Acta Universitatis Apulensis No. 25/2011
ISSN: 1582-5329 pp. 113-117

A NOTE ON GENERALIZED INTEGRAL OPERATOR

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Abstract. In this paper, we define the subclass $\mathcal{S}_g(\alpha)$ of analytic functions by using Hadamrad product. Also we investigate certain properties of the generalized integral operator $I_g(f_1, \ldots, f_m)$ for the functions belonging to the class $\mathcal{S}_g(\alpha)$.

2000 *Mathematics Subject Classification*: 30C45.

1.Introduction

Let $\mathcal{A}$ be the class of analytic functions f of the form

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j, \tag{1}$$

defined in the unit disc $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$ and satisfying the normalization condition $f(0) = f'(0) - 1 = 0$.
A function $f \in \mathcal{A}$ is said to be starlike of order α if it satisfies the inequality

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \quad (z \in \mathcal{U})$$

for some $0 \leq \alpha < 1$ and it is denoted by $\mathcal{S}^*(\alpha)$ [6].
The class of convex functions of order α, denote by $\mathcal{K}(\alpha)$ consists of function $f \in \mathcal{A}$ such that

$$\Re\left\{\frac{zf''(z)}{f'(z)} + 1\right\} > \alpha, \quad (z \in \mathcal{U})$$

for some $0 \leq \alpha < 1$ and it is denoted by $\mathcal{K}(\alpha)$ [6]. Further, $f \in \mathcal{K}(\alpha)$ if and only if $zf' \in \mathcal{S}^*(\alpha)$.
For any two functions f and g such that

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j \quad \text{and} \quad g(z) = z + \sum_{j=2}^{\infty} b_j z^j, \quad (z \in \mathcal{U})$$

the Hadamard product or convolution of f and g denoted by $f*g$ is given by

$$(f*g)(z) = z + \sum_{j=2}^{\infty} a_j b_j z^j. \tag{2}$$

For different choice of $g(z)$, the Hadamrad product $f*g$ yields the following well known operators such as Ruscheweyh derivative operator [7], *Sălăgean* operator [8], Carlson-Shaffer operator [2], Dziok-Srivasava operator [5] and Al-Oboudi differential operator [1].

We designate $\mathcal{S}_g(\alpha)$, as the class of functions $f \in \mathcal{A}$ which satisfy the condition

$$\Re\left\{\frac{z((f*g)(z))'}{(f*g)(z)}\right\} > \alpha, \ (z \in \mathcal{U})$$

for some $0 \leq \alpha < 1$.
For $g(z) = z + \sum_{j=2}^{\infty}[1+(j-1)\delta]^n z^j$, we get the class $\mathcal{S}^n(\delta,\alpha)$ introduced by Serap Bulut [9].
For $n, m \in \mathbb{N}_0$, $k_i > 0$ and $1 \leq i \leq m$ the generalized integral operator $I_g(f_1,\ldots,f_m) : \mathcal{A}^n \longrightarrow \mathcal{A}$ is defined as

$$I_g(f_1,\ldots,f_m)(z) = \int_0^z \left(\frac{(f_1*g)(t)}{t}\right)^{k_1},\ldots,\left(\frac{(f_m*g)(t)}{t}\right)^{k_m} dt, \quad z \in \mathcal{U} \tag{3}$$

where $f_i \in \mathcal{A}$ and $*$ denotes the Hadamard product.

2.Prime Results

In this section, we determine certain properties for functions belonging to the class $\mathcal{S}_g(\alpha)$ using the generalized integral operator $I_g(f_1,\ldots,f_m)$.

Theorem 2.1 *Let* $f_i \in \mathcal{S}_g(\alpha_i)$, *for* $1 \leq i \leq m$, *with* $0 \leq \alpha_i < 1$. *Let* $k_i > 0$, *with* $1 \leq i \leq m$. *If* $\sum_{i=1}^{m} k_i(1-\alpha_i) \leq 1$, *then* $I_g(f_1,\ldots,f_m) \in \mathcal{K}(\lambda)$, *where* $\lambda = 1 + \sum_{i=1}^{m} k_i(\alpha_i - 1)$.

Proof. For $1 \leq i \leq m$, using (2) we can write

$$\frac{(f_i*g)(z)}{z} = 1 + \sum_{j=2}^{\infty} a_j b_j z^{j-1}$$

and

$$\frac{(f_i * g)(z)}{z} \neq 0, \quad \text{forall} \quad z \in \mathcal{U}.$$

On the other hand, we have

$$I_g(f_1, \ldots, f_m)'(z) = \left(\frac{(f_1 * g)(z)}{z}\right)^{k_1} \cdots \left(\frac{(f_m * g)(z)}{z}\right)^{k_m} dt, \quad z \in \mathcal{U}$$

which implies,

$$\ln I_g(f_1, \ldots, f_m)'(z) = k_1 \ln \frac{(f_1 * g)(z)}{z} + \cdots + k_m \ln \frac{(f_m * g)(z)}{z}$$

or equivalently

$$\ln I_g(f_1, \ldots, f_m)'(z) = k_1[\ln(f_1 * g)(z) - \ln z] + \cdots + k_m[\ln(f_m * g)(z) - \ln z].$$

By differentiating the above equality, we get

$$\frac{I_g(f_1, \ldots, f_m)''(z)}{I_g(f_1, \ldots, f_m)'(z)} = \sum_{i=1}^{m} k_i \left[\frac{((f_i * g)(z))'}{(f_i * g)(z)} - \frac{1}{z}\right].$$

Thus,

$$\frac{zI_g(f_1, \ldots, f_m)''(z)}{I_g(f_1, \ldots, f_m)'} + 1 = \sum_{i=1}^{m} k_i \left[\frac{z((f_i * g)(z))'}{(f_i * g)(z)}\right] - \sum_{i=1}^{m} k_i + 1$$

which is equivalent to

$$\Re\left\{\frac{zI_g(f_1, \ldots, f_m)''(z)}{I_g(f_1, \ldots, f_m)'} + 1\right\} = \sum_{i=1}^{m} k_i \Re\left\{\frac{z((f_i * g)(z))'}{(f_i * g)(z)}\right\} - \sum_{i=1}^{m} k_i + 1.$$

Since $f_i \in \mathcal{S}_g(\alpha_i)$, we get

$$\Re\left\{\frac{zI_g(f_1, \ldots, f_m)''(z)}{I_g(f_1, \ldots, f_m)'} + 1\right\} > \sum_{i=1}^{m} k_i \alpha_i - \sum_{i=1}^{m} k_i + 1 = 1 + \sum_{i=1}^{m} k_i(\alpha_i - 1).$$

Hence, the integral operator $I_g(f_1, \ldots, f_m)$ is convex of order λ, where $\lambda = 1 + \sum_{i=1}^{m} k_i(\alpha_i - 1).$

For $g(z) = z + \sum_{m=2}^{\infty} [1 + (m-1)\delta]^n z^m$, we get the following results proved by Serap Bulut [9] as Corollaries to the above Theorem.

Corollary 2.2 [9]:*Let* $f_i \in \mathcal{S}^n(\delta, \alpha)$, *for* $1 \leq i \leq m$ *with* $0 \leq \alpha < 1$, $\delta \geq 0$ *and* $n \in \mathbb{N}_0$. *Also let* $k_i > 0$, $1 \leq i \leq m$. *If* $\sum_{i=1}^{m} k_i \leq \frac{1}{1-\alpha}$, *then*

$I_n(f_1, \ldots, f_m) \in \mathcal{K}(\lambda)$, *where* $\lambda = 1 + (\alpha - 1) \sum_{i=1}^{m} k_i$.

Corollary 2.3 [9]:*Let* $f \in \mathcal{S}^n(\delta, \alpha)$ *with* $0 \leq \alpha < 1$, $\delta \geq 0$ *and* $n \in \mathbb{N}_0$. *Also let* $0 < k \leq \frac{1}{1-\alpha}$. *Then,* $I_n(f)(z) \in \mathcal{K}(1 + k(\alpha - 1))$, *where*

$$I_n(f)(z) = \int_0^z \left(\frac{D^n f(t)}{t} \right)^k dt.$$

Corollary 2.4 [9]:*Let* $f \in \mathcal{S}^n(\delta, \alpha)$. *Then, the integral operator* $I_n(f)(z) \in \mathcal{K}(\alpha)$,

where

$$I_n(f)(z) = \int_0^z \left(\frac{D^n f(t)}{t} \right)^k dt.$$

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