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# A NOTE ON SUBCLASS OF ANALYTIC FUNCTIONS DEFINED BY LINEAR OPERATOR

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Abstract. In this paper, we introduce a new class $M(g,n,\gamma,\lambda,\zeta)$ of analytic functions which defined by linear operator $D_{\lambda}^{n}(f*g)(z)$ and obtain its relations with some well-known subclasses of analytic univalent functions.

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## 1. Introduction

Let $A$ denote the class of all functions of the form:

$$f(z)=z+\sum_{k=2}^{\infty}a_kz^k, \tag{1.1}$$

which are analytic in the open disc $U=\{z:z\in\mathbb{C}\ and\ |z|<1\}$ and normalized by $f(0)=0=f'(0)-1$. Also let $S$ denote the subclass of all functions in $A$ which are univalent in $U$.

A function $f(z)\in S$ is said to be starlike of order $\zeta$ $(0\leq\zeta<1)$ if and only if

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>\zeta \quad (z\in U)\,. \tag{1.2}$$

We denote by $S^*(\zeta)$ the class of all starlike functions of order $\zeta$.

A function $f(z)\in S$ is said to be convex of order $\zeta$ $(0\leq\zeta<1)$ if and only if

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\zeta \quad (z\in U)\,. \tag{1.3}$$

We denote by $K(\zeta)$ the class of all convex functions of order $\zeta$ and denote by $R\,(\zeta)$ the class of all functions in $A$ which satisfy

$$\operatorname{Re}\left\{f'(z)\right\}>\zeta \quad (z\in U)\,. \tag{1.4}$$

It is well known that $K(\zeta) \subset S^*(\zeta) \subset S$.

For functions $f$ given by (1.1) and $g \in A$ given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k, \tag{1.5}$$

the Hadamard product (or convolution) of $f$ and $g$ is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z). \tag{1.6}$$

For two analytic functions $f$ and $g$ in $U$, $f$ is subordinate to $g$, written $f \prec g$ or $f(z) \prec g(z)$, if there exists an analytic function $w(z)$ in $U$, with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$. If $g(z)$ is univalent function, then $f \prec g$ if and only if (see [15] and [16])

$$f(0) = g(0) \text{ and } f(U) \subset g(U).$$

For functions $f, g \in A$, we define the linear operator $D_\lambda^n : A \to A$ $(\lambda \geq 0, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mathbb{N} = \{1, 2, ...\})$ by:

$$D_\lambda^0 (f * g)(z) = (f * g)(z),$$

$$D_\lambda^1 (f * g)(z) = D_\lambda (f * g)(z) = (1 - \lambda)(f * g)(z) + \lambda z((f * g)(z))' \tag{1.7}$$

and ( in general )

$$\begin{aligned} D_\lambda^n (f * g)(z) &= D_\lambda (D_\lambda^{n-1} (f * g)(z)) \\ &= z + \sum_{k=2}^{\infty} [1 + \lambda(k-1)]^n a_k b_k z^k \quad (\lambda \geq 0; n \in \mathbb{N}_0). \end{aligned} \tag{1.8}$$

From (1.8), we can easily deduce that

$$\lambda z (D_\lambda^n (f * g)(z))' = D_\lambda^{n+1} (f * g)(z) - (1 - \lambda) D_\lambda^n (f * g)(z) \ (\lambda > 0). \tag{1.9}$$

The linear operator $D_\lambda^n (f * g)(z)$ was introduced by Aouf and Mostafa [3], Aouf and Seoudy [4] and Mostafa and Aouf [17] and we observe that $D_\lambda^n (f * g)(z)$ reduces to several interesting many other linear operators considered earlier for different choices of $n$, $\lambda$ and the function $g$.

**Definition 1.** *For $0 \leq \zeta < 1$, $f, g \in A$ given by (1.1) and (1.5), respectively, and $\gamma \geq 0$, a function $f$ given by (1.1), is said to be in the class $M(g,n,\gamma,\lambda,\zeta)$ if it satisfies the following condition:*

$$\left| \frac{D_{\lambda}^{n+1}(f*g)(z)}{z} \left( \frac{z}{D_{\lambda}^{n}(f*g)(z)} \right)^{\gamma} - 1 \right| < 1 - \zeta \ \ (z \in U). \tag{1.10}$$

The class $M(g,n,\gamma,\lambda,\zeta)$ includes various new subclasses of analytic univalent functions. We observe that:
(i) Putting $n = 0$, $\lambda = 1$ and $g(z) = \frac{z}{1-z}$ in (1.10), then the class $M(\frac{z}{1-z}, 0, \gamma, 1,\zeta)$ reduces to the class
$B(\zeta, \gamma)$, which was introduced by Frasin and Jahangiri [11] and Murugusundaramoorthy and Magesh [18]. Further $B(\zeta, 2)$ has been studied by Frasin and Darus [10].
(ii) Putting $n = 0$, $\lambda = 1$ and $g(z) = z + \sum_{k=2}^{\infty} \Gamma_k(a_1) z^k$, where

$$\Gamma_k(a_1) = \frac{(a_1)_{k-1}...(a_l)_{k-1}}{(b_1)_{k-1}...(b_m)_{k-1}(1)_{k-1}}, \tag{1.11}$$

$a_i \in \mathbb{C}; i = 1, ..., l; b_j \in \mathbb{C}\backslash\mathbb{Z}_0^- = \{0, -1, -2, ...\}, j = 1, ..., m,\ l \leq m+1, l, m \in \mathbb{N}_0, z \in U$ and

$$(x)_k = \begin{cases} 1 & (k = 0; x \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}) \\ x(x+1)...(x+k-1) & (k \in \mathbb{N}; x \in \mathbb{C}), \end{cases}$$

in (1.10), then the class $M(z + \sum_{k=2}^{\infty} \Gamma_k(a_1) z^k, 0, \gamma, 1,\zeta)$ reduces to the class
$K_{l,m}(a_1, b_1, \gamma, \zeta)$,
which is defined by:

$$\left| (H_{l,m}(a_1; b_1) f(z))' \left( \frac{z}{H_{l,m}(a_1; b_1) f(z)} \right)^{\gamma} - 1 \right| < 1 - \zeta \ (\gamma \geq 0; 0 \leq \zeta < 1; z \in U), \tag{1.12}$$

where the operator $H_{l,m}(a_1; b_1)$ is the Dziok-Srivastava operator introduced and studied by Dziok and Srivastava [9].

(iii) Putting $n = 0$, $\lambda = 1$ and $g(z) = z + \sum_{k=2}^{\infty} \left[\frac{\ell+1+\theta(k-1)}{\ell+1}\right]^m z^k$, where $\theta > 0$, $\ell \geq 0$ and $m \in \mathbb{N}_0$ in (1.10), then the class $M(z + \sum_{k=2}^{\infty} \left[\frac{\ell+1+\theta(k-1)}{\ell+1}\right]^m z^k, 0, \gamma, 1,\zeta)$ reduces to the class $B(\ell, m, \theta, \gamma, \zeta)$, which is defined by:

$$\left|(I^{m}(\theta,\ell)f(z))'\left(\frac{z}{I^{m}(\theta,\ell)f(z)}\right)^{\gamma}-1\right|<1-\zeta\ (\gamma\geq 0;0\leq\zeta<1;z\in U),\quad (1.13)$$

where $I^{m}(\theta,\ell)$ is the generalized multiplier transformation which was introduced and studied by Cătaş et al. [5].

(iv) Putting $n=0$, $\lambda=1$ and $g(z)=z+\sum_{k=2}^{\infty}\left[\frac{\ell+k}{\ell+1}\right]^{m}z^{k}$, where $\ell\geq 0$ and $m\in\mathbb{N}_0$ in (1.10), then the class $M(z+\sum_{k=2}^{\infty}\left[\frac{\ell+k}{\ell+1}\right]^{m}z^{k},0,\gamma,1,\zeta)$ reduces to the class $S(\ell,m,\gamma,\zeta)$, which is defined by:

$$\left|(I^{m}(\ell)f(z))'\left(\frac{z}{I^{m}(\ell)f(z)}\right)^{\gamma}-1\right|<1-\zeta\ (\gamma\geq 0;0\leq\zeta<1;z\in U),\quad (1.14)$$

where $I^{m}(\ell)$ is the multiplier transformation (see Cho and Srivastava [7] and Cho and Kim [6]).

(v) Putting $n=0$, $\lambda=1$, $g(z)=z+\sum_{k=2}^{\infty}[1+\theta(k-1)]^{m}z^{k}$, where $\theta\geq 0$ and $m\in\mathbb{N}_0$ in (1.10), then the class $M(z+\sum_{k=2}^{\infty}[1+\theta(k-1)]^{m}z^{k},0,\gamma,1,\zeta)$ reduces to the class $Q(\theta,m,\gamma,\zeta)$, which is defined by:

$$\left|(D_{\theta}^{m}f(z))'\left(\frac{z}{D_{\theta}^{m}f(z)}\right)^{\gamma}-1\right|<1-\zeta\ (\gamma\geq 0;0\leq\zeta<1;z\in U),\quad (1.15)$$

where $D_{\theta}^{m}$ is the generalized Sălăgean operator (see AL-Oboudi [1]).

(vi) Putting $n=0$, $\lambda=1$, $g(z)=z+\sum_{k=2}^{\infty}k^{m}z^{k}$, where $m\in\mathbb{N}_0$ in (1.10), then the class $M(z+\sum_{k=2}^{\infty}k^{m}z^{k},0,\gamma,1,\zeta)$ reduces to the class $\Psi(m,\gamma,\zeta)$, which is defined by:

$$\left|(D^{m}f(z))'\left(\frac{z}{D^{m}f(z)}\right)^{\gamma}-1\right|<1-\zeta\ (\gamma\geq 0;0\leq\zeta<1;z\in U),\quad (1.16)$$

where the operator $D^{m}$ is the Sălăgean operator (see Sălăgean [19]).

(vii) Putting $n=0,\lambda=1$ and $g(z)=z+\sum_{k=2}^{\infty}\left(\frac{1+b}{k+b}\right)^{s}z^{k}$ $(b\in\mathbb{C}\backslash\mathbb{Z}_0^{-},s\in\mathbb{C})$ in (1.10), then the class $M(z+\sum_{k=2}^{\infty}\left(\frac{1+b}{k+b}\right)^{s}z^{k},0,\gamma,1,\zeta)$ reduces to the class $\mathcal{G}(s,b,\gamma,\zeta)$, which is defined by:

$$\left|(J_{s,b}f(z))'\left(\frac{z}{J_{s,b}f(z)}\right)^{\gamma}-1\right|<1-\zeta\ (\gamma\geq 0;0\leq\zeta<1;z\in U),\tag{1.17}$$

where the operator $J_{s,b}$ was introduced and studied by Srivastava and Attiya [21].
(viii) Putting $n=0,\lambda=1$ and $g(z)=z+\sum_{k=2}^{\infty}\left(\frac{2}{k+1}\right)^{\alpha}z^{k}$ $(\alpha\geq 0)$ in (1.10), then the class $M(z+\sum_{k=2}^{\infty}\left(\frac{2}{k+1}\right)^{\alpha}z^{k},0,\gamma,1,\zeta)$ reduces to the class $\mathcal{H}(\alpha,\gamma,\zeta)$, which is defined by:

$$\left|(\mathrm{I}^{\alpha}f(z))'\left(\frac{z}{\mathrm{I}^{\alpha}f(z)}\right)^{\gamma}-1\right|<1-\zeta\ (\gamma\geq 0;0\leq\zeta<1;z\in U),\tag{1.18}$$

where the operator $\mathrm{I}^{\alpha}$ was introduced and studied by Jung et al. [12].
(ix) Putting $n=0,\ \lambda=1$ and $g(z)=z+\frac{\Gamma(1+\alpha+\beta)}{\Gamma(1+\beta)}\sum_{k=2}^{\infty}\frac{\Gamma(k+\beta)}{\Gamma(k+\alpha+\beta)}z^{k}$ $(\alpha\geq 0,\beta>-1)$ in (1.10), then the class $M(z+\frac{\Gamma(1+\alpha+\beta)}{\Gamma(1+\beta)}\sum_{k=2}^{\infty}\frac{\Gamma(k+\beta)}{\Gamma(k+\alpha+\beta)}z^{k},0,\gamma,1,\zeta)$ reduces to the class $\mathcal{Z}(\gamma,\zeta,\alpha,\beta)$, which is defined by:

$$\left|\left(Q_{\beta}^{\alpha}f(z)\right)'\left(\frac{z}{Q_{\beta}^{\alpha}f(z)}\right)^{\gamma}-1\right|<1-\zeta\ (\gamma\geq 0;0\leq\zeta<1;z\in U),\tag{1.19}$$

where the operator $Q_{\beta}^{\alpha}$ was introduced and studied by Jung et al. [12].
(x) Putting $n=0,\ \lambda=1,\ g(z)=z+\sum_{k=2}^{\infty}\frac{(1+\mu)^{\upsilon}}{(k+\mu)^{\upsilon}}\Gamma_{k}(a_{1})z^{k}$, where $\Gamma_{k}(a_{1})$ is given by (1.11), $\mu\neq -1$ and $\upsilon\in\mathbb{N}_{0}$ in (1.10), then the class $M(z+\sum_{k=2}^{\infty}\frac{(1+\mu)^{\upsilon}}{(k+\mu)^{\upsilon}}\Gamma_{k}(a_{1})z^{k},0,\gamma,1,\zeta)$ reduces to the class $\mathcal{L}(\gamma,\zeta,\mu,q,s,\upsilon,a_{1})$, which is defined by:

$$\left|\left(\mathcal{K}_{\mu,q,s}^{\upsilon}(a_{1})f(z)\right)'\left(\frac{z}{\mathcal{K}_{\mu,q,s}^{\upsilon}(a_{1})f(z)}\right)^{\gamma}-1\right|<1-\zeta\ (\gamma\geq 0;0\leq\zeta<1;z\in U),\tag{1.20}$$

where the operator $\mathcal{K}_{\mu,q,s}^{\upsilon}$ was introduced and studied by Selvaraj and Karthikeyan [20].
(xi) Putting $n=0,\ \lambda=1$ and $g(z)=z+\sum_{k=2}^{\infty}\left(\frac{1+b}{k+b}\right)^{s}\frac{\rho!(k+\mu-2)!}{(\mu-1)!(k+\rho-1)!}z^{k}$ $(b\in\mathbb{C}\backslash\mathbb{Z}_{0}^{-},s\in\mathbb{C},\mu>0,\rho>-1)$ in (1.10), then the class $M(z+\sum_{k=2}^{\infty}\left(\frac{1+b}{k+b}\right)^{s}\frac{\rho!(k+\mu-2)!}{(\mu-1)!(k+\rho-1)!}z^{k},0,\gamma,1,\zeta)$

reduces to the class $\mathcal{C}(\gamma,\zeta,\rho,\mu,s,b)$, which is defined by:

$$\left|\left(J_{s,b}^{\rho,\mu}(f)(z)\right)'\left(\frac{z}{J_{s,b}^{\rho,\mu}(f)(z)}\right)^{\gamma}-1\right|<1-\zeta\ (\gamma\geq 0; 0\leq\zeta<1; z\in U), \tag{1.21}$$

where the operator $J_{s,b}^{\rho,\mu}$ was introduced and studied by Al-Shaqsi and Darus [2] and Darus and Al-Shaqsi [8].

The object of the present paper is to investigate the sufficient condition for functions to be in the class $M(g,n,\gamma,\lambda,\zeta)$. Furthermore, as a special case, we show that convex functions of order 1/2 are also members of the class $M(g,n,\gamma,\lambda,\zeta)$.

## 2. Main Results

Unless otherwise mentioned, we shall assume in the reminder of this paper that the functions $f$ and $g$ are given by (1.1) and (1.5), respectively, $\lambda>0,\ \gamma\geq 0,\ n\in\mathbb{N}_0$ *and* $1/2\leq\zeta<1$.

To prove our results we need the following lemma.

**Lemma 1 [11].** *Let $p(z)$ be analytic in $U$ with $p(0)=1$ and suppose that*

$$Re\left(1+\frac{zp'(z)}{p(z)}\right)>\frac{3\zeta-1}{2\zeta}\ \ (z\in U). \tag{2.1}$$

*Then $Re\{p(z)\}>\zeta$ for $z\in U$ and $1/2\leq\zeta<1$.*

**Theorem 1.** *Let $f,g\in A$. If*

$$\mathrm{Re}\left\{1+\frac{D_{\lambda}^{n+2}(f*g)(z)}{\lambda D_{\lambda}^{n+1}(f*g)(z)}-\frac{\gamma D_{\lambda}^{n+1}(f*g)(z)}{\lambda D_{\lambda}^{n}(f*g)(z)}+\frac{1}{\lambda}(\gamma-1)\right\}>\beta, \tag{2.2}$$

*where $\beta=\frac{3\zeta-1}{2\zeta}$, then $f(z)\in M(g,n,\gamma,\lambda,\zeta)$.*

*Proof.* Define the function $p(z)$ by

$$p(z)=\frac{D_{\lambda}^{n+1}(f*g)(z)}{z}\left(\frac{z}{D_{\lambda}^{n}(f*g)(z)}\right)^{\gamma}. \tag{2.3}$$

Then the function $p(z)$ is analytic in $U$ and $p(0)=1$. Therefore, differentiating (2.3) logarithmically with respect to $z$ and using (1.9) with simple computation, then

$$\frac{zp'(z)}{p(z)}=\frac{D_{\lambda}^{n+2}(f*g)(z)}{\lambda D_{\lambda}^{n+1}(f*g)(z)}-\frac{\gamma D_{\lambda}^{n+1}(f*g)(z)}{\lambda D_{\lambda}^{n}(f*g)(z)}+\frac{1}{\lambda}(\gamma-1),$$

by the hypothesis of the theorem, we have

$$\operatorname{Re}\left\{1+\frac{zp'(z)}{p(z)}\right\}>\frac{3\zeta-1}{2\zeta}.$$

Hence by Lemma 1, we have

$$\operatorname{Re}\left\{\frac{D_\lambda^{n+1}(f*g)(z)}{z}\left(\frac{z}{D_\lambda^{n}(f*g)(z)}\right)^{\gamma}\right\}>\zeta \ (z\in U).$$

Therefore, in view of Definition 1, we have $f(z)\in M(g,n,\gamma,\lambda,\zeta)$ .

Putting $n=0,\lambda=1$ and $g(z)=z+\sum_{k=2}^{\infty}\Gamma_k\left(a_1\right)z^k$, where $\Gamma_k\left(a_1\right)$ is given by (1.11) in Theorem 1, we obtain the following corollary:

**Corollary 1.** *Let* $f\in A$ *. If*

$$\operatorname{Re}\left\{1+\frac{z\left(H_{l,m}\left(a_1;b_1\right)f(z)\right)''}{\left(H_{l,m}\left(a_1;b_1\right)f(z)\right)'}+\gamma a_1\left(1-\frac{H_{l,m}\left(a_1+1;b_1\right)f(z)}{H_{l,m}\left(a_1;b_1\right)f(z)}\right)\right\}>\beta, \tag{2.4}$$

*then* $f(z)\in K_{l,m}(a_1,b_1,\gamma,\zeta)$, *where* $\beta=\frac{3\zeta-1}{2\zeta}$ *and* $K_{l,m}(a_1,b_1,\gamma,\zeta)$ *is given by (1.12).*

Putting $n=0,\lambda=1$ and $g(z)=z+\sum_{k=2}^{\infty}\left[\frac{\ell+1+\theta(k-1)}{\ell+1}\right]^m z^k$, where $\theta>0,\ \ell\geq 0$ and $m\in\mathbb{N}_0$ in Theorem 1, we obtain the following corollary:

**Corollary 2.** *Let* $f\in A$*. If*

$$\operatorname{Re}\left\{1+\frac{z\left(I^m(\theta,\ell)f(z)\right)''}{\left(I^m(\theta,\ell)f(z)\right)'}+\gamma\left(\frac{1+\ell}{\theta}\right)\left(1-\frac{I^{m+1}(\theta,\ell)f(z)}{I^m(\theta,\ell)f(z)}\right)\right\}>\beta, \tag{2.5}$$

*then* $f(z)\in B(\ell,m,\theta,\gamma,\zeta)$, *where* $\beta=\frac{3\zeta-1}{2\zeta}$ *and* $B(\ell,m,\theta,\gamma,\zeta)$ *is given by (1.13).*

Putting $\theta=1$ in Corollary 2, we obtain the following corollary:

**Corollary 3.** *Let* $f\in A$*. If*

$$\operatorname{Re}\left\{1+\frac{z\left(I^m(\ell)f(z)\right)''}{\left(I^m(\ell)f(z)\right)'}+\gamma(1+\ell)\left(1-\frac{I^{m+1}(\ell)f(z)}{I^m(\ell)f(z)}\right)\right\}>\beta, \tag{2.6}$$

*then* $f(z)\in S(\ell,m,\gamma,\zeta)$, *where* $\beta=\frac{3\zeta-1}{2\zeta}$ *and* $S(\ell,m,\gamma,\zeta)$ *is given by (1.14).*

Putting $\ell=0$ in Corollary 2, we obtain the following corollary:

**Corollary 4.** *Let $f \in A$. If*

$$\operatorname{Re}\left\{1+\frac{z\left(D_{\theta}^{m} f(z)\right)''}{\left(D_{\theta}^{m} f(z)\right)'}+\frac{\gamma}{\theta}\left(1-\frac{D_{\theta}^{m+1} f(z)}{D_{\theta}^{m} f(z)}\right)\right\}>\beta, \tag{2.7}$$

*then $f(z) \in Q(\theta, m, \gamma, \zeta)$, where $\beta=\frac{3\zeta-1}{2\zeta}$ and $Q(\theta, m, \gamma, \zeta)$ is given by (1.15).*

Putting $\theta=1$ and $\ell=0$ in Corollary 2, we obtain the following corollary:

**Corollary 5.** *Let $f \in A$. If*

$$\operatorname{Re}\left\{1+\frac{z\left(D^{m} f(z)\right)''}{\left(D^{m} f(z)\right)'}+\gamma\left(1-\frac{D^{m+1} f(z)}{D^{m} f(z)}\right)\right\}>\beta, \tag{2.8}$$

*then $f(z) \in \Psi(m, \gamma, \zeta)$,* where $\beta=\frac{3\zeta-1}{2\zeta}$and $\Psi(m, \gamma, \zeta)$ is given by*(1.16).*

Putting $n=0$, $\lambda=1$ and $g(z)=z+\sum\limits_{k=2}^{\infty}\left(\frac{1+b}{k+b}\right)^{s} z^{k}$ $(b \in \mathbb{C} \backslash \mathbb{Z}_{0}^{-}, s \in \mathbb{C})$ in Theorem 1, we obtain the following corollary:

**Corollary 6.** *Let $f \in A$. If*

$$\operatorname{Re}\left\{1+\frac{z\left(J_{s, b} f(z)\right)''}{\left(J_{s, b} f(z)\right)'}+\gamma(1+b)\left(1-\frac{J_{s-1, b} f(z)}{J_{s, b} f(z)}\right)\right\}>\beta, \tag{2.9}$$

*then $f(z) \in \mathcal{G}(s, b, \gamma, \zeta)$, where $\beta=\frac{3\zeta-1}{2\zeta}$ and $G(s, b, \gamma, \zeta)$ is given by (1.17).*

Putting $n=0, \lambda=1$ and $g(z)=z+\sum\limits_{k=2}^{\infty}\left(\frac{2}{k+1}\right)^{\alpha} z^{k}$ $(\alpha \geq 0)$ in Theorem 1, we obtain the following corollary:

**Corollary 7.** *Let $f \in A$. If*

$$\operatorname{Re}\left\{1+\frac{z\left(\mathrm{I}^{\alpha} f(z)\right)''}{\left(\mathrm{I}^{\alpha} f(z)\right)'}+2\gamma\left(1-\frac{\mathrm{I}^{\alpha-1} f(z)}{\mathrm{I}^{\alpha} f(z)}\right)\right\}>\beta, \tag{2.10}$$

*then $f(z) \in \mathcal{H}(\alpha, \gamma, \zeta)$, where $\beta=\frac{3\zeta-1}{2\zeta}$ and $\mathcal{H}(\alpha, \gamma, \zeta)$ is given by (1.18).*

Putting $n=0$, $\lambda=1$ and $g(z)=z+\frac{\Gamma(1+\alpha+\beta)}{\Gamma(1+\beta)} \sum\limits_{k=2}^{\infty} \frac{\Gamma(k+\beta)}{\Gamma(k+\alpha+\beta)} z^{k}$ $(\alpha \geq 0, \beta>-1)$ in Theorem 1, we obtain the following corollary:

**Corollary 8.** *Let $f \in A$. If*

$$\operatorname{Re}\left\{1+\frac{z\left(Q_{\beta}^{\alpha} f(z)\right)''}{\left(Q_{\beta}^{\alpha} f(z)\right)'}+\gamma(\alpha+\beta)\left(1-\frac{Q_{\beta}^{\alpha-1} f(z)}{Q_{\beta}^{\alpha} f(z)}\right)\right\}>\beta, \tag{2.11}$$

*then* $f(z) \in \mathcal{Z}(\gamma, \zeta, \alpha, \beta)$, *where* $\beta = \frac{3\zeta-1}{2\zeta}$ *and* $\mathcal{Z}(\gamma, \zeta, \alpha, \beta)$ *is given by (1.19).*

Putting $n = 0, \lambda = 1$ and $g(z) = z + \sum_{k=2}^{\infty} \frac{(1+\mu)^{\upsilon}}{(k+\mu)^{\upsilon}} \Gamma_k(a_1) z^k$, where $\Gamma_k(a_1)$ is given by (1.11), $\mu \neq -1$ and $\upsilon \in \mathbb{N}_0$ in Theorem 1, we obtain the following corollary:

**Corollary 9.** *Let* $f \in A$. *If*

$$\operatorname{Re}\left\{1 + \frac{z\left(\mathcal{K}^{\upsilon}_{\mu,q,s}(a_1) f(z)\right)''}{\left(\mathcal{K}^{\upsilon}_{\mu,q,s}(a_1) f(z)\right)'} + \gamma\alpha_1 \left(1 - \frac{z\left(\mathcal{K}^{\upsilon}_{\mu,q,s}(a_1+1) f(z)\right)'}{\mathcal{K}^{\upsilon}_{\mu,q,s}(a_1) f(z)}\right)\right\} > \beta, \quad (2.12)$$

*then* $f(z) \in \mathcal{L}(\gamma, \zeta, \mu, q, s, \upsilon, a_1)$, *where* $\beta = \frac{3\zeta-1}{2\zeta}$ *and* $\mathcal{L}(\gamma, \zeta, \mu, q, s, \upsilon, a_1)$ *is given by (1.20).*

Putting $n = 0$, $\lambda = 1$ and $g(z) = z + \sum_{k=2}^{\infty} \left(\frac{1+b}{k+b}\right)^s \frac{\rho!(k+\mu-2)!}{(\mu-1)!(k+\rho-1)!} z^k$ $(b \in \mathbb{C}\backslash\mathbb{Z}_0^-, s \in \mathbb{C}, \mu > 0, \rho > -1)$ in Theorem 1, we obtain the following corollary:

**Corollary 10.** *Let* $f \in A$. *If*

$$\operatorname{Re}\left\{1 + \frac{z\left(J^{\rho,\mu}_{s,b}(f)(z)\right)''}{\left(J^{\rho,\mu}_{s,b}(f)(z)\right)'} + \gamma\mu \left(1 - \frac{J^{\rho,\mu+1}_{s,b}(f)(z)}{J^{\rho,\mu}_{s,b}(f)(z)}\right)\right\} > \beta, \quad (2.13)$$

*then* $f(z) \in \mathcal{C}(\gamma, \zeta, \rho, \mu, s, b)$, *where* $\beta = \frac{3\zeta-1}{2\zeta}$ *and* $\mathcal{C}(\gamma, \zeta, \rho, \mu, s, b)$ *is given by (1.21).*

Putting $n = \lambda = \gamma = 1$, $\zeta = \frac{1}{2}$ and $g(z) = \frac{z}{1-z}$ in Theorem 1, we obtain the following corollary:

**Corollary 11.** If $f \in A$ *given by (1.1) and*

$$\operatorname{Re}\left\{\frac{zf'''(z)}{f''(z)} - \frac{zf''(z)}{f'(z)}\right\} > -\frac{3}{2} \quad (z \in U),$$

then

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \frac{1}{2} \quad (z \in U).$$

**Remarks.** (i) Putting $n = 0$, $\lambda = 1$ and $g(z) = \frac{z}{1-z}$ in Theorem 1, we obtain the results obtained by Murugusundaramoorthy and Magesh [18, Corollary 4] and Frasin and Jahangiri [11, Theorem 2.3];

(ii) Putting $n = 0, \lambda = \gamma = 1, \zeta = \frac{1}{2}$ and $g(z) = \frac{z}{1-z}$ in Theorem 1, we obtain the results obtained by Murugusundaramoorthy and Magesh [18, Corollary 7] and Lupas and Catas [14, Corollary 2.7];

(iii) Putting $n = \gamma = 0$, $\lambda = 1$, $\zeta = \frac{1}{2}$ and $g(z) = \frac{z}{1-z}$ in Theorem 1, we obtain the results obtained by Murugusundaramoorthy and Magesh [18, Corollary 8] and Lupas and Catas [13, Corollary 2.6] and Lupas and Catas [14, Corollary 4].

## References


[1] F. M. Al. Oboudi, On univalent functions defined by a generalized Sălăgean operator, Internat. J. Math. Math. Sci., 27 (2004), 1429-1436.

[2] K. Al-Shaqsi and M. Darus, On certain subclasses of analytic functions defined by a multiplier transformation with two parameters, Appl. Math. Sci., 3 (2009), no. 36, 1799-1810.

[3] M. K. Aouf and A. O. Mostafa, Sandwich theorems for analytic functions defined by convolution, Acta Univ. Apulensis, (2010), no. 21 , 7-20.

[4] M. K. Aouf and T. M. Seoudy, On differential sandwich theorems of analytic functions defined by certain linear operator, Ann. Univ. Mariae Curie-Sklodowska Sect. A, 64 (2010), no. 2 , 1-14.

[5] A. Cătaş, G. I. Oros and G. Oros, Differential subordinations associated with multiplier transformations, Abstract Appl. Anal., 2008 (2008), ID 845724, 1-11.

[6] N. E. Cho and T. H. Kim, Multiplier transformations and strongly close-to-convex functions, Bull. Korean Math. Soc., 40 (2003), no. 3, 399-410.

[7] N. E. Cho and H. M. Srivastava, Argument estimates of certain analytic functions defined by a class of multiplier transformations, Math. Comput. Modelling, 37 (1-2) (2003), 39-49.

[8] M. Darus and K. Al-Shaqsi, On subordinations for certain analytic functions associated with generalized integral operator, Lobachevskii J. Math., 29 (2008), 90-97.

[9] J. Dziok and H. M. Srivastava, Classes of analytic functions associated with the generalized hypergeometric function, Appl. Math. Comput., 103 (1999), 1-13.

[10] B. A. Frasin and M. Darus, On certain analytic univalent functions, Internat. J. Math. Math. Sci., 25 (2001), no. 5, 305-310.

[11] B. A. Frasin and J. M. Jahangiri, A new and comprehensive class of analytic functions, An. Univ. Oradea Fasc. mat., 15 (2008), 59-62.

[12] I. B. Jung, Y. C. Kim and H. M. Srivastava, The Hardy space of analytic functions associated with certain one-parameter families of integral operators, J. Math. Anal. Appl., 176 (1993), 138–147.

[13] A. A. Lupas and A. A. Cătaş, A note on a subclasses of analytic functions defined by Ruscheweyh derivative, J. Math. Inequal., 4 (2010), no. 2, 233-236.

[14] A. A. Lupas and A. A. Cătaş, A note on a subclasses of analytic functions defined by a generalized Sălăgean operator, Acta Univ. Apule., (2010), no. 22 , 35-39.

[15] S. S. Miller and P. T. Mocanu, Differential Subordinations: Theory and Applications, Series on Monographs and Textbooks in Pure Appl. Math., Vol. 225, Marcel Dekker, New York and Basel, 2000.

[16] S. Miller and P. T. Mocanu, Subordinants of differential superordinations, Complex Var. Theory Appl., 48 (2003), no.10, 815–826.

[17] A. O. Mostafa and M. K. Aouf, Sandwich results for certain subclasses of analytic functions defined by convolution, General Math., (To appear).

[18] G. Murugusundaramoorthy and N. Magesh, On certain sufficient conditions for analytic univalent functions, European J. Pure Appl. Math., 4 (2011), 76-82.

[19] G. S. Sălăgean, Subclasses of univalent functions, Lecture Notes in Math. ( Springer-Verlag ) 1013 (1983), 362-372.

[20] C. Selvaraj and K. R. Karthikeyan, Subclasses of analytic functions involving a certain family of linear operators, Internat. J. Contemp. Math. Sci., 3 (2008), no. 13 , 615-627.

[21] H. M. Srivastava and A. A. Attiya, An integral operator associated with the Hurwitz–Lerch Zeta function and differential subordination, Integral Transforms Spec. Funct., 18 (2007), 207-216.

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