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PARTIAL SUMS OF CERTAIN CLASSES OF ANALYTIC FUNCTIONS DEFINED BY DZIOK-SRIVASTAVA OPERATOR

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Abstract. In this paper, we introduce the class $UT_{q,s}([\alpha_1]\,;\alpha,\beta)$ of analytic functions in the open unit disc $U = \{z \in \mathbb{C} : \ |z| < 1\}$ defined by Dziok-Srivastava operator. The object of the present paper is to determine coefficient estimates and some results concerning the partial sums for functions $f(z)$ belonging to this class.

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1.Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disc $U = \{z \in \mathbb{C} : \ |z| < 1\}$. Let $K(\alpha)$ and $S^*(\alpha)$ denote the subclasses of A which are, respectively, convex and starlike functions of order α, $0 \le \alpha < 1$. For convenience, we write $K(0) = K$ and $S^*(0) = S^*$ (see [17]).

The Hadamard product (or convolution) $(f*g)(z)$ of the functions $f(z)$ and $g(z)$, that is, if $f(z)$ is given by (1.1) and $g(z)$ is given by $g(z) = z + \sum\limits_{n=2}^{\infty} b_n z^n$, is defined by

$$(f*g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g*f)(z).$$

For positive real parameters $\alpha_1, ..., \alpha_q$ and $\beta_1, ..., \beta_s$ ($\beta_j \in \mathbb{C}\backslash\mathbb{Z}_0^-, \mathbb{Z}_0^- = 0, -1, ...; j$

$= 1, 2, ..., s)$, the generalized hypergeometric function ${}_qF_s(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s;\ z)$ is defined by

$$ {}_qF_s(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n...(\alpha_q)_n}{(\beta_1)_n...(\beta_s)_n n!} z^n $$

$$ (q \leq s+1; s, q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mathbb{N} = \{1, 2, ...\};\ z \in U), $$

where $(\theta)_n$, is the Pochhammer symbol defined in terms of the Gamma function Γ, by

$$ (\theta)_n = \frac{\Gamma(\theta+n)}{\Gamma(\theta)} = \begin{cases} 1 & (n=0) \\ \theta(\theta+1)....(\theta+n-1) & (n \in \mathbb{N}). \end{cases} $$

For the function $h(\alpha_1, ..., \alpha_q; \beta_1, ...\beta_s; z) = z_qF_s(\alpha_1, ..., \alpha_q;\ \beta_1, ..., \beta_s;\ z)$, the Dziok-Srivastava linear operator (see [5] and [6]) $H_{q,s}(\alpha_1, ..., \alpha_q;\ \beta_1, ..., \beta_s) : A \longrightarrow A$, is defined by the Hadamard product as follows:

$$ \begin{aligned} H_{q,s}(\alpha_1, ..., \alpha_q;\ \beta_1, ..., \beta_s)f(z) &= h(\alpha_1, ..., \alpha_q; \beta_1, ...\beta_s; z) * f(z) \\ &= z + \sum_{n=2}^{\infty} \Psi_n(\alpha_1)\, a_n z^n \quad (z \in U), \end{aligned} \tag{1.2} $$

where

$$ \Psi_n(\alpha_1) = \frac{(\alpha_1)_{n-1}.......(\alpha_q)_{n-1}}{(\beta_1)_{n-1}...(\beta_s)_{n-1}\ (n-1)!}. \tag{1.3} $$

For brevity, we write

$$ H_{q,s}(\alpha_1, ..., \alpha_q;\ \beta_1, ..., \beta_s; z)f(z) = H_{q,s}(\alpha_1)f(z). \tag{1.4} $$

For $0 \leq \alpha < 1, \beta \geq 0$ and for all $z \in U$, let $US_{q,s}([\alpha_1]; \alpha, \beta)$ denote the subclass of A consisting of functions $f(z)$ of the form (1.1) and satisfying the analytic criterion

$$ \operatorname{Re}\left\{ \frac{H_{q,s}(\alpha_1)f(z)}{z(H_{q,s}(\alpha_1)f(z))'} - \alpha \right\} > \beta \left| \frac{H_{q,s}(\alpha_1)f(z)}{z(H_{q,s}(\alpha_1)f(z))'} - 1 \right|. \tag{1.5} $$

We note that for suitable choices of q, s, α and β, we obtain the following subclasses studied by various authors.

(1) Putting $q = 2, s = 1, \alpha_1 = a\,(a > 0), \alpha_2 = 1$ *and* $\beta_1 = c\,(c > 0)$ in (1.5), the class $US_{2,1}([a, 1; c]; \alpha, \beta)$ reduces to the class $\mathcal{LS}(a, c; \alpha, \beta)$

$$= \left\{ f \in A : \mathrm{Re}\left\{ \frac{L(a,c)f(z)}{z(L(a,c)f(z))'} - \alpha \right\} > \beta \left| \frac{L(a,c)f(z)}{z(L(a,c)f(z))'} - 1 \right|, \ 0 \leq \alpha < 1, \beta \geq 0, a > 0, c > 0, z \in U \right\},$$

where $L(a,c)$ is the Carlson - Shaffer operator (see [2]);

(2) Putting $q = 2, s = 1, \alpha_1 = v + 1\,(v > -1)\,, \alpha_2 = 1$ *and* $\beta_1 = v + 2$ in (1.5), the class $US_{2,1}([v+1, 1; v+2]\,; \alpha, \beta)$ reduces to the class $\zeta S(v; \alpha, \beta)$

$$= \left\{ f \in A : \mathrm{Re}\left\{ \frac{J_v f(z)}{z(J_v f(z))'} - \alpha \right\} > \beta \left| \frac{J_v f(z)}{z(J_v f(z))'} - 1 \right|, \ 0 \leq \alpha < 1, \beta \geq 0, v > -1, z \in U \right\},$$

where $J_v f(z)$ is the generalized Bernardi - Libera - Livingston operator (see [1], [8] and [9]);

(3) Putting $q = 2, s = 1, \alpha_1 = 2, \alpha_2 = 1$ *and* $\beta_1 = 2 - \mu\,(\mu \neq 2, 3, ...)$ in (1.5), the class $US_{2,1}([2, 1; 2-\mu]; \alpha, \beta)$ reduces to the class $\mathcal{F}S(\mu; \alpha, \beta)$

$$= \left\{ f \in A : \mathrm{Re}\left\{ \frac{\Omega_z^\mu f(z)}{z(\Omega_z^\mu f(z))'} - \alpha \right\} > \beta \left| \frac{\Omega_z^\mu f(z)}{z(\Omega_z^\mu f(z))'} - 1 \right|, \ 0 \leq \alpha < 1, \beta \geq 0, \mu \neq 2, 3, ..., z \in U \right\},$$

where $\Omega_z^\mu f(z)$ is the Srivastava - Owa fractional derivative operator (see [12] and [13]);

(4) Putting $q = 2, s = 1, \alpha_1 = \mu\,(\mu > 0)\,, \alpha_2 = 1$ *and* $\beta_1 = \lambda + 1\,(\lambda > -1)$ in (1.5), the class $US_{2,1}([\mu, 1; \lambda+1]; \alpha, \beta)$ reduces to the class $\pounds S(\mu, \lambda; \alpha, \beta)$

$$= \left\{ f \in A : \mathrm{Re}\left\{ \frac{I_{\lambda,\mu} f(z)}{z(I_{\lambda,\mu} f(z))'} - \alpha \right\} > \beta \left| \frac{I_{\lambda,\mu} f(z)}{z(I_{\lambda,\mu} f(z))'} - 1 \right|, \ -1 \leq \alpha < 1, \beta \geq 0, \mu > 0, \lambda > -1, z \in U \right\},$$

where $I_{\lambda,\mu} f(z)$ is the Choi-Saigo-Srivastava operator (see [4]);

(5) Putting $q = 2, s = 1, \alpha_1 = 2, \alpha_2 = 1$ *and* $\beta_1 = k + 1\,(k > -1)$ in (1.5), the class $US_{2,1}([2, 1; k+1]; \alpha, \beta)$ reduces to the class $AS(k; \alpha, \beta)$

$$= \left\{ f \in A : \mathrm{Re}\left\{ \frac{I_k f(z)}{z(I_k f(z))'} - \alpha \right\} > \beta \left| \frac{I_k f(z)}{z(I_k f(z))'} - 1 \right|, \ 0 \leq \alpha < 1, \beta \geq 0, k > -1, z \in U \right\},$$

where $I_k f(z)$ is the Noor integral operator (see [11]);

(6) Putting $q = 2, s = 1, \alpha_1 = c\,(c > 0)\,, \alpha_2 = \lambda + 1\,(\lambda > -1)$ *and* $\beta_1 = a\,(a > 0)$ in (1.5), the class $US_{2,1}([c, \lambda + 1; a]\,; \alpha, \beta)$ reduces to the class $FS(c, a, \lambda; \alpha, \beta)$

$$= \left\{ f \in A : \operatorname{Re}\left\{ \frac{I^{\lambda}(a,c)f(z)}{z(I^{\lambda}(a,c)f(z))'} - \alpha \right\} > \beta \left| \frac{I^{\lambda}(a,c)f(z)}{z(I^{\lambda}(a,c)f(z))'} - 1 \right|,\ 0 \leq \alpha < 1, \beta \geq 0, a > 0, c > 0, \lambda > -1, z \in U \right\},$$

where $I^{\lambda}(a,c)f(z)$ is the Cho-Kwon-Srivastava operator (see [3]).

Denote by T the subclass of A consisting of functions of the form:

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \ (a_n \geq 0), \tag{1.6}$$

which are analytic in U. We define the class $UT_{q,s}([\alpha_1]\,; \alpha, \beta)$ by:

$$UT_{q,s}([\alpha_1]\,; \alpha, \beta) = US_{q,s}([\alpha_1]\,; \alpha, \beta) \cap T. \tag{1.7}$$

We note that for suitable choices of q, s, α and β, we obtain the following subclasses studied by various authors.

(1) Putting $q = 2$ *and* $s = \alpha_1 = \alpha_2 = \beta_1 = 1$ in (1.5), the class $UT_{2,1}([1]\,; \alpha, \beta)$ reduces to the class $ST(\alpha, \beta)$

$$= \left\{ f \in T : \operatorname{Re}\left\{ \frac{f(z)}{zf'(z)} - \alpha \right\} > \beta \left| \frac{f(z)}{zf'(z)} - 1 \right|,\ 0 \leq \alpha < 1, \beta \geq 0, z \in U \right\}$$

and the class $ST(\alpha, 0) = ST(\alpha)$ is the family of functions $f(z) \in T$ which satisfy the following condition (see [7] and [18])

$$ST(\alpha) = \operatorname{Re}\left\{ \frac{f(z)}{zf'(z)} \right\} > \alpha \quad (0 \leq \alpha < 1);$$

(2) Putting $q = 2, s = 1, \alpha_1 = \lambda + 1 (\lambda > -1)$ *and* $\alpha_2 = \beta_1 = 1$ in (1.5), the class $UT_{2,1}([\lambda + 1]; \alpha, \beta)$ reduces to the class $W_{\lambda}(\alpha, \beta)$

$$= \left\{ f \in T : \operatorname{Re}\left\{ \frac{D^{\lambda}f(z)}{z(D^{\lambda}f(z))'} - \alpha \right\} > \beta \left| \frac{D^{\lambda}f(z)}{z(D^{\lambda}f(z))'} - 1 \right|,\ 0 \leq \alpha < 1, \beta \geq 0, \lambda > -1, z \in U \right\} \ (\ \textit{see}\ [10]\ ),$$

where $D^{\lambda}(\lambda > -1)$ is the Ruscheweyh derivative operator (see [14]);

In this paper, applying methods used by Silverman [15] and Silvia [16], we investigate the ratio of a function of the form (1.6) to its sequence of partial sums $f_m(z) = z + \sum\limits_{n=2}^{m} a_n\, z^n$. More precisely, we will determine sharp lower bounds for $\mathrm{Re}\left\{\dfrac{f(z)}{f_m(z)}\right\}$, $\mathrm{Re}\left\{\dfrac{f_m(z)}{f(z)}\right\}$, $\mathrm{Re}\left\{\dfrac{f'(z)}{f'_m(z)}\right\}$ and $\mathrm{Re}\left\{\dfrac{f'_m(z)}{f'(z)}\right\}$. In the sequel, we will make use of the well-known result that $\mathrm{Re}\left\{\dfrac{1+w(z)}{1-w(z)}\right\} > 0\,(z \in U)$ if and only if $w(z) = \sum\limits_{n=1}^{\infty} c_n z^n$ satisfies the inquality $|w(z)| \leq |z|$.

2. Coefficient estimates

Unless otherwise mentioned, we shall assume in the reminder of this paper that, the parameters $\alpha_1, ..., \alpha_q$ and $\beta_1, ..., \beta_s$ are positive real numbers, $-1 \leq \alpha < 1,\ \beta \geq 0,\ n \geq 2,\ z \in U$ *and* $\Psi_n(\alpha_1)$ is defined by (1.3).

Using the technique used by Yamakawa [18, Lemma 2.1] we prove the following theorem:

Theorem 1. *A function $f(z)$ of the form (1.1) is in the class $US_{q,s}([\alpha_1]; \alpha, \beta)$ if*

$$\sum_{n=2}^{\infty} [2n - n(\alpha-\beta) - (\beta+1)]\Psi_n(\alpha_1)\,|a_n| \leq 1-\alpha. \tag{2.1}$$

Proof. Suppose that (2.1) is true. Since

$$\frac{[2n - n(\alpha-\beta) - (\beta+1)]\Psi_n(\alpha_1)}{1-\alpha} - n\Psi_n(\alpha_1) = \frac{(n-1)(1+\beta)\Psi_n(\alpha_1)}{1-\alpha} > 0$$

we deduce

$$\sum_{n=2}^{\infty} n\Psi_n(\alpha_1)\,|a_n| < \sum_{n=2}^{\infty} \frac{[2n - n(\alpha-\beta) - (\beta+1)]\Psi_n(\alpha_1)}{1-\alpha}\,|a_n| \leq 1.$$

It suffices to show that

$$\beta\left|\frac{H_{q,s}(\alpha_1)f(z)}{z(H_{q,s}(\alpha_1)f(z))'} - 1\right| - \mathrm{Re}\left(\frac{H_{q,s}(\alpha_1)f(z)}{z(H_{q,s}(\alpha_1)f(z))'} - 1\right) \leq 1-\alpha,$$

we have

$$\begin{aligned}
&\beta\left|\frac{H_{q,s}(\alpha_1)f(z)}{z(H_{q,s}(\alpha_1)f(z))'}-1\right|-\operatorname{Re}\left(\frac{H_{q,s}(\alpha_1)f(z)}{z(H_{q,s}(\alpha_1)f(z))'}-1\right)\\
&\leq (1+\beta)\left|\frac{H_{q,s}(\alpha_1)f(z)}{z(H_{q,s}(\alpha_1)f(z))'}-1\right|\\
&\leq \frac{(1+\beta)\sum\limits_{n=2}^{\infty}(n-1)\Psi_n(\alpha_1)|a_n|}{1-\sum\limits_{n=2}^{\infty}n\Psi_n(\alpha_1)|a_n|}.
\end{aligned}$$

which yields

$$\begin{aligned}
&(1-\alpha)-(1+\beta)\left|\frac{H_{q,s}(\alpha_1)f(z)}{z(H_{q,s}(\alpha_1)f(z))'}-1\right|\\
&> \frac{(1-\alpha)-\sum\limits_{n=2}^{\infty}[2n-n(\alpha-\beta)-(\beta+1)]\Psi_n(\alpha_1)|a_n|}{1-\sum\limits_{n=2}^{\infty}n\Psi_n(\alpha_1)|a_n|}\geq 0.
\end{aligned}$$

This completes the proof of Theorem 1.

Unfortunately, the converse of the above Theorem 1 is not true. So we define the subclass $G_{q,s}([\alpha_1];\alpha,\beta)$ of $US_{q,s}([\alpha_1];\alpha,\beta)$ consisting of functions $f(z)$ which satisfy (2.1).

3. Partial sums

Thorem 2. *If f of the form (1.1) satisfies the condition (2.1) and* $\frac{f(z)}{z}\neq 0$ $(0<|z|<1)$, *then*

$$\operatorname{Re}\left\{\frac{f(z)}{f_m(z)}\right\}\geq 1-\frac{1}{C_{m+1}} \tag{3.1}$$

and

$$C_n\geq\begin{cases}1 & n=2,3,..............,m\\ C_{m+1} & n=m+1,m+2,...........\end{cases}, \tag{3.2}$$

where

$$C_n=\frac{[2n-n(\alpha-\beta)-(\beta+1)]\Psi_n(\alpha_1)}{(1-\alpha)}. \tag{3.3}$$

The result in (3.1) is sharp for every m, with the extremal function

$$f(z) = z + \frac{z^{m+1}}{C_{m+1}}. \tag{3.4}$$

Proof. We may write

$$\begin{aligned} \frac{1+w(z)}{1-w(z)} &= C_{m+1}\left\{\frac{f(z)}{f_m(z)} - (1 - \frac{1}{C_{m+1}})\right\} \\ &= \left\{\frac{1+\sum\limits_{n=2}^{m} a_n z^{n-1} + C_{m+1}\sum\limits_{n=m+1}^{\infty} a_n z^{n-1}}{1+\sum\limits_{n=2}^{m} a_n z^{n-1}}\right\}. \end{aligned} \tag{3.5}$$

Then

$$w(z) = \frac{C_{m+1}\sum\limits_{n=m+1}^{\infty} a_n z^{n-1}}{2+2\sum\limits_{n=2}^{m} a_n z^{n-1} + C_{m+1}\sum\limits_{n=m+1}^{\infty} a_n z^{n-1}}$$

and

$$|w(z)| \leq \frac{C_{m+1}\sum\limits_{n=m+1}^{\infty} |a_n|}{2-2\sum\limits_{n=2}^{m} |a_n| - C_{m+1}\sum\limits_{n=m+1}^{\infty} |a_n|}.$$

Now $|w(z)| \leq 1$ if

$$2C_{m+1}\sum_{n=m+1}^{\infty} |a_n| \leq 2 - 2\sum_{n=2}^{m} |a_n|,$$

which is equivalent to

$$\sum_{n=2}^{m} |a_n| + C_{m+1}\sum_{n=m+1}^{\infty} |a_n| \leq 1. \tag{3.6}$$

It is suffices to show that the left hand side of (3.6) is bounded above by $\sum\limits_{n=2}^{\infty} C_n |a_n|$,which is equivalent to

$$\sum_{n=2}^{m} (C_n - 1)|a_n| + \sum_{n=m+1}^{\infty} (C_n - C_{m+1})|a_n| \geq 0.$$

To see that the function f given by (3.4) gives the sharp result, we observe for $z = re^{i\pi/n}$ that

$$\frac{f(z)}{f_m(z)} = 1 + \frac{z^m}{C_{m+1}}. \tag{3.7}$$

Letting $z \longrightarrow 1^-$, we have

$$\frac{f(z)}{f_m(z)} = 1 - \frac{1}{C_{m+1}}.$$

This completes the proof of Theorem 2.

Thorem 3. *If f of the form (1.1) satisfies the condition (2.1) and $\frac{f(z)}{z} \neq 0$ $(0 < |z| < 1)$, then*

$$\operatorname{Re}\left\{\frac{f_m(z)}{f(z)}\right\} \geq \frac{C_{m+1}}{1 + C_{m+1}}.$$

The result is sharp for every m, with the extremal function $f(z)$ given by (3.4).

Proof. We may write

$$\begin{aligned} \frac{1+w(z)}{1-w(z)} &= (1 + C_{m+1})\left\{\frac{f_m(z)}{f(z)} - \frac{C_{m+1}}{1+C_{m+1}}\right\} \\ &= \left\{\frac{1 + \sum\limits_{n=2}^{m} a_n z^{n-1} - C_{m+1}\sum\limits_{n=m+1}^{\infty} a_n z^{n-1}}{1 + \sum\limits_{n=2}^{\infty} a_n z^{n-1}}\right\}, \end{aligned} \tag{3.8}$$

where

$$w(z) = \frac{(1 + C_{m+1})\sum\limits_{n=m+1}^{\infty} a_n z^{n-1}}{-\left(2 + 2\sum\limits_{n=2}^{m} a_n z^{n-1} - (1 - C_{m+1})\sum\limits_{n=m+1}^{\infty} a_n z^{n-1}\right)},$$

and

$$|w(z)| \leq \frac{(1 + C_{m+1})\sum\limits_{n=m+1}^{\infty} |a_n|}{2 - 2\sum\limits_{n=2}^{m} |a_n| + (1 - C_{m+1})\sum\limits_{n=m+1}^{\infty} |a_n|}.$$

Now $|w(z)| \leq 1$ if and only if

$$2C_{m+1} \sum_{n=m+1}^{\infty} |a_n| \leq 2 - 2 \sum_{n=2}^{m} |a_n| ,$$

which is equivalent to

$$\sum_{n=2}^{m} |a_n| + C_{m+1} \sum_{n=m+1}^{\infty} |a_n| \leq 1. \tag{3.9}$$

It is suffices to show that the left hand side of (3.9) is bounded above by $\sum_{n=2}^{\infty} C_n |a_n|$,which is equivalent to

$$\sum_{n=2}^{m} (C_n - 1) |a_n| + \sum_{n=m+1}^{\infty} (C_n - C_{m+1}) |a_n| \geq 0.$$

This completes the proof of Theorem 3.

Putting $q = 2, s = 1, \alpha_1 = a(a > 0), \alpha_2 = 1$ and $\beta_1 = c(c > 0)$, in Theorems 2 and 3, respectively, we obtain the following corollary.

Corollary 1. *If f of the form (1.1) satisfies the condition (2.1) and* $\frac{f(z)}{z} \neq 0$ $(0 < |z| < 1)$, *then*

$$\operatorname{Re}\left\{\frac{f(z)}{f_m(z)}\right\} \geq \frac{[2m - (m+1)(\alpha - \beta) - (\beta - 1)](a)_m - (c)_m(1 - \alpha)}{[2m - (m+1)(\alpha - \beta) - (\beta - 1)](a)_m} \tag{3.10}$$

and

$$\operatorname{Re}\left\{\frac{f_m(z)}{f(z)}\right\} \geq \frac{[2m - (m+1)(\alpha - \beta) - (\beta - 1)](a)_m}{(c)_m(1 - \alpha) + [2m - (m+1)(\alpha - \beta) - (\beta - 1)](a)_m}. \tag{3.11}$$

Theorem 4. *If f of the form (1.1) satisfies the condition (2.1) and* $\frac{f(z)}{z} \neq 0$ $(0 < |z| < 1)$, *then*

$$(a) \operatorname{Re}\left\{\frac{f'(z)}{f'_m(z)}\right\} \geq 1 - \frac{m+1}{C_{m+1}} \tag{3.12}$$

and

$$(b) \operatorname{Re}\left\{\frac{f'_m(z)}{f'(z)}\right\} \geq \frac{C_{m+1}}{1 + m + C_{m+1}}, \tag{3.13}$$

where

$$C_n \geq \left\{ \begin{array}{ll} 1 & n = 1, 2, 3, ..., m \\ n\dfrac{C_{m+1}}{m+1} & n = m+1, m+2, ... \end{array} \right\}$$

and C_n is defined by(3.3).The estimates in (3.12) and (3.13) are sharp with the extremal function given by (3.4).

Proof. We prove only (a), which is similar in spirit of the proof of Theorem 2. The proof of (b) follows the pattern of that in Theorem 3. We write

$$\begin{aligned} \frac{1+w(z)}{1-w(z)} &= C_{m+1}\left\{\frac{f'(z)}{f'_m(z)} - (1 - \frac{1+m}{C_{m+1}})\right\} \\ &= \left\{\frac{1 + \sum\limits_{n=2}^{m} na_n z^{n-1} + \frac{C_{m+1}}{m+1}\sum\limits_{n=m+1}^{\infty} na_n z^{n-1}}{1 + \sum\limits_{n=2}^{m} na_n z^{n-1}}\right\}, \end{aligned}$$

where

$$w(z) = \frac{\frac{C_{m+1}}{m+1}\sum\limits_{n=m+1}^{\infty} na_n z^{n-1}}{2 + 2\sum\limits_{n=2}^{m} na_n z^{n-1} + \frac{C_{m+1}}{m+1}\sum\limits_{n=m+1}^{\infty} na_n z^{n-1}}$$

and

$$|w(z)| \leq \frac{\frac{C_{m+1}}{m+1}\sum\limits_{n=m+1}^{\infty} n\,|a_n|}{2 - 2\sum\limits_{n=2}^{m} n\,|a_n| - \frac{C_{m+1}}{m+1}\sum\limits_{n=m+1}^{\infty} n\,|a_n|}.$$

Now $|w(z)| \leq 1$ if and only if

$$\sum_{n=2}^{m} n\,|a_n| + \frac{C_{m+1}}{m+1}\sum_{n=m+1}^{\infty} n\,|a_n| \leq 1, \tag{3.14}$$

since the left hand side of (3.14) is bounded above by $\sum\limits_{n=2}^{\infty} C_n\,|a_n|$, this completes the proof of Theorem 4.

Putting $q = 2, s = 1, \alpha_1 = a(a > 0), \alpha_2 = 1$ and $\beta_1 = c(c > 0)$, in Theorem 4, we obtain the following corollary.

Corollary 2. If f *of the form (1.1) satisfies the condition (2.1) and* $\frac{f(z)}{z} \neq 0$ $(0 < |z| < 1)$, then

$$\operatorname{Re}\left\{\frac{f'(z)}{f'_m(z)}\right\} \geq \frac{[2m-(m+1)(\alpha-\beta)-(\beta-1)](a)_m-(m+1)(c)_m(1-\alpha)}{[2m-(m+1)(\alpha-\beta)-(\beta-1)](a)_m} \tag{3.15}$$

and

$$\operatorname{Re}\left\{\frac{f'_m(z)}{f'(z)}\right\} \geq \frac{[2m-(m+1)(\alpha-\beta)-(\beta-1)](a)_m}{(m+1)(c)_m(1-\alpha)+[2m-(m+1)(\alpha-\beta)-(\beta-1)](a)_m}. \tag{3.16}$$

Remark. Specializing the parameters $q, s, \alpha_1, ..., \alpha_q$ and $\beta_1, ..., \beta_s$, in Theorems 2, 3 and 4, respectively, we obtain the corresponding results for the corresponding classes defined in the introduction.

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