MAPPINGS ON S-PARACOMPACT SPACES

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Abstract. In this paper, we prove that open, perfect mappings both preserve and inversely preserve S-paracompact spaces. As some applications these results, some sum-theorems and product-theorems for S-paracompact spaces are obtained.

2000 *Mathematics Subject Classification*: 54D20, 54C10, 54C05, 54B10, 54B99.

Keywords and phrases: S-paracompact space, open perfect mapping, product space, sum space

1. Introduction

In [2], K. Y. Al-Zoubi introduced S-paracompact spaces, and obtained many interesting properties of S-paracompact spaces. Recently X. Ge also gave some investigations for S-paracompact spaces [6]. The purpose of this paper is to investigate open perfect mappings on S-paracompact spaces. We prove that open perfect mappings both preserve and inversely preserve S-paracompact spaces. As some applications of these results, we obtain some sum-theorems and product-theorems for S-paracompact spaces, which generalize related results in [2].

Throughout this paper, all mappings are continuous and onto. $\mathbf{N}$ denotes the set of all natural numbers, ω denotes the first infinite cardinal, X and Y denote topological spaces. For a subset P of a space X, $\overline{P}$ denotes the closure of P in X. Let $\mathcal{U}$ and $\mathcal{V}$ be two covers of semi-open subsets of a space X. We say that $\mathcal{V}$ is a semi-open refinement of $\mathcal{U}$, if for each $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ such that $V \subset U$. Let $\mathcal{U}$ be a collection of subsets of a space X and $F \subset X$. $\bigcup\mathcal{U}$ and $\mathcal{U} \bigwedge F$ denote the union $\bigcup\{U : U \in \mathcal{U}\}$ and the collection $\{U \bigcap A : U \in \mathcal{U}\}$, respectively. Let $f : X \longrightarrow Y$ be a mapping, and let $\mathcal{U}$ and $\mathcal{V}$ are two collections of subsets of X and Y, respectively, then $f(\mathcal{U}) = \{f(U) : U \in \mathcal{U}\}$ and $f^{-1}(\mathcal{V}) = \{f^{-1}(V) : V \in \mathcal{V}\}$. The term "clopen" means "both open and closed". One may refer to [5] for undefined notations and terminology.

2. Invariance of Images of S-paracompact Spaces

Definition 2.1. *Let X be a space.*

(1) A collection $\mathcal{F} = \{F_\alpha : \alpha \in I\}$ of subsets of a space X is called to be locally finite [4], if for each $x \in X$, there exists an open neighborhood U_x of x such that U_x intersects at most finitely many members of $\mathcal{F}$.

(2) A subset B of X is called a semi-open subset of X [7] if there exists an open set U of X such that $U \subset B \subset \overline{U}$.

(3) A space X is called S-paracompact [2] if each open cover of X has a locally finite semi-open refinement.

Definition 2.2[5]. *Let $f : X \longrightarrow Y$ be a mapping.*

(1) f is called a compact mapping if $f^{-1}(y)$ is a compact subset of X for each $y \in Y$.

(2) f is called an open (resp. closed) mapping if $f(U)$ is an open (resp. closed) subset of Y for each open (resp. closed) subset U of X.

(3) f is called a perfect mapping if f is a closed and compact mapping.

(4) f is called an open perfect mapping if f is an open and perfect mapping.

The following lemma comes from [5, Theorem 1.4.13]

Lemma 2.3. *A mapping $f : X \longrightarrow Y$ is closed if and only if for each $y \in Y$ and each open subset U in X which contains $f^{-1}(y)$, there exists an open neighborhood V of y in Y such that $f^{-1}(V) \subset U$.*

Theorem 2.4. *Let $f : X \longrightarrow Y$ be an open perfect mapping. If X is S-paracompact, then Y is S-paracompact.*

Proof. Assume X is an S-paracompact space. Let $\mathcal{U}$ be an open cover of Y. Then $f^{-1}(\mathcal{U})$ is an open cover of X, and so $f^{-1}(\mathcal{U})$ has a locally finite semi-open refinement $\mathcal{V}$. It is easy to see that $f(\mathcal{V})$ is a semi-open refinement of $\mathcal{U}$ because f is clopen. It suffices to prove that $f(\mathcal{V})$ is locally-finite in Y.

Let $y \in Y$. For each $x \in f^{-1}(y)$, since $\mathcal{V}$ is locally-finite, there exists an open neighborhood G_x of x such that G_x intersects at most finitely many members of $\mathcal{V}$. Note that f is a compact mapping, there exists a finite subcollection $\mathcal{W}_y$ of $\{G_x : x \in f^{-1}(y)\}$ such that $f^{-1}(y) \subset \bigcup \mathcal{W}_y$. It is clear that $\bigcup \mathcal{W}_y$ intersects at most finitely many members of $\mathcal{V}$. By Lemma 2.3, there exists an open neighborhood O_y of y in Y such that $f^{-1}(O_y) \subset \bigcup \mathcal{W}_y$, then $f^{-1}(O_y)$ intersects at most finitely many members of $\mathcal{V}$. Therefore O_y intersects at most finitely many members of $f(\mathcal{V})$. This proves that $f(\mathcal{V})$ is locally-finite in Y.

As an application of Theorem 2.4, we give the following two sum theorems for S-paracompactness, which generalize [2, Theorem 4.1].

Theorem 2.5. *Let $\{X_\alpha : \alpha \in I\}$ be a locally finite clopen cover of a space X. Then X is S-paracompact if and only if X_α is S-paracompact for each $\alpha \in I$.*

Proof. Necessity: It follows from [2, Corollary 3.5].

sufficiency: The proof is based on a construction which is essentially due to K.Morita [8]. For each $\alpha \in I$, let Y_α denote a copy of X_α and let f_α be this homeomorphism. Let Y be the disjoint topological sum of $\{Y_\alpha : \alpha \in I\}$. By [2, Theorem 4.1], Y is S-paracompact. Let $f : Y \longrightarrow X$ be the mapping defined as follows:

For each $x \in Y$, $f(x) = f_\alpha(x)$ if $x \in Y_\alpha$.

By Theorem 2.4, we only need to prove that f is open perfect.

(1) f is compact: Let $x \in X$. Since $\{X_\alpha : \alpha \in I\}$ is locally finite, x belongs to at most finitely many members of $\{X_\alpha : \alpha \in I\}$. It follows that $f^{-1}(x)$ is finite. So f is compact.

(2) f is open: Let U be an open subset of Y. Then $U = \bigcup\{U_\alpha : \alpha \in I'\}$, where U_α is an open subset Y_α for each $\alpha \in I'$, and $I' \subset I$. Further, $f(U) = \bigcup\{f(U_\alpha) : \alpha \in I'\}$. For each $\alpha \in I'$, Since $f(U_\alpha) = f_\alpha(U_\alpha)$, $f(U_\alpha)$ is an open subset of X_α. Note that X_α is an open subset of X, $f(U_\alpha)$ is also an open subset of X. It follows that $f(U)$ is an open subset of X. So f is open.

(3) f is closed: Let F be a closed subset of Y. Then $F = \bigcup\{F_\alpha : \alpha \in I'\}$, where F_α is a closed subset F_α for each $\alpha \in I'$, and $I' \subset I$. Further, $f(F) = \bigcup\{f(F_\alpha) : \alpha \in I'\}$. For each $\alpha \in I'$, Since $f(F_\alpha) = f_\alpha(F_\alpha)$, $f(F_\alpha)$ is a closed subset of X_α. Note that X_α is an closed subset of X, $f(F_\alpha)$ is also a closed subset of X. Since $\{X_\alpha : \alpha \in I\}$ is locally finite and each $f(F_\alpha) \subset X_\alpha$, $\{f(F_\alpha) : \alpha \in I'\}$ is locally finite. By [4, Proposition 1.1(iv)], $f(F) = \bigcup\{f(F_\alpha) : \alpha \in I'\} = \overline{\bigcup\{f(F_\alpha) : \alpha \in I'\}}$. It follows that $f(F)$ is a closed subset of X. So f is closed.

By the above (1), (2) and (3), f is open perfect.

For a set B, we denote the cardinal of B by $|B|$. Recall a mapping $f : X \longrightarrow Y$ is a k-to-1 mapping [3], if $|f^{-1}(x)| = k < \omega$ for each $x \in X$.

Theorem 2.6. *Let $\{X_\alpha : \alpha \in I\}$ be an open cover of a space X, and let $|\{\alpha \in I : x \in X_\alpha\}| = k < \omega$ for each $x \in X$. Then X is S-paracompact if and only if X_α is S-paracompact for each $\alpha \in I$.*

Proof. Necessity: Assume X is S-paracompact. Let $\alpha \in I$. By [2, Corollary 3.5], it suffices to prove that X_α is a closed subset of X. Let $x \notin X_\alpha$. Put $\{\beta \in I :$

$x \in X_\beta\} = I'$, and put $U = \bigcap\{X_\beta : \beta \in I'\}$. Then U is an open neighborhood of x. We claim that $U \bigcap X_\alpha = \emptyset$. In fact, if there exists $x' \in U \bigcap X_\alpha$, then $|\{\beta \in I : x' \in X_\beta\}| > k$. This is a contradiction.

sufficiency: Assume X_α is S-paracompact for each $\alpha \in I$. By a similar way as in the proof of the sufficiency of Theorem 2.4, we construct an open and compact mapping $f : Y \longrightarrow X$, where Y is S-paracompact. It is not difficult to discover that f is a k-to-1 mapping. By [3, Lemma 1 and Lemma 2], each open and k-to-1 mapping is a closed mapping. Thus f is open perfect. By Theorem 2.4, X is S-paracompact.

3. Invariance of Inverse Images of S-paracompact Spaces

Lemma 3.1. *Let $f : X \longrightarrow Y$ be an open mapping. If U is a semi-open subset of Y and V is an open subset of X, then $f^{-1}(U) \bigcap V$ is a semi-open subset of X.*

Proof. Let U be a semi-open subset of Y and V be an open subset of X. Since U is a semi-open subset of Y, there exists an open subset G of Y such that $G \subset U \subset \overline{G}$, and so $f^{-1}(G) \subset f^{-1}(U) \subset f^{-1}(\overline{G})$. By [5, 1.4.C], $\overline{f^{-1}(G)} = f^{-1}(\overline{G})$ because f is an open mapping. Thus $f^{-1}(G) \subset f^{-1}(U) \subset \overline{f^{-1}(G)}$. Note that $f^{-1}(G)$ is an open subset of X, so $f^{-1}(U)$ is a semi-open subset of X. By [2, Lemma 1.5(a)], $f^{-1}(U) \bigcap V$ is a semi-open subset of X.

Theorem 3.2. *Let $f : X \longrightarrow Y$ be an open perfect mapping. If Y is S-paracompact, then X is S-paracompact.*

Proof. Assume Y is an S-paracompact space. Let $\mathcal{U}$ be an open cover of X. For each $y \in Y$, there exists a finite subcollection $\mathcal{U}_y$ of $\mathcal{U}$ such that $f^{-1}(y) \subset \bigcup \mathcal{U}_y$ because f is a compact mapping. By Lemma 2.3, there exists an open neighborhood V_y of y in Y such that $f^{-1}(V_y) \subset \bigcup \mathcal{U}_y$ Put $\mathcal{V} = \{V_y : y \in Y\}$, then $\mathcal{V}$ is an open cover of Y. Since Y is S-paracompact, $\mathcal{V}$ has a locally finite semi-open refinement $\mathcal{W}$. Without loss of generality, we may assume $\mathcal{W} = \{W_y : y \in Y\}$, where $W_y \subset V_y$ for each $y \in Y$. Put $\mathcal{F}_y = \mathcal{U}_y \bigwedge f^{-1}(W_y)$ for each $y \in Y$. By Lemma 3.1, each member of $\mathcal{F}_y$ is a semi-open subset of X. Put $\mathcal{F} = \bigcup\{\mathcal{F}_y : y \in Y\}$, then $\mathcal{F}$ is a semi-open refinement of $\mathcal{U}$. It suffices to prove that $\mathcal{F}$ is locally finite.

Let $x \in X$. Because $\mathcal{W}$ is locally finite in Y and f inversely preserves locally finite collections, $f^{-1}(\mathcal{W})$ is locally finite in X. So there exists a neighborhood U_x of x in X and a finite subset Y_0 of Y such that for each $y \in Y - Y_0$, U_x misses $f^{-1}(W_y)$. Further, U_x misses each member of $\mathcal{F}_y$ for each $y \in Y - Y_0$. Thus $\{F \in \mathcal{F} : U_x \bigcap F \neq \emptyset\} \subset \bigcup\{\mathcal{F}_y : y \in Y_0\}$. Note that $\mathcal{F}_y$ is finite for each $y \in Y_0$. $\{F \in \mathcal{F} : U_x \bigcap F \neq \emptyset\}$ is finite. This proves that $\mathcal{F}$ is locally finite.

As an application of Theorem 3.2, we give a proof of [2, Theorem 4.2] by mappings.

Theorem 3.3. *Let X be a compact space and let Y be an S-paracompact space. Then $X \times Y$ is S-paracompact.*

Proof. Let $f : X \times Y \longrightarrow Y$ be the projection. By Theorem 3.2, We only need to prove that the projection $f : X \times Y \longrightarrow Y$ is open perfect.

It is well known that each projection is an open mapping. For each $y \in Y$, it is easy to see that $f^{-1}(y) = X \times \{y\}$, which is homeomorphous to X, so $f^{-1}(y)$ is a compact subset of $X \times Y$. Thus, f is a compact mapping. It suffices to prove that f is a closed mapping.

Let F is a closed subset $X \times Y$ and let $y \notin f(F)$. Then for each $x \in X$, $(x, y) \notin F$, and so there exist an open neighborhood U_x of x in X and an open neighborhood V_x of y in Y such that $(U_x \times V_x) \bigcap F = \emptyset$. Put $\mathcal{U} = \{U_x : x \in X\}$, then $\mathcal{U}$, which is an open cover of the compact space X, has a finite subcover $\mathcal{U}'$ of X. Let $\mathcal{U}' = \{U_x : x \in X'\}$, where X' is a finite subset of X. Put $V_y = \bigcap\{V_x : x \in X'\}$, then V_y is an open neighborhood of y in Y. It is easy to see that $(X \times V_y) \bigcap F = \emptyset$, and so $V_y \bigcap f(F) = \emptyset$, thus $f(F)$ is a closed subset of Y. This proves that f is a closed mapping.

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