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PROPERTIES OF STRONGLY θ-β-$\mathcal{I}$-CONTINUOUS FUNCTIONS

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Abstract: In this paper, we investigate several properties of strongly θ-β-$\mathcal{I}$-continuous functions due to Yuksel et. al. [5].

Keywords: Ideal topological spaces, β-$\mathcal{I}$-open sets, strongly θ-β-$\mathcal{I}$-continuous functions.

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1. Introduction

The subject of ideals in topological spaces has been intruduced and studied by Kuratowski [3] and Vaidyanathasamy [4]. An ideal $\mathcal{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies (i) $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$ and (ii) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$. Given a topological space (X, τ) with an ideal $\mathcal{I}$ on X and if $\mathcal{P}(X)$ is the set of all subsets of X, a set operator $(.)^*$: $\mathcal{P}(X) \to \mathcal{P}(X)$, called the local function [4] of A with respect to τ and $\mathcal{I}$, is defined as follows: for $A \subset X$, $A^*(\tau, \mathcal{I}) = \{x \in X|\ U \cap A \notin \mathcal{I}$ for each neighbourhood U of $x\}$ where $\tau(x) = \{U \in \tau | x \in U\}$. A Kuratowski closure operator $\mathrm{Cl}^*(.)$ for a topology τ^* called the $*$-topology, finer than τ is defined by $\mathrm{Cl}^*(A) = A \cup A^*(\tau, \mathcal{I})$, When there is no chance of confusion, $A^*(\mathcal{I})$ is denoted by A^*. If $\mathcal{I}$ is an ideal on X, then $(X, \tau, \mathcal{I})$ is called an ideal space. By a space, we always mean a topological space (X, τ) with no separation properties are assumed. If $A \subset X$, $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$ will denote the closure and interior of A in (X, τ), repectively. In this paper we obtain several properties of strongly θ-β-$\mathcal{I}$-continuous functions due to Yuksel et. al.[5].

2. Preliminaries

A subset S of an ideal topological space $(X, \tau, \mathcal{I})$ is β-$\mathcal{I}$-open [2] (resp. α-$\mathcal{I}$-open [2]) if $S \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}^*(S)))$ (resp. $S \subset \mathrm{Int}(\mathrm{Cl}^*(\mathrm{Int}(S)))$). The complement of a β-$\mathcal{I}$-open set is called β-$\mathcal{I}$-closed [2]. The intersection of all β-$\mathcal{I}$-closed sets containing S is called the β-$\mathcal{I}$-closure of S and is denoted by ${}_{\beta\mathcal{I}}\mathrm{Cl}(S)$. The β-$\mathcal{I}$-Interior of S is defined by the union of all β-$\mathcal{I}$-open sets contained in S and is denoted by ${}_{\beta\mathcal{I}}\mathrm{Int}(S)$.

A subset S of an ideal space $(X, \tau, \mathcal{I})$ is said to be β-$\mathcal{I}$-regular [6] if it is both β-$\mathcal{I}$-open and β-$\mathcal{I}$-closed. The family of all β-$\mathcal{I}$-regular (resp. β-$\mathcal{I}$-open, β-$\mathcal{I}$-closed) sets of $(X, \tau, \mathcal{I})$ is denoted by $\beta\mathcal{I}R(X)$ (resp. $\beta\mathcal{I}O(X)$, $\beta\mathcal{I}C(X)$). The family of all β-$\mathcal{I}$-regular (resp. β-$\mathcal{I}$-open, β-$\mathcal{I}$-closed) sets of $(X, \tau, \mathcal{I})$ containing a point $x \in X$ is denoted by $\beta\mathcal{I}R(X, x)$ (resp. $\beta\mathcal{I}O(X, x)$, $\beta\mathcal{I}C(X, x)$). A point $x \in X$ is called the β-$\mathcal{I}$-θ-cluster point [6] of S if ${}_{\beta\mathcal{I}}\operatorname{Cl}(U) \cap S \neq \emptyset$ for every β-$\mathcal{I}$-open set U of $(X, \tau, \mathcal{I})$ containing x. The set of all β-$\mathcal{I}$-θ-cluster points of S is called the β-$\mathcal{I}$-closure [6] of S and is denoted by ${}_{\beta\mathcal{I}}\operatorname{Cl}_\theta(S)$. A subset A is said to be β-$\mathcal{I}$-θ-closed [6] if $A = {}_{\beta\mathcal{I}}\operatorname{Cl}_\theta(A)$. A point $x \in X$ is called the β-$\mathcal{I}$-θ-interior point of S if there exists a β-$\mathcal{I}$-regular set U of X containing x such that $x \in U \subset S$. The set of all β-$\mathcal{I}$-θ-interior points of S and is denoted by ${}_{\beta\mathcal{I}}\operatorname{Int}_\theta(\mathrm{S})$. A subset A of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be β-$\mathcal{I}$-θ-open if $A = {}_{\beta\mathcal{I}}\operatorname{Int}_\theta(A)$. Equivalently, the complement of β-$\mathcal{I}$-θ-closed set is β-$\mathcal{I}$-θ-open.

3. Strongly θ-β-$\mathcal{I}$-continuous functions

Definition 0.1. *A function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is said to be strongly θ-β-$\mathcal{I}$-continuous [5] (resp. β-$\mathcal{I}$-continuous [2]) at a point $x \in X$ if for each open set V of Y containing $f(x)$, there exists $U \in \beta\mathcal{I}O(X, x)$ such that $f({}_{\beta\mathcal{I}}\operatorname{Cl}(U)) \subset V$ (resp. $f(U) \subset V$). If f has this property at each point of X, then it is said to be strongly θ-β-$\mathcal{I}$-continuous (resp. β-$\mathcal{I}$-continuous [2]) function.*

Theorem 0.2. *If a function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is strongly θ-β-$\mathcal{I}$-continuous, then it is β-$\mathcal{I}$-continuous.*

Proof. Let f be a strongly θ-β-$\mathcal{I}$-continuous function on X. Then for each $x \in X$ and V be an open set of Y containing $f(x)$, there exists $U \in \beta\mathcal{I}O(X, x)$ such that $f({}_{\beta\mathcal{I}}\operatorname{Cl}(U)) \subset V$. Since $U \subset_{\beta\mathcal{I}} \operatorname{Cl}(U)$, we have $f(U) \subset_{\beta\mathcal{I}} \operatorname{Cl}(U)$. Hence $f(U) \subset V$. Thus, there exists a β-$\mathcal{I}$-open set U of X containing x and $f(U) \subset V$. Therefore, f is β-$\mathcal{I}$-continuous. □

Remark 0.3. *The function in Example 3.2 of [5] is β-$\mathcal{I}$-continuous but not strongly θ-β-$\mathcal{I}$-continuous.*

Theorem 0.4. *Let Y be a regular space. Then $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is strongly θ-β-$\mathcal{I}$-continuous if and only if f is β-$\mathcal{I}$-continuous.*

Proof. Let $x \in X$ and V be an open subset of Y containing $f(x)$. Since Y is regular, there exists an open set W such that $f(x) \in W \subset \operatorname{Cl}(W) \subset V$. If f is β-$\mathcal{I}$-continuous, there exists $U \in \beta\mathcal{I}O(X, x)$ such that $f(U) \subset W$. We shall show that $f({}_{\beta\mathcal{I}}\operatorname{Cl}(U)) \subset \operatorname{Cl}(W)$. Suppose that $y \notin \operatorname{Cl}(W)$. There exists an open set G containing y such that $G \cap W = \emptyset$. Since f is β-$\mathcal{I}$-continuous, $f^{-1}(G) \in \beta\mathcal{I}O(X)$ and $f^{-1}(G) \cap U =$

$\emptyset$ and hence $f^{-1}(G) \cap_{\beta\mathcal{I}} \mathrm{Cl}(U) = \emptyset$. Therefore, we obtain $G \cap f({}_{\beta\mathcal{I}}\mathrm{Cl}(U)) = \emptyset$ and $y \notin f({}_{\beta\mathcal{I}}\mathrm{Cl}(U))$. Consequently, we have $f({}_{\beta\mathcal{I}}\mathrm{Cl}(U)) \subset \mathrm{Cl}(W) \subset V$. The converse is obvious. □

Theorem 0.5. *For a function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$, the following properties are equivalent:*

(i) f is strongly θ-β-$\mathcal{I}$-continuous;

(ii) for each $x \in X$ and each open set V of Y containing $f(x)$, there exists $U \in \beta\mathcal{I}R(X, x)$ such that $f(U) \subset V$;

(iii) $f^{-1}(V)$ is β-$\mathcal{I}$-θ-open in X for each open set V of Y;

(iv) $f^{-1}(F)$ is β-$\mathcal{I}$-θ-closed in X for each closed set F of Y;

(v) $f({}_{\beta\mathcal{I}}\mathrm{Cl}_\theta(A)) \subset \mathrm{Cl}(f(A))$ for each subset A of X;

(vi) ${}_{\beta\mathcal{I}}\,\mathrm{Cl}_\theta(f^{-1}(A)) \subset f^{-1}(\mathrm{Cl}(B))$ for each subset B of Y.

Proof. (i)⇒(ii): It follows from Theorem 4.1 of [6]. (ii)⇒(iii): Let V be any open subset of Y and $x \in f^{-1}(V)$. There exists $U \in \beta\mathcal{I}R(X, x)$ such that $f(U) \subset V$. Therefore, we have $x \in U \subset f^{-1}(V)$. Therefore, $f^{-1}(V) = \bigcup_{x\in f^{-1}(V)} U_x$. Since any union of β-$\mathcal{I}$-open sets is β-$\mathcal{I}$-open ([5]), $f^{-1}(V)$ is β-$\mathcal{I}$-open in X. (iii)⇒(iv): This is obvious. (iv)⇒(v): Let A be any subset of X. Since $\mathrm{Cl}(f(A))$ is closed in Y, by (iv), $f^{-1}(\mathrm{Cl}(f(A)))$ is β-$\mathcal{I}$-θ-closed and we have ${}_{\beta\mathcal{I}}\,\mathrm{Cl}_\theta(A) \subset_{\beta\mathcal{I}} \mathrm{Cl}_\theta(f^{-1}(f(A))) \subset_{\beta\mathcal{I}} \mathrm{Cl}_\theta(f^{-1}(\mathrm{Cl}(f(A)))) = f^{-1}(\mathrm{Cl}(f(A)))$. Therefore, we obtain $f({}_{\beta\mathcal{I}}\mathrm{Cl}_\theta(A)) \subset \mathrm{Cl}(f(A))$. (v)⇒(vi): Let B be any subset of Y. By (v), we obtain $f({}_{\beta\mathcal{I}}\mathrm{Cl}_\theta(f^{-1}(B))) \subset \mathrm{Cl}(f(f^{-1}(B))) \subset \mathrm{Cl}(B)$ and hence ${}_{\beta\mathcal{I}}\,\mathrm{Cl}_\theta(f^{-1}(B)) \subset f^{-1}(\mathrm{Cl}(B))$. (vi)⇒(i): Let $x \in X$ and V be any open set of Y containing $f(x)$. Since $Y\backslash V$ is closed in Y, we have ${}_{\beta\mathcal{I}}\,\mathrm{Cl}_\theta(f^{-1}(Y\backslash V)) \subset f^{-1}(\mathrm{Cl}(Y\backslash V)) = f^{-1}(Y\backslash V)$. Therefore, $f^{-1}(Y\backslash V)$ is β-$\mathcal{I}$-θ-closed in X and $f^{-1}(V)$ is a β-$\mathcal{I}$-θ-open set of X containing x. There exists $U \in \beta\mathcal{I}O(X, x)$ such that ${}_{\beta\mathcal{I}}\,\mathrm{Cl}(U) \subset f^{-1}(V)$ and hence $f({}_{\beta\mathcal{I}}\mathrm{Cl}(U)) \subset V$. This shows that f is strongly θ-β-$\mathcal{I}$-continuous. □

Definition 0.6. *A sequence (x_n) is said to be β-$\mathcal{I}$-θ-convergent to a point x if for every β-$\mathcal{I}$-θ-open set V containing x, there exists an index x_0 such that for $n \geq n_0$, $x_n \in V$.*

Theorem 0.7. *For a function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$, the following properties are equivalent:*

(i) f is strongly θ-β-$\mathcal{I}$-continuous on X;

(ii) for each $x \in X$ and each sequence (x_n) in X. If (x_n) β-$\mathcal{I}$-θ-converges to x, then the sequence $(f(x_n))$ convergs to $f(x)$.

Proof. (i)⇒(ii): Let $x \in X$ and (x_n) be a sequence in X such that (x_n) β-$\mathcal{I}$-θ-converges to x. Let V be an open set containing $f(x)$. Since f is strongly θ-β-$\mathcal{I}$-continuous, there exists a β-$\mathcal{I}$-open set U of X containing x such that $f({}_{\beta\mathcal{I}}\mathrm{Cl}(U)) \subset V$. Since (x_n) β-$\mathcal{I}$-θ-converges to x, there exists n_0 such that $n_0 \in {}_{\beta\mathcal{I}}\mathrm{Cl}(U)$ for all $n \geq n_0$. Hence $f(x_n) \in f({}_{\beta\mathcal{I}}\mathrm{Cl}(U))$ for all $n \geq n_0$. Since $f({}_{\beta\mathcal{I}}\mathrm{Cl}(U)) \subset V$, hence $f(x_n) \subset V$ for all $n \geq n_0$. Thus, the sequence $(f(x_n))$ convergs to $f(x)$. (ii)⇒(i): Suppose that f is not strongly θ-β-$\mathcal{I}$-continuous on X. Then there exists $x \in X$ and an open set V containing $f(x)$ such that $f({}_{\beta\mathcal{I}}\mathrm{Cl}(U))$ is not a subset of V for all β-$\mathcal{I}$-open sets U containing x. Thus, there exists $x_U \in {}_{\beta\mathcal{I}}\mathrm{Cl}(U)$ such that $f(x_U) \notin V$. Consider, the sequence $\{x_U : U \in \beta\mathcal{I}O(X, x)\}$. Then, (x_U) β-$\mathcal{I}$-θ-converges to x but $(f(x_U))$ does not converges to $f(x_0)$, which is contradict to (ii), and hence f is strongly θ-β-$\mathcal{I}$-continuous on X. □

Definition 0.8. *By a strongly θ-β-$\mathcal{I}$-continuous retraction, we mean a strongly θ-β-$\mathcal{I}$-continuous function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$, where $Y \subset X$ and $f_{|_Y}$ is the identity function on Y.*

Theorem 0.9. *Let (X, τ) be a Hausdorff space and $\mathcal{I}$ is an ideal on X. If A is a strongly θ-β-$\mathcal{I}$-continuous retraction of X, then ${}_{\beta\mathcal{I}}\mathrm{Cl}_\theta(A) = A$.*

Proof. Suppose that ${}_{\beta\mathcal{I}}\mathrm{Cl}_\theta(A) \neq A$. Then there exists $x \in {}_{\beta\mathcal{I}}\mathrm{Cl}_\theta(A) - A$. Since A is a strongly θ-β-$\mathcal{I}$-continuous retract of X, we have $f(x) \neq x$ for some $x \in X$. Since X is a Hausdorff space, there exists disjoint open sets U and V such that $x \in U$ and $f(x) \in V$. Thus, $U \subset X - V$ and hence ${}_{\beta\mathcal{I}}\mathrm{Cl}(U) \cap V = \emptyset$. Since $U \subset_{\beta\mathcal{I}} \mathrm{Cl}(U)$, hence $x \in_{\beta\mathcal{I}} \mathrm{Cl}(U)$ for open set U containing x. Let W be an open set containing x. Since $W \subset_{\beta\mathcal{I}} \mathrm{Cl}(W)$, hence $x \in_{\beta\mathcal{I}} \mathrm{Cl}(W)$. Since $(U \cap W) \subset_{\beta\mathcal{I}} \mathrm{Cl}(U \cap W)$, hence $x \in_{\beta\mathcal{I}} \mathrm{Cl}(U \cap W)$ for open set $U \cap W$ containing x. Since $x \in_{\beta\mathcal{I}} \mathrm{Cl}_\theta(A)$ such that ${}_{\beta\mathcal{I}}\mathrm{Cl}_\theta(A)$ $= \{x \in X :_{\beta\mathcal{I}} \mathrm{Cl}(U) \cap A \neq \emptyset foreachU \in \beta\mathcal{I}O(X, x)\}$, hence ${}_{\beta\mathcal{I}}\mathrm{Cl}(U \cap W) \cap A \neq \emptyset$. Since ${}_{\beta\mathcal{I}}\mathrm{Cl}(U \cap W) \subset_{\beta\mathcal{I}} \mathrm{Cl}(U) \cap_{\beta\mathcal{I}} \mathrm{Cl}(W)$, hence $({}_{\beta\mathcal{I}}\mathrm{Cl}(U) \cap_{\beta\mathcal{I}} \mathrm{Cl}(W) \cap A = \emptyset$. Let $a \in ({}_{\beta\mathcal{I}}\mathrm{Cl}(U) \cap_{\beta\mathcal{I}} \mathrm{Cl}(W)) \cap A = \emptyset$. We have $a \in_{\beta\mathcal{I}} \mathrm{Cl}(U)$, $a \in_{\beta\mathcal{I}} \mathrm{Cl}(W)$ and $a \in A$. Since $a \in A$ hence $f(a) = a$, $a \in_{\beta\mathcal{I}} \mathrm{Cl}(W)$ hence $f(a) \in f({}_{\beta\mathcal{I}}\mathrm{Cl}(W))$ and $a \in_{\beta\mathcal{I}} \mathrm{Cl}(U)$ hence $a \notin V$. Thus, $f(a) \notin V$, we have $f({}_{\beta\mathcal{I}}\mathrm{Cl}(W))$ is not a subset of V for $W \in \beta\mathcal{I}O(X, x)$. Thus this contradicition f is strongly θ-β-$\mathcal{I}$-continuous on X. Therefore, ${}_{\beta\mathcal{I}}\mathrm{Cl}_\theta(A) = A$. □

Definition 0.10. *An ideal topological space $(X, \tau, \mathcal{I})$ is said to be $\mathcal{I}$-β-regular if for each closed set F and each point $x \in X \backslash F$, there exist disjoint β-$\mathcal{I}$-open sets U and V such that $x \in U$ and $F \subset V$.*

Lemma 0.11. *For an ideal topological space $(X, \tau, \mathcal{I})$, the following properties are equivalent:*

(i) X *is* $\mathcal{I}$-β-*regular;*

(ii) *for each point* $x \in X$ *and for each open set* U *of* X *containing* x*, there exists* $V \in \beta\mathcal{I}O(X)$ *such that* $x \in V \subset {}_{\beta\mathcal{I}}\,\mathrm{Cl}(V) \subset U$*;*

(iii) *for each subset* A *of* X *and each closed set* F *such that* $A \cap F = \emptyset$*, there exist disjoint* $U, V \in \beta\mathcal{I}O(X)$ *such that* $A \cap U \neq \emptyset$ *and* $F \subset V$*;*

(iv) *for each closed set* F *of* X*,* $F = \bigcap\{{}_{\beta\mathcal{I}}\mathrm{Cl}(V) : F \subset V$ *and* $V \in \beta\mathcal{I}O(X)$*.*

Proof. Clear. □

Theorem 0.12. *A continuous function* $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ *is strongly* θ-β-$\mathcal{I}$-*continuous if and only if* $(X, \tau, \mathcal{I})$ *is* $\mathcal{I}$-β-*regular.*

Proof. Necessity. Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be the identity function. Then f is continuous and strongly θ-β-$\mathcal{I}$-continuous by our hypothesis. For any open set U of X and any point $x \in U$, we have $f(x) = x \in U$ and there exists $G \in \beta\mathcal{I}O(X, x)$ such that $f({}_{\beta\mathcal{I}}\mathrm{Cl}(G)) \subset U$. Therefore, we have $x \in G \subset {}_{\beta\mathcal{I}}\,\mathrm{Cl}(G) \subset U$. It follows from Lemma 0.11 that $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-β-regular. Sufficiency. Suppose that $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is continuous and X is $\mathcal{I}$-β-regular. For any $x \in X$ and an open set V containing $f(x)$, $f^{-1}(V)$ is an open set containing x. Since X is $\mathcal{I}$-β-regular, there exists $U \in \beta\mathcal{I}O(X)$ such that $X \in U \subset {}_{\beta\mathcal{I}}\,\mathrm{Cl}(U) \subset f^{-1}(V)$. Therefore, we have $f({}_{\beta\mathcal{I}}\mathrm{Cl}(U)) \subset V$. This shows that f is strongly θ-β-$\mathcal{I}$-continuous. □

Definition 0.13 (1). *Let* A *and* X_0 *be subsets of an ideal topological space* $(X, \tau, \mathcal{I})$ *such that* $A \subset X_0 \subset X$*. Then* $(X_0, \tau_{|X_0}, \mathcal{I}_{|X_0})$ *is an ideal topological space with an ideal* $\mathcal{I}_{|X_0} = \{\mathcal{I} \in \mathcal{I} | \mathcal{I} \subset X_0\} = \{\mathcal{I} \bigcap X_0 | \mathcal{I} \in \mathcal{I}\}$*.*

Lemma 0.14. *[5] Let* A *and* X_0 *be subsets of an ideal topological space* $(X, \tau, \mathcal{I})$*. Then,*

(i) If $A \in \beta\mathcal{I}O(X)$ *and* X_0 *is* α-$\mathcal{I}$-*open in* $(X, \tau, \mathcal{I})$*, then* $A \cap X_0 \in \beta\mathcal{I}O(X_0)$*;*

(ii) If $A \in \beta\mathcal{I}O(X_0)$ *and* $X_0 \in \beta\mathcal{I}O(X)$*, then* $A \in \beta\mathcal{I}O(X)$*.*

Theorem 0.15. *If a function* $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ *is strongly* θ-β-$\mathcal{I}$-*continuous and* A *is an* α-$\mathcal{I}$-*open subset of* $(X, \tau, \mathcal{I})$*, then* $f_{|A} : (A, \tau_{|A}, \mathcal{I}_{|A}) \to (Y, \sigma)$ *is strongly* θ-$\beta - \mathcal{I}_{|A}$-*continuous.*

Proof. For any $x \in X_0$ and any open set V of Y containing $f(x)$, there exists $U \in \beta\mathcal{I}O(X,x)$ such that $f({}_{\beta\mathcal{I}}\mathrm{Cl}(U)) \subset V$ since f is strongly θ-β-$\mathcal{I}$-continuous. Put $U_0 = U \cap A$, then by Lemma 0.14, $U_0 \in \beta\mathcal{I}_{|A}O(A,x)$ and ${}_{\beta\mathcal{I}}\mathrm{Cl}_A(U_0) \subset_{\beta\mathcal{I}} \mathrm{Cl}(U_0)$. Therefore, we obtain $(f_{|A})({}_{\beta\mathcal{I}}\mathrm{Cl}_A(U_0)) = f({}_{\beta\mathcal{I}}\mathrm{Cl}_A(U_0)) \subset f({}_{\beta\mathcal{I}}\mathrm{Cl}(U_0)) \subset f({}_{\beta\mathcal{I}}\mathrm{Cl}(U)) \subset V$. This shows that $f_{|A}$ is strongly θ-β-$\mathcal{I}_{|A}$-continuous. □

Definition 0.16. *An ideal topological space $(X, \tau, \mathcal{I})$ is said to be β-$\mathcal{I}$-T_2 [5] if and only if for each pair of distinct points $x, y \in X$, there exist $U \in \beta\mathcal{I}O(X,x)$ and $V \in \beta\mathcal{I}O(X,y)$ such that $U \cap V = \emptyset$.*

Lemma 0.17. *An ideal topological space $(X, \tau, \mathcal{I})$ is said to be β-$\mathcal{I}$-T_2 [5] if and only if for each pair of distinct points $x, y \in X$, there exist $U \in \beta\mathcal{I}O(X,x)$ and $V \in \beta\mathcal{I}O(X,y)$ such that ${}_{\beta\mathcal{I}}\mathrm{Cl}(U) \cap {}_{\beta\mathcal{I}}\mathrm{Cl}(V) = \emptyset$.*

Theorem 0.18. *If $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is a strongly θ-β-$\mathcal{I}$-continuous injection and (Y, σ) is T_0, then X is β-$\mathcal{I}$-T_2.*

Proof. Let x and y be any distinct points of X. Since f is injective, $f(x) \neq f(y)$ and there exists an open set V containing $f(x)$ not containing $f(y)$ or an open set W containing $f(y)$ not containing $f(x)$. If the first case holds, then there exists $U \in \beta\mathcal{I}O(X,x)$ such that $f({}_{\beta\mathcal{I}}\mathrm{Cl}(U)) \subset V$. Therefore, we obtain $f(y) \notin f({}_{\beta\mathcal{I}}\mathrm{Cl}(U))$ and hence $X \backslash_{\beta\mathcal{I}} \mathrm{Cl}(U) \in \beta\mathcal{I}O(X,y)$. If the second case holds, then we obtain a similar result. Therefore, X is β-$\mathcal{I}$-T_2. □

Lemma 0.19. *The product of two β-$\mathcal{I}$-open sets is β-$\mathcal{I}$-open.*

Proof. Simillar to the proof of Lemma 3.4 of [7]. □

Theorem 0.20. *If a function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is strongly θ-β-$\mathcal{I}$-continuous and Y is Hausdorff, then the subset $\{(x_1, x_2) | f(x_1)=f(x_2)\}$ is $\beta\mathcal{I}\theta$-closed in the product space $X \times X$.*

Proof. Let $A = \{(x_1, x_2) | f(x_1)=f(x_2)\}$. If $(x_1, x_2) \notin A$, then we have $f(x_1) \neq f(x_2)$. Since Y is Hausdorff, there exist disjoint open sets V_1 and V_2 in Y such that $f(x_1) \in V_1$ and $f(x_2) \in V_2$. Since f is strongly θ-β-$\mathcal{I}$-continuous, there exist $U_1 \in \beta\mathcal{I}O(X,x_1)$ and $U_2 \in \beta\mathcal{I}O(X,x_2)$ such that $f({}_{\beta\mathcal{I}}\mathrm{Cl}(U_1)) \subset V_1$ and $f({}_{\beta\mathcal{I}}\mathrm{Cl}(U_2)) \subset V_2$. Put $U = {}_{\beta\mathcal{I}}\mathrm{Cl}(U_1) \times_{\beta\mathcal{I}} \mathrm{Cl}(U_2)$. Then by Lemma 0.19 U is β-$\mathcal{I}$-open in $X \times X$. Since every β-$\mathcal{I}$-open is β-$\mathcal{I}$-regular, U is β-$\mathcal{I}$-regular in $X \times X$ containing (x_1, x_2) and $A \cap U = \varnothing$. Therefore, we have $(x_1, x_2) \in {}_{\beta\mathcal{I}}\mathrm{Cl}(A)$. This shows that, A is β-$\mathcal{I}$-θ closed in $X \times X$. □

Recall that for a function $f : X \to Y$, the subset $\{(x, f(x)) : x \in X\}$ of $X \to Y$ is called the graph of f and is denoted by $G(f)$.

Definition 0.21. *The graph $G(f)$ of a function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is said to be β-$\mathcal{I}$-closed if for each $(x, y) \in (X \times Y)\backslash G(f)$, there exist $U \in \beta\mathcal{I}O(X, x)$ and an open set V in Y containing y such that $({}_{\beta\mathcal{I}}\mathrm{Cl}(U) \times V) \cap G(f) = \emptyset$.*

Lemma 0.22. *The graph $G(f)$ of a function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is said to be β-$\mathcal{I}$-closed if and only if for each $(x, y) \in (X \times Y)\backslash G(f)$, there exist $U \in \beta\mathcal{I}O(X, x)$ and an open set V in Y containing y such that $f({}_{\beta\mathcal{I}}\mathrm{Cl}(U)) \cap V = \emptyset$.*

Proof. It is an immediate consequnece of Definition 0.21. □

Theorem 0.23. *If $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is strongly θ-β-$\mathcal{I}$-continuous and Y is Hausdorff, then $G(f)$ is β-$\mathcal{I}$-closed in $X \times Y$.*

Proof. Let $(x, y) \in (X \times Y)\backslash G(f)$. Then $f(x) \neq y$. Since Y is Hausdorff, there exist open sets V and W in Y containing $f(x)$ and y, respectively, such that $V \cap W = \varnothing$. Since f is strongly θ-β-$\mathcal{I}$-continuous, there exists $U \in \beta\mathcal{I}O(X, x)$ such that $f({}_{\beta\mathcal{I}}\mathrm{Cl}(U)) \subset V$. Therefore, $f({}_{\beta\mathcal{I}}\mathrm{Cl}(U)) \cap W = \emptyset$ and then by Lemma 0.22, $G(f)$ is β-$\mathcal{I}$-closed in $X \times Y$. □

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