AN APPROACH FOR SOLVING FUNCTIONAL INTEGRAL EQUATIONS

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ABSTRACT. In this paper, the Lagrange interpolation method has been used to numerically solve the nonlinear Fredholm and Volterra functional integral equations of the second kind. For approximating the integrals in discretizing the equations we used Clenshaw-Curtis rule. The method transforms any integral equation into a system of nonlinear algebraic equations. These equations then can be solved using the Newton method. Some numerical examples presented to illustrate the method and show the efficiency.

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1. INTRODUCTION

Consider the nonlinear functional integral equation of the type

$$y(t) = f(t) + \sum_{i=1}^{3} \int_{0}^{\sigma_i(t)} k_i(t,s)g_i(s,y(\eta_i(s)))ds, \quad t \in I \tag{1}$$

where $I := [0,1]$, $f : I \to \mathbb{R}$, $k_i : I \times I \to \mathbb{R}$, $g_i : I \times \mathbb{R} \to \mathbb{R}$, and $\sigma_i, \eta_i : I \to I$ for $i = 1, 2, 3$.

y is a solution for the functional integral equation (FIE) (1), if y is a bounded and measurable real-value function on I which satisfies (1).

The FIE (1) is new to the theory of nonlinear integral equations and in this paper we solve the specials case of the form:

$$y(t) = f(t) + \int_{0}^{1} k(t,s)g(s,y(s))ds, \tag{2}$$

and

$$y(t) = f(t) + \int_{0}^{t} k(t,s)g(s,y(s))ds, \quad t \in I. \tag{3}$$

In this paper, we consider the numerical solution of the functional integral equations of the form (2) and (3), where f, k and g are given functions and y is the unknown function to be determined.

In [4] the existence of extremal solutions of nonlinear discontinuous functional integral equations of mixed type (1) was proved under mixed Lipschitz, Caratheodory and monotonicity conditions. In [8], an application of He's homotopy perturbation method was applied to solve functional integral equations of similar types as (2) and (3). Also, in [2] an expansion method was introduced to treat similar functional equations. In [9] the Chebyshev collocation method was used to solve the functional integral equation of the first and second kind. For solving (2) and (3) in [5] a new Collocation-type method was introduced in which the Collocation is applied to an equivalent equation for the Hammerstein integral equations.

For solving the (2) and (3), we suppose that:

$$z(t) = g(t, y(t)), \quad t \in I. \tag{4}$$

with substituting (4) into (2) and (3) we obtain

$$y(t) = f(t) + \int_0^1 k(t, s) z(s) ds,$$

and

$$y(t) = f(t) + \int_0^t k(t, s) z(s) ds, \quad t \in I$$

so that the new unknown $z(t)$ satisfies the following equations:

$$z(t) = g\left(t, f(t) + \int_0^1 k(t, s) z(s) ds\right)$$

and

$$z(t) = g\left(t, f(t) + \int_0^t k(t, s) g(s, y(s)) ds\right), \quad t \in I.$$

2. Discretizing the equations

Here we use Lagrange interpolation for $y(t)$, $k(t, s) g(s, y(s))$ as follows to solve the mentioned FIEs of the forms (2) and (3). For this, we can write:

$$y_n(t) = \sum_{j=0}^n l_n(t, t_j) y(t_j), \tag{5}$$

$$k(t, s) z(s) = \sum_{j=0}^n l_n(s, t_j) k(t, t_j) z(t_j). \tag{6}$$

where

$$l_n(t, t_j) = \prod_{k=0, k \neq j}^{n} \frac{t - t_k}{t_j - t_k}$$

substituting from (5),(6) in (2) and (3) we get from each functional integral equation, a system of equation respectively:

$$\sum_{j=0}^{n} l_n(t_i, t_j) y(t_j) = f(t_i) + \int_0^1 \sum_{j=0}^{n} l_n(s, t_j) k(t_i, t_j) z(t_j) ds, \quad i = 0, 1, \ldots, n \ (7)$$

$$\sum_{j=0}^{n} l_n(t_i, t_j) y(t_j) = f(t_i) + \int_0^t \sum_{j=0}^{n} l_n(s, t_j) k(t_i, t_j) z(t_j) ds, \quad i = 0, 1, \ldots, n \ (8)$$

where

$$s_i = \frac{(b - a)}{2} \cos \frac{(i - 0.45)\pi}{n} + \frac{b + a}{2}, \quad i = 0, 1, \ldots, n$$

and

$$t_i = a + \frac{i(b - a)}{n}, \quad i = 0, 1, \ldots, n.$$

Now, we approximate the integrals in (7) and (8) by means of Clenshaw-Curtis rule [6]:

$$\int_0^1 l_n(s, t_j) ds = \frac{1}{2} \sum_{k=0}^{n} d_k \prod_{k=0, k \neq j}^{n} \frac{\frac{1}{2} \cos(\frac{k\pi}{n}) + \frac{1}{2} - t_k}{t_j - t_k},$$

and

$$\int_0^{t_i} l_n(s, t_j) ds = \int_0^{t_i} \prod_{k=0, k \neq j}^{n} \frac{s - t_k}{t_j - t_k} ds$$

$$= \frac{t_i}{2} \int_{-1}^{1} \prod_{k=0, k \neq j}^{n} \frac{\frac{t_i}{2} s + \frac{t_i}{2} - t_k}{t_j - t_k} ds$$

$$= \frac{t_i}{2} \sum_{k=0}^{n} d_k \prod_{k=0, k \neq j}^{n} \frac{\frac{t_i}{2} \cos(\frac{k\pi}{n}) + \frac{t_i}{2} - t_k}{t_j - t_k},$$

where $i, j = 0, 1, \ldots, n$,

$$d_k = \frac{2\delta_k}{n} \sum_{m=0}^{n''} I_m \cos \frac{mk\pi}{n},$$

$$\delta_k = \left\{ \begin{array}{ll} \frac{1}{2}, & k = 0, n; \\ 1, & 0 < k < n. \end{array} \right.$$

$$I_m = \left\{ \begin{array}{ll} 2, & \text{m=0}; \\ 0, & \text{m is odd}; \\ \frac{2}{1-m^2}, & \text{m is even}. \end{array} \right.$$

where the double prime on the above summation sign implies that the first and latest terms are halved.

2.1. Solving the system

For solving the nonlinear equations system, let

$$z_n(s_i) = \sum_{j=0}^{n} l_n(s_i, t_j) z(t_j) = g\left(s_i, f(s_i) + \sum_{j=0}^{n} z(t_j) \int_0^1 k(s_i, s) l_n(s, t_j) ds \right) \quad (9)$$

and

$$z_n(s_i) = \sum_{j=0}^{n} l_n(s_i, t_j) z(t_j) = g\left(s_i, f(s_i) + \sum_{j=0}^{n} z(t_j) \int_0^{t_i} k(s_i, s) l_n(s, t_j) ds \right) \quad (10)$$

The integral in the relations (9)and (10)are approximated as:

$$\int_0^1 k(s_i, s) l_n(s, t_j) ds = \frac{1}{2} \sum_{k=0}^{n} d_k k(s_i, \frac{1}{2} \cos(\frac{k\pi}{n}) + \frac{1}{2}) \prod_{k=0, k \neq j}^{n} \frac{\frac{1}{2} \cos(\frac{k\pi}{n}) + \frac{1}{2} - t_k}{t_j - t_k},$$

and

$$\int_0^{t_i} k(s_i, s) l_n(s, t_j) ds = \frac{t_i}{2} \sum_{k=0}^{n} d_k k(s_i, \frac{1}{2} \cos(\frac{k\pi}{n}) + \frac{t_i}{2}) \prod_{k=0, k \neq j}^{n} \frac{\frac{t_i}{2} \cos(\frac{k\pi}{n}) + \frac{t_i}{2} - t_k}{t_j - t_k},$$

let

$$l_n(s_i, t_j) = l_{ij}$$

and

$$\int_0^1 k(s_i, s) l(s, t_j) ds = L_{ij}$$

we use Newton method, as described below for solving the system, let

$$h_i(z_0, z_1, \ldots, z_n) = \sum_{j=0}^{n} l_{ij} z_j - g\left(t_i, f(t_i) + \sum_{j=0}^{n} L_{ij} z_j \right), \quad i = 0, 1, \ldots, n$$

then

$$\frac{\partial h_i}{\partial z_k} = l_{ik} - L_{ik}\frac{\partial g}{\partial y}\left(t_i, f(t_i) + \sum_{j=0}^{n} L_{ij}z_j\right), \quad i = 0, 1, \ldots, n$$

let

$$\bar{z}_p = (z_{0,p}, z_{1,p}, \ldots, z_{n,p})^T,$$
$$F(\bar{z}_p) = (h_0(\bar{z}_p), h_1(\bar{z}_p), \ldots, h_n(\bar{z}_p))^T$$
$$J(\bar{z}_p) = \left(\frac{\partial h_i}{\partial z_k}(\bar{z}_p)\right)_{i,k=0}^{n}$$

Now, for given $(\bar{z}_0 \in \mathbb{R})$, and applying the Newton method, we have the approximate solution from:

$$\bar{z}_{p+1} = \bar{z}_p - J(\bar{z}_p)^{-1}F(\bar{z}_p), \quad p = 0, 1, 2, \ldots.$$

3. NUMERICAL EXAMPLES

In this section, we illustrate the method discussed above with some numerical examples to show efficiently and accuracy of the discussed method, all computations were carried out using Mathematica 5.

Example 1. In this example we solve the following equation

$$y(t) = f(t) + \int_0^1 \frac{1}{s + t^2}(2s + y(s))^2 ds,$$

where $f(t)$ is so chosen that $y(t) = e^t$ is the exact solution for the equation. Numerical results are presented in Table 1.

t	n=4	n=8	n=12
0.0	1.43e-04	1.23e-10	3.12e-14
0.2	2.01e-05	3.56e-11	3.26e-15
0.4	1.46e-05	2.79e-12	3.28e-15
0.6	1.98e-05	0.98e-12	6.25e-14
0.8	1.47e-05	1.31e-11	4.26e-13
1.0	0.45e-04	2.68e-10	5.24e-13

Table 1: The values of absolute error for example 1.

Example 2. In this example we solve the following equation

$$y(t) = f(t) + \int_0^t (e^{s-t})\sin(2s + y(s))ds, \quad 1 \le s, t \le 1$$

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where $f(t)$ is so chosen that $y(t) = \sin t$ is the exact solution for the equation. Numerical results are presented in Table 2.

t	n=4	n=8	n=12
0.0	1.26e-04	1.42e-11	2.89e-14
0.2	1.86e-05	1.32e-11	7.25e-14
0.4	2.00e-05	2.11e-12	4.56e-14
0.6	3.86e-05	1.12e-11	9.26e-14
0.8	0.25e-04	2.31e-10	1.79e-14
1.0	2.15e-04	2.58e-10	5.69e-13

Table 2: The values of absolute error for example 2.

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