UNIVALENCE CRITERION FOR AN INTEGRAL OPERATOR

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Abstract.We consider the integral operator denoted by $T_{\alpha,\beta}$ and for the function $f \in \mathcal{A}$ we proved a sufficient condition for univalence from this integral operator.

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1.Introduction

Let $\mathcal{A}$ be the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let S denote the subclass of $\mathcal{A}$ consisting of all univalent functions f in $\mathcal{U}$.

For $f \in \mathcal{A}$, the integral operator G_α is defined by

$$G_\alpha(z) = \int_0^z \left(\frac{f(u)}{u}\right)^{\frac{1}{\alpha}} du \tag{1.1}$$

for some complex numbers $\alpha(\alpha \neq 0)$.

In [1] Kim-Merkes prove that the integral operator G_α is in the class $\mathcal{S}$ for $\frac{1}{|\alpha|} \leq \frac{1}{4}$ and $f \in \mathcal{S}$.

Also, the integral operator J_γ for $f \in \mathcal{A}$ is given by

$$M_\gamma(z) = \left\{\frac{1}{\gamma}\int_0^z u^{-1}\left(f(u)\right)^{\frac{1}{\gamma}} du\right\}^\gamma \tag{1.2}$$

γ be a complex number, $\gamma \neq 0$.

Miller and Mocanu [3] have studied that the integral operator M_γ is in the class S for $f \in \mathcal{S}^*, \gamma > 0, \mathcal{S}^*$ is the subclass of $\mathcal{S}$ consisting of all starlike functions f in $\mathcal{U}$.

We consider the integral operator $T_{\alpha,\beta}$ defined by

$$T_{\alpha,\beta} = \left[\beta \int_0^z u^{\beta-1} \left(\frac{f(u)}{u}\right)^{\frac{1}{\alpha}} du\right]^{\frac{1}{\beta}} \tag{1.3}$$

for $f \in \mathcal{A}$ and α, β be complex numbers, $\alpha \neq 0$, $\beta \neq 0$.

We need the following lemmas.

Lemma 1.1.[6].*Let α be a complex number, $Re\ \alpha > 0$ and $f \in \mathcal{A}$. If*

$$\frac{1-|z|^{2Re\ \alpha}}{Re\ \alpha} \left|\frac{zf''(z)}{f'(z)}\right| \leq 1 \tag{1.4}$$

for all $z \in \mathcal{U}$, then for any complex number β, $Re\ \beta \geq Re\ \alpha$ the function

$$F_\beta(z) = \left[\beta \int_0^z u^{\beta-1} f'(u) du\right]^{\frac{1}{\beta}} \tag{1.5}$$

is in the class S.

Lemma 1.2.(Schwarz [2]). *Let f the function regular in the disk $\mathcal{U}_R = \{z \in \mathbb{C} : |z| < R\}$ with $|f(z)| < M$, M fixed. If $f(z)$ has in $z = 0$ one zero with multiply $\geq m$, then*

$$|f(z)| \leq \frac{M}{R^m}|z|^m,\ z \in \mathcal{U}_R \tag{1.6}$$

the equality (in the inequality (1.6) for $z \neq 0$) can hold only if

$$f(z) = e^{i\theta}\frac{M}{R^m}z^m,$$

where θ is constant.

2. Main results

Theorem 2.1.*Let α be a complex number, $a = Re\ \frac{1}{\alpha} > 0$ and $f \in \mathcal{A}$, $f(z) = z + a_2z^2 + \ldots$.*

If

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{(2a+1)^{\frac{2a+1}{2a}}}{2}|\alpha| \tag{2.1}$$

for all $z \in \mathcal{U}$, then for any complex number β, Re $\beta \geq$ Re $\frac{1}{\alpha}$, the function

$$T_{\alpha,\beta}(z) = \left[\beta \int_0^z u^{\beta-1}\left(\frac{f(u)}{u}\right)^{\frac{1}{\alpha}} du\right]^{\frac{1}{\beta}} \tag{2.2}$$

is in the class $\mathcal{S}$.

Proof. Let us consider the function

$$g(z) = \int_0^z \left(\frac{f(u)}{u}\right)^{\frac{1}{\alpha}} du \tag{2.3}$$

The function f is regular in $\mathcal{U}$. From (2.3) we have

$$g'(z) = \left(\frac{f(z)}{z}\right)^{\frac{1}{\alpha}},$$

$$g''(z) = \frac{1}{\alpha}\left(\frac{f(z)}{z}\right)^{\frac{1}{\alpha}-1}\frac{zf'(z)-f(z)}{z^2}$$

We define the function $h(z) = \frac{zg''(z)}{g'(z)}$, $z \in \mathcal{U}$ and we obtain

$$h(z) = \frac{zg''(z)}{g'(z)} = \frac{1}{\alpha}\left(\frac{zf'(z)}{f(z)} - 1\right), \quad z \in \mathcal{U} \tag{2.4}$$

The function h satisfies the condition $h(0) = 0$. From (2.1) and (2.4) we have

$$|h(z)| \leq \frac{(2a+1)^{\frac{2a+1}{2a}}}{2} \tag{2.5}$$

for all $z \in \mathcal{U}$. Applying Lemma 1.2 we get

$$|h(z)| \leq \frac{(2a+1)^{\frac{2a+1}{2a}}}{2}|z| \tag{2.6}$$

for all $z \in \mathcal{U}$.

From (2.4) and (2.6) we obtain

$$\frac{1-|z|^{2a}}{a}\left|\frac{zg''(z)}{g'(z)}\right| \leq \frac{(2a+1)^{\frac{2a+1}{2a}}}{2}\,\frac{1-|z|^{2a}}{a}|z| \tag{2.7}$$

for all $z \in \mathcal{U}$.

Because

$$\max_{|z|\leq 1}\left\{\frac{1-|z|^{2a}}{a}|z|\right\} = \frac{2}{(2a+1)^{\frac{2a+1}{2a}}}$$

from (2.7) we have

$$\frac{1-|z|^{2a}}{a}\left|\frac{zg''(z)}{g'(z)}\right| \leq 1 \tag{2.8}$$

for all $z \in \mathcal{U}$.

From (2.3) we have $g'(z) = \left(\frac{f(z)}{z}\right)^{\frac{1}{\alpha}}$, and by Lemma 1.1 we obtain that the integral operator $T_{\alpha,\beta}$ define by (2.2) is in the class $\mathcal{S}$.

Corollary. 2.2. *Let α be a complex number, $a = Re\ \frac{1}{\alpha} > 0$ and $f \in \mathcal{A}$,*

$$f(z) = z + a_2z^2 + a_3z^3 + \dots$$

If

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{(2a+1)^{\frac{2a+1}{2a}}}{2}|\alpha| \tag{2.9}$$

for all $z \in \mathcal{U}$, then the integral operator M_α given by (1.2) belongs to class $\mathcal{S}$.

Proof. For $\beta = \frac{1}{\alpha}$, from Theorem 2.1 we obtain Corollary 2.2.

Corollary. 2.3. *Let α be a complex number, $a = Re\ \frac{1}{\alpha} \in (0,1]$ and $f \in \mathcal{A}$, $f(z) = z + a_2z^2 + \dots$.*

If

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{(2a+1)^{\frac{2a+1}{2a}}}{2}|\alpha| \tag{2.10}$$

for all $z \in \mathcal{U}$, then the integral operator G_α, is in the class $\mathcal{S}$.

Proof. We take $\beta = 1$ in Theorem 2.1.

References

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