_1), \\ d_n(x_n, A_{n-1}A_{n-2}...A_1A_nx_n) \end{pmatrix} \leq 0 \qquad (2.n)$$

for all $x_1 \in X_1$, $x_n \in X_n$, where $\phi_i \in \Phi$, for $i = 1, \ldots, n$. Then

$$A_{i-1}A_{i-2}...A_1A_nA_{n-1}...A_i$$

has a unique fixed point $p_i \in X_i$ for $i = 1, \ldots, n$. Further, $A_ip_i = p_{i+1}$ for $i = 1, \ldots, n-1$ and $A_np_n = p_1$.

Proof. Let $\left\{x_r^{(1)}\right\}_{r\in N}, \left\{x_r^{(2)}\right\}_{r\in N}, \ldots, \left\{x_r^{(i)}\right\}_{r\in N}, \ldots, \left\{x_r^{(r)}\right\}_{r\in N}$ be sequences in X_1, $X_2, \ldots, X_i, \ldots, X_n$, respectively.

Now let $x_0^{(1)}$ be an arbitrary point in X_1. We define the sequences $\left\{x_r^{(i)}\right\}_{r\in N}$ for $i = 1, \ldots, n$ by

$$x_r^{(1)} = (A_nA_{n-1}...A_1)^r x_0^{(1)},$$

$$x_r^{(i)} = A_{i-1}A_{i-2}...A_1 (A_nA_{n-1}...A_1)^r x_0^{(1)}$$

for $i = 2, \ldots, n$.

For $n = 1, 2, \ldots$, we assume that $x_n^{(1)} \neq x_{n+1}^{(1)}$. Applying the inequality (2.1) for $x_2 = A_1 \left(A_n A_{n-1} ... A_1\right)^{r-1} x_0^{(1)}$, $x_1 = \left(A_n A_{n-1} ... A_1\right)^{r} x_0^{(1)}$ we get

$$\phi_1 \begin{pmatrix} d_1\left(\left(A_nA_{n-1}...A_1\right)^r x_0^{(1)}, \left(A_nA_{n-1}...A_1\right)^{r+1} x_0^{(1)}\right), \\ d_2\left(A_1\left(A_nA_{n-1}...A_1\right)^{r-1} x_0^{(1)}, A_1\left(A_nA_{n-1}...A_1\right)^{r} x_0^{(1)}\right), \\ d_1\left(\left(A_nA_{n-1}...A_1\right)^r x_0^{(1)}, \left(A_nA_{n-1}...A_1\right)^{r} x_0^{(1)}\right), \\ d_1\left(\left(A_nA_{n-1}...A_1\right)^r x_0^{(1)}, \left(A_nA_{n-1}...A_1\right)^{r+1} x_0^{(1)}\right) \end{pmatrix}$$
$$= \phi_1 \begin{pmatrix} d_1\left(x_r^{(1)}, x_{r+1}^{(1)}\right), & d_2\left(x_{r-1}^{(2)}, x_r^{(2)}\right), \\ 0, & d_1\left(x_r^{(1)}, x_{r+1}^{(1)}\right) \end{pmatrix} \leq 0.$$

From the implicit relation we have

$$d_1\left(x_r^{(1)}, x_{r+1}^{(1)}\right) \leq c d_2\left(x_{r-1}^{(2)}, x_r^{(2)}\right), \tag{3.1}$$

for $r = 1, 2, \ldots$.

Applying the inequality (2.i) for $x_{i+1} = A_i ... A_1 \left(A_n ... A_1\right)^{r-1} x_0^{(1)}$ and $x_i = A_{i-1} ... A_1 \left(A_n ... A_1\right)^{r} x_0^{(1)}$, we obtain

$$\phi_i \begin{pmatrix} d_i\left(A_{i-1}...A_1\left(A_n...A_1\right)^r x_0^{(1)}, A_{i-1}\left(A_n...A_1\right)^{r+1} x_0^{(1)}\right), \\ d_{i+1}\left(A_i...A_1\left(A_n...A_1\right)^{r-1} x_0^{(1)}, A_i...A_1\left(A_n...A_1\right)^{r} x_0^{(1)}\right), \\ d_i\left(A_{i-1}...A_1\left(A_n...A_1\right)^r x_0^{(1)}, A_{i-1}...A_1\left(A_n...A_1\right)^{r} x_0^{(1)}\right), \\ d_i\left(A_{i-1}...A_1\left(A_{i-1}...A_1\right)^r x_0^{(1)}, A_{i-1}...A_1\left(A_n...A_1\right)^{r+1} x_0^{(1)}\right) \end{pmatrix}$$
$$= \phi_i \begin{pmatrix} d_i\left(x_r^{(i)}, x_{r+1}^{(i)}\right), & d_{i+1}\left(x_{r-1}^{(i+1)}, x_r^{(i+1)}\right), \\ 0, & d_{i+1}\left(x_r^{(i+1)}, x_{r+1}^{(i+1)}\right) \end{pmatrix} \leq 0$$

and so

$$d_i\left(x_r^{(i)}, x_{r+1}^{(i)}\right) \leq c d_{i+1}\left(x_{r-1}^{(i+1)}, x_r^{(i+1)}\right), \tag{3.i}$$

for $i = 2, \ldots, n-1$ and $r = 1, 2, \ldots$.

Now applying the inequality (2.n) for $x_n = A_{n-1} ... A_1 \left(A_n ... A_1\right)^r x_0^{(1)}$ and $x_1 =$

$(A_n ... A_1)^r x_0^{(1)}$, we have

$$\phi_n \begin{pmatrix} d_n\left(A_{n-1}...A_1 (A_n...A_1)^r x_0^{(1)}, A_{n-1}...A_1 (A_n...A_1)^{r+1} x_0^{(1)}\right), \\ d_1\left((A_n...A_1)^r x_0^{(1)}, (A_n...A_1)^{r+1} x_0^{(1)}\right), \\ d_n\left(A_{n-1}...A_1 (A_n...A_1)^r x_0^{(1)}, A_{n-1}...A_1 (A_n...A_1)^r x_0^{(1)}\right), \\ d_n\left(A_{n-1}...A_1 (A_n...A_1)^r x_0^{(1)}, A_{n-1}...A_1 (A_n...A_1)^r x_0^{(1)}\right) \end{pmatrix}$$

$$= \phi_n \begin{pmatrix} d_n\left(x_n^{(n)}, x_{n+1}^{(n)}\right), & d_1\left(x_{n-1}^{(1)}, x_n^{(1)}\right), \\ 0, & d_n\left(x_n^{(n)}, x_{n+1}^{(n)}\right) \end{pmatrix} \leq 0$$

and so

$$d_n\left(x_r^{(n)}, x_{r+1}^{(n)}\right) \leq c d_1\left(x_{r-1}^{(1)}, x_r^{(1)}\right), \tag{3.n}$$

for $r = 1, 2, \ldots$.

It now follows from (3.1), (3.i) and (3.n) that for large enough n

$$\begin{aligned} d_i\left(x_r^{(i)}, x_{r+1}^{(i)}\right) &\leq c d_{i+1}\left(x_{r-1}^{(i+1)}, x_r^{(i+1)}\right) \\ &\leq \ldots \\ &\leq c^{n-i} d_n\left(x_{r+i-n}^{(n)}, x_{r+i-n+1}^{(n)}\right) \\ &\leq c^{n-i+1} d_1\left(x_{r+i-n-1}^{(1)}, x_{r+i-n}^{(1)}\right) \\ &\leq \ldots \\ &\leq c^{2n-i+1} d_1\left(x_{r+i-2n-1}^{(1)}, x_{r+i-2n}^{(1)}\right) \\ &\leq \ldots \\ &\leq c^{mn-i+1} d_1\left(x_{r+i-mn-1}^{(1)}, x_{r+i-mn}^{(1)}\right) \\ &\leq c^{mn} \max\left\{d_1\left(x_1^{(1)}, x_2^{(1)}\right), \ldots, d_n\left(x_1^{(n)}, x_2^{(n)}\right)\right\}. \end{aligned}$$

Since $c < 1$, it follows that $\left\{x_r^{(i)}\right\}$ is Cauchy sequences in X_i with a limit p_i in X_i for $i = 1, 2, \ldots, n$.

To prove that p_i is a fixed point of $A_{i-1}...A_1 A_n...A_i p_i$ for $i = 2, \ldots, n-1$, suppose that $A_{i-1}...A_1 A_n...A_i p_i \neq p_i$. Using the inequality (2.i) for $x_i = p_i$ and $x_{i+1} = x_r^{(i+1)}$, we obtain

$$\phi_i \begin{pmatrix} d_i\left(x_{r+1}^{(i)}, A_{i-1}...A_1 A_n...A_i p_i\right), \\ d_{i+1}\left(x_r^{(i+1)}, x_{r+1}^{(i+1)}\right), d_i\left(p_i, x_r^{(i)}\right), \\ d_i\left(p_i, A_{i-1}...A_1 A_n...A_i p_i\right) \end{pmatrix} \leq 0.$$

Letting $r \to \infty$, we have

$$\phi_i \begin{pmatrix} d_i(p_i, A_{i-1}...A_1A_n...A_ip_i), 0, 0, \\ d_i(p_i, A_{i-1}...A_1A_n...A_ip_i) \end{pmatrix} \leq 0.$$

It follows from (ii) that $p_i = A_{i-1}...A_1A_n...A_ip_i$ for $i = 2, \ldots, n-1$.

For the case $i = 1$, we use (2.1) with $x_1 = p_1$ and $x_2 = A_1(A_n...A_1)^{r-1}x_0^{(1)}$, giving

$$\phi_1 \begin{pmatrix} d_1\left(x_r^{(1)}, A_n...A_1p_1\right), \\ d_2\left(x_r^{(2)}, x_{r+1}^{(2)}\right), d_1\left(p_1, x_r^{(1)}\right), \\ d_1(p_1, A_n...A_1p_1) \end{pmatrix} \leq 0.$$

Letting $r \to \infty$, we have

$$\phi_i \begin{pmatrix} d_1(p_1, A_n...A_1p_1), 0, 0, \\ d_1(p_1, A_n...A_1p_1) \end{pmatrix} \leq 0.$$

It follows from (ii) that $p_1 = A_n...A_1p_1$.

Finally, if $i = n$, using the inequality (2.n) for $x_n = p_n$ and $x_1 = x_n^{(1)}$ we get

$$\phi_n \begin{pmatrix} d_n\left(x_{r+1}^{(n)}, A_{n-1}...A_1A_np_n\right), \\ d_1\left(x_r^{(1)}, x_{r+1}^{(1)}\right), d_n\left(p_n, x_{r+1}^{(n)}\right), \\ d_n(p_n, A_{n-1}...A_1A_np_n) \end{pmatrix} \leq 0.$$

Letting $r \to \infty$, we have

$$\phi_n \begin{pmatrix} d_n(p_n, A_{n-1}...A_1A_np_n), 0, 0, \\ d_n(p_n, A_{n-1}...A_1A_np_n) \end{pmatrix} \leq 0$$

and by (ii), $p_n = A_{n-1}...A_1A_np_n$.

To prove the uniqueness, suppose that $A_{i-1}...A_1A_n...A_i$ has a second fixed point $z_i \neq p_i$ in X_i.

Using the inequality (2.i) for $x_{i+1} = A_iz_i$ and $x_i = p_i$ we get

$$\begin{aligned} &\phi_1 \begin{pmatrix} d_1(A_i...A_1A_n...A_iz_i, A_{i-1}...A_1A_n...A_ip_i), \\ d_{i+1}(A_iz_i, A_i...A_1A_n...A_{i+1}z_i) \\ d_i(p_i, A_{i-1}...A_1A_n...A_{i+1}z_i), d_i(p_i, A_{i-1}...A_1A_n...A_ip_i) \end{pmatrix} \\ = \;& \phi_1(d_i(z_i, p_i), 0, d_i(p_i, z_i), 0) \leq 0 \end{aligned}$$

which implies that $z_i = p_i$, proving the uniqueness of p_i for $i = 2, \ldots, n-1$. The uniqueness of p_1 in X_1 and p_n in X_n follow similarly.

We finally note that

$$A_i p_i = A_i ... A_1 A_n ... A_{i+1}(A_i p_i),$$

so that $A_i p_i$ is a fixed point of $A_i ... A_1 A_n ... A_{i+1}$. Since the fixed point is unique, it follows that $A_i p_i = p_{i+1}$ for $i = 1, \ldots, n-1$. It follows similarly that $A_n p_n = p_1$. This completes the proof of the theorem.

Example 2.2. Let (X_i, d) for $i = 1, \ldots, n$ be n complete metric spaces where $X_i = \{x_i : i-1 \leq x_i \leq i\}$ for $i = 1, \ldots, n$ and d is the usual metric for the real numbers. Define $A_i : X_i \to X_{i+1}$ for $i = 1, \ldots, n-1$ and $A_n : X_n \to X_1$ by

$$\begin{aligned} A_1 x_1 &= \begin{cases} 5/4 & \text{if } 0 \leq x_1 < 1/2, \\ 3/2 & \text{if } 1/2 \leq x_1 \leq 1, \end{cases} \\ A_i x_i &= \begin{cases} i + 1/4 & \text{if } i-1 \leq x_i < i - 3/4, \\ i + 1/2 & \text{if } i - 3/4 \leq x_i \leq i \end{cases} \end{aligned}$$

for $i = 2, \ldots, n-1$,

$$A_n x_n = \begin{cases} 3/4 & \text{if } n-1 \leq x_n < n - 3/4, \\ 1 & \text{if } n - 3/4 \leq x_n \leq n \end{cases}$$

and $\phi_1 = \phi_2 = ... \phi_n = \phi \in \Phi$ such that $\phi(t_1, t_2, t_3, t_4) = t_1 - c \max\{t_2, t_3, t_4\}$ and $0 \leq c < 1$

Note that there exists p_i in X_i such that $A_{i-1} ... A_1 A_n ... A_i p_i = p_i$ for $i = 1, \ldots, n$.

For example if we put

(a) $i = n$, we get $A_{n-1} ... A_1 A_n p_n = p_n$ if $w_n = n - \dfrac{1}{2}$ because

$$\begin{aligned} A_{n-1} ... A_1 A_n (n - 1/2) &= A_{n-1} ... A_1 (1) \\ &= A_{n-1} A_{n-2} .. A_2 (3/2) \\ &= A_{n-1} ... A_{i+1} (i + 1/2) \\ &= \ldots \\ &= A_{n-1} (n - 5/2) \\ &= A_{n-1} (n - 3/2) = n - 1/2 \end{aligned}$$

and $n \leq n - 3/2 \leq n - 3/4$.

(b) Note that for $i = 1, \ldots, n-1$ and $i - 3/4 \leq x_i < i$, $(i+1) - 3/4 \leq A_i x_i < i+1$ and $1/2 \leq A_n x_n \leq 1$ with $n - 3/4 \leq x_n < n$, there exists $p_i = i - 1/2$ such that $A_{i-1} ... A_1 A_n ... A_i (i - 1/2) = i - 1/2$ for $i = 1, \ldots, n-1$. Further,

$$d(A_{i-1} ... A_1 A_n ... A_{i+1} x_{i+1}, A_{i-1} ... A_1 A_n ... A_i x_i) = 0$$

for $i = 1, \dots, n$ and

$$\phi_i \begin{pmatrix} d\left(A_{i-1}...A_1A_n...A_{i+1}x_{i+1}, A_{i-1}...A_1A_n...A_ix_i\right), \\ d\left(x_{i+1}, A_i...A_1A_n...A_{i+1}x_{i+1}\right), \\ d\left(x_i, A_{i-1}...A_1A_n...A_{i+1}x_{i+1}\right), \\ d\left(x_i, A_{i-1}...A_1A_n...A_ix_i\right) \end{pmatrix}$$

$$= -c\max\left\{ \begin{array}{c} d\left(x_{i+1}, A_i...A_1A_n...A_{i+1}x_{i+1}\right), \\ d\left(x_i, A_{i-1}...A_1A_n...A_{i+1}x_{i+1}\right), \\ d\left(x_i, A_{i-1}...A_1A_n...A_ix_i\right) \end{array} \right\} \leq 0$$

which is true for all $x_i \in X_i$, $x_{i+1} \in X_{i+1}$ and $0 \leq c < 1$. Thus all conditions of Theorem 2.1 are satisfied.

If we take $n = 5$ in Theorem 2.1, we get the following Corollary.

Corollary 2.3. *Let* (X_i, d_i), $i = 1, \dots, 5$ *be* 5 *complete metric spaces,* $A_i : X_i \to X_{i+1}$, $i = 1, 2, 3, 4$ *and* $A_5 : X_5 \to X_1$ *be* 5 *mappings satisfying*

$$\phi_1 \begin{pmatrix} d_1\left(A_5A_4A_3A_2x_2, A_5A_4A_3A_2A_1x_1\right), \\ d_2\left(x_2, A_1A_5A_4A_3A_2x_2\right) \\ d_1\left(x_1, A_5A_4A_3A_2x_2\right), d_1\left(x_1, A_5A_4A_3A_2A_1x_1\right) \end{pmatrix} \leq 0 \tag{7.1}$$

for all $x_1 \in X_1$ *and* $x_2 \in X_{\ 2}$ *with* $x_2 \neq A_1x_1$,

$$\phi_2 \begin{pmatrix} d_2\left(A_1A_5A_4A_3x_3, A_1A_5A_4A_3A_2x_2\right), \\ d_3\left(x_3, A_2A_1A_5A_4A_3x_3\right) \\ d_2\left(x_2, A_1A_5A_4A_3x_3\right), d_2\left(x_2, A_1A_5A_4A_3A_2x_2\right) \end{pmatrix} \leq 0 \tag{7.2}$$

for all $x_2 \in X_2$ *and* $x_3 \in X_3$ *with* $x_3 \neq A_2x_2$,

$$\phi_3 \begin{pmatrix} d_3\left(A_2A_1A_nA_{n-1}...A_4x_4, A_2A_1A_nA_{n-1}...A_3x_3\right), \\ d_4\left(x_4, A_3A_2A_1A_nA_{n-1}..A_4x_4\right) \\ d_3\left(x_3, A_2A_1A_nA_{n-1}...A_4x_4\right), \\ d_3\left(x_3, A_2A_1A_nA_{n-1}...A_3x_3\right) \end{pmatrix} \leq 0 \tag{7.3}$$

for all $x_3 \in X_2$ *and* $x_4 \in X_3$ *with* $x_4 \neq A_3x_3$,

$$\phi_4 \begin{pmatrix} d_4\left(A_3A_2A_1A_5x_5, A_3A_2A_1A_5A_4x_4\right), \\ d_5\left(x_5, A_4A_3A_2A_1A_5x_5\right) \\ d_4\left(x_4, A_3A_2A_1A_5x_5\right), \\ d_4\left(x_4, A_3A_2A_1A_5x_5A_4x_4\right) \end{pmatrix} \leq 0$$

for all $x_4 \in X_4$ *and* $x_5 \in X_5$ *with* $x_5 \neq A_4x_4$.

$$\phi_5 \begin{pmatrix} d_5\left(A_4A_3A_2A_1x_1, A_4A_3A_2A_1A_5x_5\right), \\ d_1\left(x_1, A_5A_4A_3A_2A_1x_1\right) \\ d_5\left(x_5, A_4A_3A_2A_1x_1\right), \\ d_5\left(x_5, A_4A_3A_2A_1A_5x_5\right) \end{pmatrix} \leq 0 \tag{7.5}$$

for all $x_1 \in X_1$ *and* $x_5 \in X_5$ *with* $x_1 \neq A_5x_5$. *Then* $A_{i-1}A_{i-2}...A_1A_nA_{n-1}..A_i$ *has a unique fixed point* $p_i \in X_i$, *for all* $i = 1, \ldots, 5$

The following example illustrates our Corollary 2.3.

Example 2.4. Let (X_i, d) for $i = 1, \ldots, 5$ be 5 metric spaces where $X_i = \{x_i : i-1 \leq i \leq i\}$ for $i = 1, \ldots, 5$ and d is the usual metric for the real numbers. Define $A_i : X_i \to X_{i+1}$, for $i = 1, \ldots, 4$ and $A_5 : X_5 \to X_1$ by

$$\begin{aligned} A_ix_1 &= \begin{cases} 1 \text{ if } 0 \leq x_1 < 3/4, \\ 3/2 \text{ if } 3/4 \leq x_1 \leq 1 \end{cases}, A_2x_2 = \begin{cases} 5/2 \text{ if } 1 \leq x_2 < 3/2, \\ 3 \text{ if } 3/2 \leq x_2 \leq 2 \end{cases}, \\ A_3x_3 &= \begin{cases} 13/4 \text{ if } 2 \leq x_3 < 5/2, \\ 7/2 \text{ if } 5/2 \leq x_3 \leq 3 \end{cases}, A_4x_4 = \begin{cases} 17/4 \text{ if } x_4 \leq 3 < 7/2 \\ \frac{9}{2} \text{ if } x_4 \in [\frac{7}{2}, 4[\end{cases}, \\ A_5x_5 &= \begin{cases} 3/4 \text{ if } 4 \leq x_5 < 9/2 \\ 1 \text{ if } 9/2 \leq x_5 \leq 5 \end{cases}. \end{aligned}$$

Let $\phi_1\left(t_1, t_2, t_3, t_4\right) = t_1 - c\max\left\{t_2, t_3, t_4\right\}$ and $\phi_1 = \phi_2 = \phi_3 = \phi_4 = \phi_5$. Then

$$\begin{aligned} A_5A_4A_3A_2A_1\left(\frac{3}{2}\right) &= \frac{3}{2}, \\ A_1A_5A_4A_3A_2\left(3\right) &= 3, \\ A_2A_1A_5A_4A_3\left(\frac{7}{2}\right) &= \frac{7}{2}, \\ A_3A_2A_1A_5A_4\left(\frac{9}{2}\right) &= \frac{9}{2}, \\ A_4A_3A_2A_1A_5\left(1\right) &= 1. \end{aligned}$$

The inequalities $(7,1)$, (7.2), (7.3), (7.4) and (7.5) are satisfied for $i = 1, \ldots, 5$ since the value of the left hand side of each inequality is 0. Hence, all the conditions of Corollary 2.3 are satisfied.

We finally note that if we take $n = 2$ in Theorem 2.1, we obtain Theorem 2.3 of [1].

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Fayçal Merghadi
Department of Mathematics
University of Larbi Tebessa Tebessa, 12000, Algeria.
email: *faycel_mr@yahoo.fr*

Abdelkrim Aliouche
Department of Mathematics
University of Larbi Ben M'Hidi Oum-El-Bouaghi, 04000, Algeria.
email: *alioumath@yahoo.fr*

Brian Fisher
Department of Mathematics
University of Leicester, Leicester, LE8 5PJ, UK.
email: *fbr@le.ac.uk*