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AN INTEGRAL FORMULA FOR WILLMORE SURFACES IN AN N-DIMENSIONAL SPHERE

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Abstract. A surface $x : M \to S^n$ is called a Willmore surface if it is a critical surface of the Willmore functional. In this paper, we obtain an integral formula using $\square$ self-adjoint operator for compact Willmore surfaces in S^n.

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1. Introduction

We use the same notations and terminologies as in [2], [5], [6]. Let $x : M \to S^n$ be a surface in an n-dimensional unit sphere space S^n. If h_{ij}^{α} denotes the second fundamental form of M, S denotes the square of the length of the second fundamental form, $\mathbf{H}$ denotes the mean curvature vector, and H denotes the mean curvature of M, then we have

$$S = \sum_{\alpha} \sum (h_{ij}^{\alpha})^2, \quad \mathbf{H} = \sum_{\alpha} H^{\alpha} e_{\alpha}, \quad H^{\alpha} = \frac{1}{2} \sum_{k} h_{kk}^{\alpha}, \quad H = |\mathbf{H}|,$$

where $e_{\alpha} (3 \leq \alpha \leq n)$ are orthonormal vector fields of M in S^n.

We define the following nonnegative function on M:

$$\rho^2 = S - 2H^2, \tag{1.1}$$

which vanishes exactly at the umbilic points of M.

The Willmore functional is the following non-negative functional (see[1])

$$w(x) = \int_M \rho^2 dv = \int_M (S - 2H^2) dv, \tag{1.2}$$

that this functional is an invariant under conformal transformations of S^n.

Ximin [8] studied compact space-like submanifolds in a de Sitter space $M_p^{n+p}(c)$. Furthermore, in [9], the authors studied Willmore submanifolds in a sphere.

In this paper we studied Willmore surfaces in S^n and using the method of proof which is given in [4], [8], [9], we obtained an integral formula.

1. Local Formulas

Let $x : M \to S^n$ be a surface in an n-dimensional unit sphere. We choose an orthonormal basis $e_1, ..., e_n$ of S^n such that $\{e_1, e_2\}$ are tangent to $x(M)$ and $\{e_3, ..., e_n\}$ is a local frame in the normal bundle. Let $\{w_1, w_2\}$ be the dual forms of $\{e_1, e_2\}$. We use the following convention on the ranges of indices:

$$1 \leq i, j, k, ... \leq 2 \; ; \; 3 \leq \alpha, \beta, \gamma, ... \leq n.$$

Then we have the structure equations

$$dx = \sum_i w_i e_i, \tag{2.1}$$

$$de_i = \sum_j w_{ij} e_j + \sum_{\alpha,j} h_{ij}^\alpha w_j e_\alpha - w_i x, \tag{2.2}$$

$$de_\alpha = -\sum_{i,j} h_{ij}^\alpha w_j e_i + \sum_\beta w_{\alpha\beta} e_\beta \ , \ h_{ij}^\alpha = h_{ji}^\alpha \tag{2.3}$$

The Gauss equations and Ricci equations are

$$R_{ijkl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \sum_\alpha (h_{ik}^\alpha h_{jl}^\alpha - h_{il}^\alpha h_{jk}^\alpha), \tag{2.4}$$

$$R_{ik} = \delta_{ik} + 2\sum_\alpha H^\alpha h_{ik}^\alpha - \sum_{\alpha,j} h_{ij}^\alpha \, h_{jk}^\alpha, \tag{2.5}$$

$$2K = 2 + 4H^2 - S, \tag{2.6}$$

$$R_{\beta\alpha 12} = \sum_i (h_{1i}^\beta h_{i2}^\alpha - h_{2i}^\beta h_{i1}^\alpha), \tag{2.7}$$

where K is the Gauss curvature of M and $S = \sum\limits_{\alpha,i,j} (h_{ij}^\alpha)^2$ is the norm of the square of the second fundamental form, $\mathbf{H} = \sum\limits_\alpha H^\alpha e_\alpha = (\frac{1}{2})\sum\limits_\alpha (\sum\limits_\alpha h_{kk}^\alpha) e_\alpha$

$= \frac{1}{2}\sum\limits_{\alpha} tr(h_\alpha)e_\alpha$ is the mean curvature vector and $H = |\mathbf{H}|$ is the mean curvature of M.

We have the following Codazzi equations and Ricci identities:

$$
\begin{aligned}
(2.8)\qquad h^\alpha_{ijk} - h^\alpha_{ikj} &= 0, \\
(2.9)\qquad h^\alpha_{ijkl} - h^\alpha_{ijlk} &= h^\alpha_{ijkl}R_{mikl} + \sum_m h^\alpha_{im}R_{mjkl} + \sum_\beta h^\beta_{ij}R_{\beta\alpha kl}
\end{aligned}
$$

where h^α_{ijk} and h^α_{ijkl} are defined by

$$
\begin{aligned}
(2.10)\qquad \sum_k h^\alpha_{ijk}w_k &= dh^\alpha_{ij} + \sum_k h^\alpha_{kj}w_{ki} + \sum_k h^\alpha_{ik}w_{kj} + \sum_\beta h^\beta_{ij}w_{\beta\alpha}, \\
(2.11)\qquad \sum_l h^\alpha_{ijkl}w_l &= dh^\alpha_{ijk} + \sum_l h^\alpha_{ljk}w_{li} + \sum_l h^\alpha_{ilk}w_{lj} + \sum_l h^\alpha_{ijl}w_{lk} \\
&\quad + \sum_\beta h^\beta_{ijk}w_{\beta\alpha}.
\end{aligned}
$$

As M is a two-dimensional surface, we have from (2.6) and (1.1)

$$
\begin{aligned}
(2.12)\qquad 2K &= 2 + 4H^2 - S = 2 + 2H^2 - \rho^2, \\
(2.13)\qquad R_{ijkl} &= K(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})\ ,\ R_{ik} = K\delta_{ik}.
\end{aligned}
$$

By a simple calculation, we have the following calculations [3]:

$$
(2.14)\qquad \frac{1}{2}\triangle S = \sum_{\alpha,i,j,k}(h^\alpha_{ijk})^2 + \sum_{\alpha,i,j} h^\alpha_{ij}tr(h_\alpha)_{ij} + 2K\rho^2 - \sum_{\alpha,\beta}(R_{\beta\alpha 12})^2.
$$

1. Proof Of The Theorem

Theorem. *Let M be a compact Willmore surface in an n-dimensional unit sphere S^n. Then, we have*

$$
\begin{aligned}
0 &= \int [|\nabla S| + \sum_{\alpha,i,j} h^\alpha_{ij}tr(h_\alpha)_{ij} + 2K\rho^2 - \sum_{\alpha,\beta}(R_{\beta\alpha 12})^2 \\
&\quad -4\,|\nabla H|^2 - \sum_i \lambda^\alpha_i(2H)_{ii}]dv.
\end{aligned}
$$

Proof.
We know from (2.6) that

$$4H^2 - S = 2K - 2. \tag{2.15}$$

Taking the covariant derivative of (2.15) and using the fact that $K = const.$, we obtain

$$4HH_k = \sum_{i,j,\alpha} h_{ij}^{\alpha} . h_{ijk}^{\alpha},$$

and hence, by Cauchy-Schwarz inequality, we have

$$\sum_k 16 \quad H^2 (H_k)^2 = \sum_k (\sum_{i,j,\alpha} h_{ij}^{\alpha} . h_{ijk}^{\alpha})^2 \leq \sum_{i,j,\alpha} (h_{ij}^{\alpha})^2 . \sum_{i,j,k,\alpha} (h_{ijk}^{\alpha})^2$$

that is

$$16H^2 \left\| \nabla H \right\|^2 \leq S. \left\| \nabla S \right\| . \tag{2.16}$$

On the other hand, the Laplacian $\triangle h_{ij}^{\alpha}$ of the fundamental form h_{ij}^{α} is defined to be $\sum\limits_k h_{ijkk}^{\alpha}$, and hence using (2.8), (2.9) and the assumption that M has flat normal bundle, we have

$$\triangle h_{ij}^{\alpha} = \sum_m h_{im}^{\alpha} R_{mjkk} + \sum_m h_{mk}^{\alpha} R_{mijk} + tr(h_{\alpha})_{ij}.$$

Since the normal bundle of M is flat, we choose $e_3, ...e_n$ such that

$$h_{ij}^{\alpha} = \lambda_i^{\alpha} \delta_{ij}.$$

We define an operator $\Box$ acting on f by [7]:

$$\Box f = \sum_{i,j} (2H^{\alpha} \delta_{ij} - h_{ij}^{\alpha}) f_{ij}. \tag{2.17}$$

Since $(2H^{\alpha}\delta_{ij} - h^{\alpha}_{ij})$ is trace-free it follows from [4] that the operator $\Box$ is self-adjoint to the L^2-inner product of M, i.e.,

$$\int_M f\Box g = \int_M g\Box f.$$

Thus we have the following computation by use of (2.17) and (2.14)

$$\begin{aligned} \Box 2H &= 2H\triangle(2H) - \sum_i \lambda^{\alpha}_i (2H)_{ii} \\ &= \frac{1}{2}\triangle(2H)^2 - \sum_i (2H)^2_i - \sum_i \lambda^{\alpha}_i (2H)_{ii} \end{aligned}$$

$$\Box 2H = \frac{1}{2}\triangle S + \triangle K - 4\left|\nabla H\right|^2 - \sum_i \lambda^{\alpha}_i (2H)_{ii} \tag{2.18}$$

Putting (2.14) in (2.18), we have

$$\begin{aligned} \Box 2H &= \sum_{\alpha,i,j,k} (h^{\alpha}_{ijk})^2 + \sum_{\alpha,i,j,k} h^{\alpha}_{ij} tr(h_{\alpha})_{ij} + 2K\rho^2 - \sum_{\alpha,\beta} (R_{\beta\alpha 12})^2 \\ &\quad + \triangle K - 4\left|\nabla H\right|^2 - \sum_i \lambda^{\alpha}_i (2H)_{ii}. \end{aligned} \tag{2.19}$$

Now we assume that M is compact and we obtain the following key formula by integrating (2.19) and noting $\int_M \triangle K dv = 0$ and $\int_M \Box(2H)dv = 0$,

$$\begin{aligned} 0 &= \int [\left|\nabla S\right| + \sum_{\alpha,i,j} h^{\alpha}_{ij} tr(h_{\alpha})_{ij} + 2K\rho^2 - \sum_{\alpha,\beta} (R_{\beta\alpha 12})^2 \\ &\quad - 4\left|\nabla H\right|^2 - \sum_i \lambda^{\alpha}_i (2H)_{ii}] dv. \end{aligned}$$

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