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SUBORDINATION RESULTS AND INTEGRAL MEANS FOR $K-$UNIFORLY STARLIKE FUNCTIONS

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Abstract. In this paper, we introduce a generalized class of $k-$uniformly starlike functions and obtain the subordination results and integral means inequalities . Some interesting consequences of our results are also pointed out.

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1. Introduction and Preliminaries

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic and univalent in the open disc $U = \{z : |z| < 1\}$. For functions $f \in A$ given by (1) and $g \in A$ given by $g(z) = z + \sum\limits_{n=2}^{\infty} b_n z^n$, we define the Hadamard product (or Convolution) of f and g by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in U. \tag{2}$$

For complex parameters $\alpha_1, \ldots, \alpha_l$ and $\beta_1, \ldots, \beta_m$ ($\beta_j \neq 0, -1, \ldots; j = 1, 2, \ldots, m$) the *generalized hypergeometric function* ${}_lF_m(z)$ is defined by

$$\begin{aligned} {}_lF_m(z) \equiv {}_lF_m(\alpha_1, \ldots \alpha_l; \beta_1, \ldots, \beta_m; z) \; &:= \; \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_l)_n}{(\beta_1)_n \ldots (\beta_m)_n} \frac{z^n}{n!} \\ (l \leq m+1;\; l, m \in N_0 \; &:= \; N \cup \{0\}; z \in U) \end{aligned} \tag{3}$$

where N denotes the set of all positive integers and $(\lambda)_n$ is the Pochhammer symbol defined by

$$(\lambda)_n = \begin{cases} 1, & n = 0 \\ \lambda(\lambda+1)(\lambda+2)\dots(\lambda+n-1), & n \in N. \end{cases} \tag{4}$$

The notation ${}_lF_m$ is quite useful for representing many well-known functions such as the exponential, the Binomial, the Bessel, the Laguerre polynomial and others; for example see [6] and [23].

Let $H(\alpha_1, \dots \alpha_l; \beta_1, \dots, \beta_m) : A \to A$ be a linear operator defined by

$$\begin{aligned} [(H(\alpha_1, \dots \alpha_l; \beta_1, \dots, \beta_m))(f)](z) &:= z \, {}_lF_m(\alpha_1, \alpha_2, \dots \alpha_l; \beta_1, \beta_2 \dots, \beta_m; z) * f(z) \\ &= z + \sum_{n=2}^{\infty} \Gamma_n \, a_n z^n \end{aligned} \tag{5}$$

where

$$\Gamma_n = \frac{(\alpha_1)_{n-1} \dots (\alpha_l)_{n-1}}{(n-1)!(\beta_1)_{n-1} \dots (\beta_m)_{n-1}} \ . \tag{6}$$

For notational simplicity, we can use a shorter notation $H^l_m[\alpha_1, \beta_1]$ for $H(\alpha_1, \dots \alpha_l; \beta_1, \dots, \beta_m)$ in the sequel.

The linear operator $H^l_m[\alpha_1, \beta_1]$ is called Dziok-Srivastava operator (see [8]), includes (as its special cases) various other linear operators introduced and studied by Bernardi [4], Carlson and Shaffer [7], Libera [15], Livingston [17], Owa [22], Ruscheweyh [27] and Srivastava-Owa [23].

For $0 \le \gamma < 1$ and $k \ge 0$, we let $\mathcal{H}^l_m(\gamma, k)$ be the subclass of A consisting of functions of the form (1) and satisfying the analytic criterion

$$\text{Re} \left\{ \frac{z(H^l_m[\alpha_1, \beta_1]f(z))'}{H^l_m[\alpha_1, \beta_1]f(z)} - \gamma \right\} > k \left| \frac{z(H^l_m[\alpha_1, \beta_1]f(z))'}{H^l_m[\alpha_1, \beta_1]f(z)} - 1 \right|, \quad z \in U, \tag{7}$$

where $H^l_m[\alpha_1, \beta_1]f(z)$ is given by (5). We further let $T\mathcal{H}^l_m(\gamma, k) = \mathcal{H}^l_m(\gamma, k) \cap T$, where

$$T \ := \ \left\{ f \in A : f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \quad z \in U \right\} \tag{8}$$

is a subclass of A introduced and studied by Silverman [29].

By suitably specializing the values of l, m, $\alpha_1, \alpha_2, \dots, \alpha_l$, $\beta_1, \beta_2, \dots, \beta_m$, γ and k in the class $\mathcal{H}^l_m(\gamma, k)$, we obtain the various subclasses, we present some examples.

Example 1. If $l = 2$ and $m = 1$ with $\alpha_1 = 1$, $\alpha_2 = 1$, $\beta_1 = 1$ then $\mathcal{H}^2_1(\gamma, k) \equiv S(\gamma, k) := \left\{ f \in A : \text{Re} \left\{ \frac{zf'(z)}{f(z)} - \gamma \right\} > k \left| \frac{zf'(z)}{f(z)} - 1 \right|, \ z \in U \right\}$.

Further $TS(\gamma,k) = S(\gamma,k) \cap T$, where T is given by (8). The class $TS(\gamma,k) \equiv UST(\gamma,k)$. A function in $UST(\gamma,k)$ is called $k-$uniformly starlike of order γ, $0 \leq \gamma < 1$ was introduced in [5]. Note that the classes $UST(\gamma,0)$ and $UST(0,0)$ were first introduced in [29]. We also observe that $UST(\gamma,0) \equiv T^*(\gamma)$ is well-known subclass of starlike functions of order γ.

Example 2. If $l = 2$ and $m = 1$ with $\alpha_1 = \delta + 1\,(\delta > -1)$, $\alpha_2 = 1$, $\beta_1 = 1$, then $\mathcal{H}_1^2(\gamma,k) \equiv R_\delta(\gamma,k) := \left\{ f \in A : \mathrm{Re}\ \left\{\frac{z(D^\delta f(z))'}{D^\delta f(z)} - \gamma\right\} > k\left|\frac{z(D^\delta f(z))'}{D^\delta f(z)} - 1\right|,\ \ z \in U\right\}$, where D^δ is called Ruscheweyh derivative of order δ $(\delta > -1)$ defined by

$$D^\delta f(z) := \frac{z}{(1-z)^{\delta+1}} * f(z) \equiv H_1^2(\delta+1, 1; 1) f(z).$$

Also $TR_\delta(\gamma,k) = R_\delta(\gamma,k) \cap T$, where T is given by (8).

The class $TR_\delta(\gamma,0)$ was studied in [26,28]. Earlier, this class was introduced and studied by Ahuja in [1,2].

Example 3. If $l = 2$ and $m = 1$ with $\alpha_1 = c + 1(c > -1)$, $\alpha_2 = 1$, $\beta_1 = c + 2$, then

$$\mathcal{H}_1^2(\gamma,k) \equiv B_c(\gamma,k) = \left\{ f \in A : \mathrm{Re}\ \left(\frac{z(J_c f(z))'}{J_c f(z)} - \gamma\right) > k\left|\frac{z(J_c f(z))'}{J_c f(z)} - 1\right|,\ \ z \in U\right\},$$

where J_c is a Bernardi operator [4] defined by

$$J_c f(z) := \frac{c+1}{z^c}\int_0^z t^{c-1} f(t)\,dt \equiv H_1^2(c+1, 1; c+2) f(z).$$

Note that the operator J_1 was studied earlier by Libera [15] and Livingston [17]. Further, $TB_c(\gamma,k) = B_c(\gamma,k) \cap T$, where T is given by (8).

Example 4. If $l = 2$ and $m = 1$ with $\alpha_1 = a\,(a > 0)$, $\alpha_2 = 1$, $\beta_1 = c\,(c > 0)$, then

$$\mathcal{H}_1^2(\gamma,k) \equiv L_c^a(\gamma,k) = \left\{ f \in A : \mathrm{Re}\left(\frac{z(L(a,c)f(z))'}{L(a,c)f(z)} - \gamma\right) > k\left|\frac{z(L(a,c)f(z))'}{L(a,c)f(z)} - 1\right|\right\},$$

where $z \in U$ and $L(a,c)$ is a well-known Carlson-Shaffer linear operator [7] defined by

$$L(a,c)f(z) := \left(\sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} z^{k+1}\right) * f(z) \equiv H_1^2(a, 1; c) f(z).$$

The class $L_c^a(\gamma,k)$ was introduced in [19] and also $TL_c^a(\gamma,k) = L_c^a(\gamma,k) \cap T$, where T is given by (8)was introduced and studied in [20, 21].

Remark 1.1. Observe that, specializing the parameters l, m, $\alpha_1, \alpha_2, \ldots, \alpha_l$, and $\beta_1, \beta_2, \ldots, \beta_m$, γ, k in the class $\mathcal{H}_m^l(\gamma, k)$, we obtain various classes introduced and studied by Goodman [10,11], Kanas et.al., [12, 13. 14], Ma and Minda [18], Rønning [24, 25] and others.

Now we state the results due to Aouf and Murugusundaramoorthy [3].

Theorem 1.1. *A function $f(z)$ of the form (1) is in $\mathcal{H}_m^l(\gamma, k)$ if*

$$\sum_{n=2}^{\infty}[n(1+k)-(\gamma+k)]\Gamma_n \ |a_n| \leq 1-\gamma, \tag{9}$$

where $0 \leq \lambda < 1$, $0 \leq \gamma < 1$, $k \geq 0$, Γ_n is given by (6) and suppose that the parameters $\alpha_1, \ldots, \alpha_l$ and $\beta_1, \ldots, \beta_m$ are positive real numbers.

Theorem 1.2. *Let $0 \leq \gamma < 1$, $k \geq 0$ and suppose that the parameters $\alpha_1, \ldots, \alpha_l$ and $\beta_1, \ldots, \beta_m$ are positive real numbers. Then a function f of the form (8) to be in the class $T\mathcal{H}_m^l(\gamma, k)$ if and only if*

$$\sum_{n=2}^{\infty}[n(1+k)-(\gamma+k)]\Gamma_n \ |a_n| \leq 1-\gamma, \tag{10}$$

where Γ_n is given by (6).

Corollary 1.1. *If $f \in T\mathcal{H}_m^l(\gamma, k)$, then*

$$|a_n| \leq \frac{1-\gamma}{[n(1+k)-(\gamma+k)]\Gamma_n}, \ \ , \ 0 \leq \gamma < 1, k \geq 0, \tag{11}$$

where Γ_n is given by (6) and suppose the parameters $\alpha_1, \ldots, \alpha_l$ and $\beta_1, \ldots, \beta_m$ are positive real numbers .

Equality holds for the function $f(z) = z - \frac{1-\gamma}{[n(1+k)-(\gamma+k)]\Gamma_n} z^n$.

Theorem 1.3.(Extreme Points) *Let*

$$f_1(z) \ = \ z \quad \text{and} \quad f_n(z) = z - \frac{1-\gamma}{[n(1+k)-(\gamma+k)]\Gamma_n} z^n, \ \ n \geq 2,$$

for $0 \leq \gamma < 1$, $0 \leq \lambda < 1$, $k \geq 0$, suppose that the parameters $\alpha_1, \ldots, \alpha_l$ and $\beta_1, \ldots, \beta_m$ are positive real numbers and Γ_n is given by (6). Then $f(z)$ is in the class $T\mathcal{H}_m^l(\gamma, k)$ if and only if it can be expressed in the form $f(z) = \sum\limits_{n=1}^{\infty} \mu_n f_n(z)$, where $\mu_n \geq 0$ and $\sum\limits_{n=1}^{\infty} \mu_n = 1$.

Let $\mathcal{H}_m^{*l}(\gamma, k)$ denote the subclass of functions f in A whose Taylor-Maclaurin coefficients a_n satisfy the condition (9). We note that $\mathcal{H}_m^{*l}(\gamma, k) \subseteq \mathcal{H}_m^{l}(\gamma, k)$.

To prove our results we need the following definitions and lemmas.

Definition 1.1. For analytic functions g and h with $g(0) = h(0)$, g is said to be subordinate to h, denoted by $g \prec h$, if there exists an analytic function w such that $w(0) = 0$, $|w(z)| < 1$ and $g(z) = h(w(z))$, for all $z \in U$.

Definition 1.2. A sequence $\{b_n\}_{n=1}^{\infty}$ of complex numbers is said to be a subordinating sequence if, whenever $f(z) = \sum\limits_{n=1}^{\infty} a_n z^n$, $a_1 = 1$ is regular, univalent and convex in U, we have

$$\sum_{n=1}^{\infty} b_n a_n z^n \prec f(z), \quad z \in U. \tag{12}$$

In 1961, Wilf [34] proved the following subordinating factor sequence.

Lemma 1.1. *The sequence $\{b_n\}_{n=1}^{\infty}$ is a subordinating factor sequence if and only if*

$$Re \left\{1 + 2\sum_{n=1}^{\infty} b_n z^n\right\} > 0, \quad z \in U. \tag{13}$$

Motivated by above results, in this paper, we obtain the subordination results and integral means inequalities for the generalized class $k-$ uniformly starlike functions. Some interesting consequences of our results are also pointed out.

2.Subordination Results

Theorem 2.1. *Let $f \in \mathcal{TH}_m^l(\gamma, k)$ and $g(z)$ be any function in the usual class of convex functions C, then*

$$\frac{(2 + k - \gamma))\Gamma_2}{2[1 - \gamma + (2 + k - \gamma)\Gamma_2]}(f * g)(z) \prec g(z) \tag{14}$$

where $0 \leq \gamma < 1$; $k \geq 0$ with

$$\Gamma_2 = \frac{\alpha_1 \ldots \alpha_l}{\beta_1 \ldots \beta_m} \tag{15}$$

and

$$Re \ \{f(z)\} > -\frac{[1 - \gamma + (2 + k - \gamma)\Gamma_2]}{(2 + k - \gamma)\Gamma_2}, \quad z \in U. \tag{16}$$

The constant factor $\frac{(2+k-\gamma)\Gamma_2}{2[1-\gamma+(2+k-\gamma)\Gamma_2]}$ *in (14) cannot be replaced by a larger number.*

Proof. Let $f \in \mathcal{TH}_m^l(\gamma, k)$ and suppose that $g(z) = z + \sum\limits_{n=2}^{\infty} c_n z^n \in C$. Then

$$\frac{(2+k-\gamma)\Gamma_2}{2[1-\gamma+(2+k-\gamma)\Gamma_2]}(f * g)(z)$$
$$= \frac{(2+k-\gamma)\Gamma_2}{2[1-\gamma+(2+k-\gamma)\Gamma_2]}\left(z + \sum_{n=2}^{\infty} c_n a_n z^n\right). \tag{17}$$

Thus, by Definition 1.2, the subordination result holds true if

$$\left\{\frac{(2+k-\gamma)\Gamma_2}{2[1-\gamma+(2+k-\gamma)\Gamma_2]}\right\}_{n=1}^{\infty}$$

is a subordinating factor sequence, with $a_1 = 1$. In view of Lemma 1.1, this is equivalent to the following inequality

$$\mathrm{Re}\left\{1 + \sum_{n=1}^{\infty} \frac{(2+k-\gamma)\Gamma_2}{[1-\gamma+(2+k-\gamma)\Gamma_2]} a_n z^n\right\} > 0, \quad z \in U. \tag{18}$$

Since $\frac{(n(1+k)-(\gamma+k))\Gamma_n}{(1-\gamma)} \geq \frac{(2+k-\gamma)\Gamma_2}{(1-\gamma)} > 0$, for $n \geq 2$ we have

$$\mathrm{Re}\left\{1 + \frac{(2+k-\gamma)\Gamma_2}{[1-\gamma+(2+k-\gamma)\Gamma_2]}\sum_{n=1}^{\infty} a_n z^n\right\}$$
$$= \mathrm{Re}\left\{1 + \frac{(2+k-\gamma)\Gamma_2}{[1-\gamma+(2+k-\gamma)\Gamma_2]} z + \frac{\sum\limits_{n=2}^{\infty}(2+k-\gamma)\Gamma_2 a_n z^n}{[1-\gamma+(2+k-\gamma)\Gamma_2]}\right\}$$
$$\geq 1 - \frac{(2+k-\gamma)\Gamma_2}{[1-\gamma+(2+k-\gamma)\Gamma_2]} r$$
$$- \frac{1}{[1-\gamma+(2+k-\gamma)\Gamma_2]}\sum_{n=2}^{\infty} |[n(1+k)-(\gamma+k)(1+n\lambda-\lambda)]\Gamma_n a_n| r^n$$
$$\geq 1 - \frac{(2+k-\gamma)\Gamma_2}{[1-\gamma+(2+k-\gamma)\Gamma_2]} r - \frac{1-\gamma}{[1-\gamma+(2+k-\gamma)\Gamma_2]} r$$
$$> 0, \quad |z| = r < 1,$$

where we have also made use of the assertion (9) of Theorem 1.1. This evidently proves the inequality (18) and hence the subordination result (14) asserted by Theorem 2.1. The inequality (16) follows from (14) by taking

$$g(z) = \frac{z}{1-z} = z + \sum_{n=2}^{\infty} z^n \in C.$$

Next we consider the function

$$F(z) := z - \frac{1-\gamma}{(2+k-\gamma)\Gamma_2} z^2$$

where $0 \leq \gamma < 1,\ k \geq 0$, and Γ_2 is given by (15). Clearly $F \in \mathcal{TH}_m^l(\gamma, k)$. For this function ,(14)becomes

$$\frac{(2+k-\gamma)\Gamma_2}{2[1-\gamma+(2+k-\gamma)\Gamma_2]} F(z) \prec \frac{z}{1-z}.$$

It is easily verified that

$$\min\left\{\text{Re}\ \left(\frac{(2+k-\gamma)\Gamma_2}{2[1-\gamma+(2+k-\gamma)\Gamma_2]} F(z)\right)\right\} = -\frac{1}{2}, \quad z \in U.$$

This shows that the constant $\frac{(2+k-\gamma)\Gamma_2}{2[1-\gamma+(2+k-\gamma)\Gamma_2]}$ cannot be replaced by any larger one.

By taking different choices of l, m, $\alpha_1, \alpha_2, \ldots, \alpha_l$, $\beta_1, \beta_2, \ldots, \beta_m$, γ and k in the above theorem and in view of Examples 1 to 4 in Section 1, we state the following corollaries for the subclasses defined in those examples.

Corollary 2.1. *If $f \in S^*(\gamma, k)$, then*

$$\frac{2+k-\gamma}{2[3+k-\gamma]}(f * g)(z) \prec g(z), \tag{19}$$

where $0 \leq \gamma < 1,\ ,\ k \geq 0,\ g \in C$ and

$$Re\{f(z)\} > -\frac{3+k-2\gamma}{2+k-\gamma}, \ \ z \in U.$$

The constant factor

$$\frac{2+k-\gamma}{2[3+k-2\gamma]}$$

in (19) cannot be replaced by a larger one.

Remark 2.1. Corollary 2.1, yields the result obtained by Singh [32] when $\gamma = k = 0$.

Remark 2.2. Corollary 2.1, yields the results obtained by Frasin [9] for the special values of γ and k.

Let $R^*_\delta(\gamma, k)$ denote the subclass of functions f in A we note that $R^*_\delta(\gamma, k) \subseteq R_\delta(\gamma, k)$.

Corollary 2.2. *If $f \in R^*_\delta(\gamma, k)$, then*

$$\frac{(\delta+1)(2+k-\gamma)}{2[(1-\gamma)+(\delta+1)(2+k-\gamma)]}(f*g)(z) \prec g(z), \tag{20}$$

where $0 \leq \gamma < 1,\ ,\ k \geq 0,\ \delta > -1,\ g \in C$ and

$$Re\{f(z)\} > -\frac{[(1-\gamma)+(\delta+1)(2+k-\gamma)]}{(\delta+1)(2+k-\gamma)},\ \ z \in U.$$

The constant factor

$$\frac{(\delta+1)[(2+k-\gamma)]}{2[(1-\gamma)+(\delta+1)(2+k-\gamma)]}$$

in (20) cannot be replaced by a larger one.

Let $B^*_c(\gamma, k)$ denote the subclass of functions f in A we note that $B^*_c(\gamma, k) \subseteq B_c(\gamma, k)$.

Corollary 2.3. *If $f \in B^*_c(\gamma, k)$, then*

$$\frac{(c+1)(2+k-\gamma)}{2[(c+2)(1-\gamma)+(c+1)(2+k-\gamma)]}(f*g)(z) \prec g(z), \tag{21}$$

where $0 \leq \gamma < 1,\ ,\ k \geq 0,\ c > -1,\ g \in C$ and

$$Re\{f(z)\} > -\frac{[(c+2)(1-\gamma)+(c+1)(2+k-\gamma)]}{(c+1)(2+k-\gamma)},\ \ z \in U.$$

The constant factor

$$\frac{(c+1)(2+k-\gamma)}{2[(c+2)(1-\gamma)+(c+1)(2+k-\gamma)]}$$

in (21) cannot be replaced by a larger one.

Let $L^{*a}_c(\gamma, k)$ denote the subclass of functions f in A we note that $L^{*a}_c(\gamma, k) \subseteq L^a_c(\gamma, k)$.

Corollary 2.4. *If $f \in L^{*a}_c(\gamma, k)$, then*

$$\frac{a(2+k-\gamma)}{2[c(1-\gamma)+a(2+k-\gamma)]}(f*g)(z) \prec g(z), \tag{22}$$

where $0 \leq \gamma < 1,\ ,\ k \geq 0,\ a > 0, c > 0,\ g \in C$ *and*

$$Re\{f(z)\} > -\frac{[c(1-\gamma)+a(2+k-\gamma)]}{a(2+k-\gamma)},\ \ z \in U.$$

The constant factor

$$\frac{a(2+k-\gamma)}{2[c(1-\gamma)+a(2+k-\gamma)]}$$

in (22) cannot be replaced by a larger one.

3.Integral Means Inequalities

In 1925, Littlewood [16] proved the following subordination theorem.

Lemma 3.1.*If the functions* f *and* g *are analytic in* U *with* $g \prec f$, *then for* $\eta > 0$, *and* $0 < r < 1$,

$$\int_0^{2\pi} \left|g(re^{i\theta})\right|^{\eta} d\theta \leq \int_0^{2\pi} \left|f(re^{i\theta})\right|^{\eta} d\theta. \tag{23}$$

In [29], Silverman found that the function $f_2(z) = z - \frac{z^2}{2}$ is often extremal over the family T. He applied this function to resolve his integral means inequality, conjectured in [30] and settled in [31], that

$$\int_0^{2\pi} \left|f(re^{i\theta})\right|^{\eta} d\theta \leq \int_0^{2\pi} \left|f_2(re^{i\theta})\right|^{\eta} d\theta,$$

for all $f \in T$, $\eta > 0$ and $0 < r < 1$. In [31], he also proved his conjecture for the subclasses $T^*(\gamma)$ and $C(\gamma)$ of T.

Applying Lemma 3.1, Theorem 1.2 and Theorem 1.3, we obtain integral means inequalities for the functions in the family $T\mathcal{H}^l_m(\gamma, k)$. By taking appropriate choices of the parameters $l, m, \alpha_1, \ldots, \alpha_l, \beta_1, \ldots, \beta_m,\ \gamma,\ k$, we obtain the integral means inequalities for several known as well as new subclasses.

Theorem 3.1.*Suppose* $f \in T\mathcal{H}^l_m(\gamma, k)$, $\eta > 0$, $0 \leq \gamma < 1$, $k \geq 0$ *and* $f_2(z)$ *is defined by*

$$f_2(z) = z - \frac{1-\gamma}{(2+k-\gamma)\Gamma_2} z^2,$$

where Γ_2 *is given by (15). Then for* $z = re^{i\theta}$, $0 < r < 1$, *we have*

$$\int_0^{2\pi} |f(z)|^\eta \, d\theta \le \int_0^{2\pi} |f_2(z)|^\eta \, d\theta. \tag{24}$$

Proof. For $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$, (24) is equivalent to proving that

$$\int_0^{2\pi} \left| 1 - \sum_{n=2}^{\infty} |a_n| z^{n-1} \right|^\eta d\theta \le \int_0^{2\pi} \left| 1 - \frac{(1-\gamma)}{(2+k-\gamma)\Gamma_2} z \right|^\eta d\theta.$$

By Lemma 3.1, it suffices to show that

$$1 - \sum_{n=2}^{\infty} |a_n| z^{n-1} \prec 1 - \frac{1-\gamma}{(2+k-\gamma)\Gamma_2} z.$$

Setting

$$1 - \sum_{n=2}^{\infty} |a_n| z^{n-1} = 1 - \frac{1-\gamma}{(2+k-\gamma)\Gamma_2} w(z), \tag{25}$$

and using (10), we obtain

$$\begin{aligned} |w(z)| &= \left| \sum_{n=2}^{\infty} \frac{[n(1+k) - (\gamma+k)]\Gamma_n}{1-\gamma} |a_n| z^{n-1} \right| \\ &\le |z| \sum_{n=2}^{\infty} \frac{[n(1+k) - (\gamma+k)]\Gamma_n}{1-\gamma} |a_n| \\ &\le |z|, \end{aligned}$$

where Γ_n is given by (6). This completes the proof by Theorem 1.2.

In view of the Examples 1 to 4 in Section 1 and Theorem 3.1, we can state the following corollaries without proof for the classes defined in those examples.

Corollary 3.1.*If* $f \in TS(\gamma, k)$, $0 \le \gamma < 1$, $k \ge 0$ *and* $\eta > 0$, *then the assertion (24) holds true where*

$$f_2(z) = z - \frac{1-\gamma}{[2+k-\gamma)]} z^2.$$

Remark 3.1. Fixing $k = 0$, Corollary 3.1, leads the integral means inequality for the class $T^*(\gamma)$ obtained in [31].

Corollary 3.2. *If $f \in TR_\delta(\gamma, k)$, $\delta > -1$, $0 \le \gamma < 1$, $k \ge 0$ and $\eta > 0$, then the assertion (24) holds true where*

$$f_2(z) = z - \frac{(1-\gamma)}{(\delta+1)[2+k-\gamma]} z^2 \ .$$

Corollary 3.3. *If $f \in TB_c(\gamma, k)$, $c > -1$, $0 \le \gamma < 1$, $k \ge 0$ and $\eta > 0$, then the assertion (24) holds true where*

$$f_2(z) = z - \frac{(1-\gamma)(c+2)}{(c+1)[2+k-\gamma]} z^2 \ .$$

Corollary 3.4. *If $f \in TL_c^a(\gamma, k)$, $a > 0$, $c > 0$, $0 \le \gamma < 1$, $k \ge 0$ and $\eta > 0$, then the assertion (24) holds true where*

$$f_2(z) = z - \frac{c(1-\gamma)}{a[2+k-\gamma]} z^2.$$

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