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ON A CLASS OF SUMMATION INTEGRAL TYPE OPERATORS

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Abstract. In this paper we study a mixed summation-integral type of linear positive operators that approximate certain functions defined on $[0,\infty)$. The pointwise rate of convergence is obtained. In our construction particular cases are outlined.

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1. Introduction

The approximation of functions by using linear positive operators is currently under intensive research. In the following we focus on approximate real valued functions defined on the unbounded interval $\mathbb{R}_+ = [0,\infty)$. Set $\mathbb{N}_0 = \mathbb{N}\cup\{0\}$. Usually, two types of approximation linear positive operators are used: the discrete respectively continuous form.

In the first case, they are often designed as follows

$$(\Lambda_n f)(x) = \sum_{k=0}^{\infty} a_{n,k}(x) f(x_{n,k}),\ n\in\mathbb{N},\ x\in\mathbb{R}_+, \tag{1}$$

where $a_{n,k}\in C(\mathbb{R}_+)$ for each $k\in\mathbb{N}_0$ and $(x_{n,k})_{k\geq 0}$ represents a net on $\mathbb{R}_+$. Classic examples in this direction are Szász-Mirakjan operators, where

$$a_{n,k}(x)\equiv s_{n,k}(x) = e^{-nx}\frac{(nx)^k}{k!}, \tag{2}$$

respectively Baskakov operators, where

$$a_{n,k}(x)\equiv v_{n,k}(x) = \frac{1}{(1+x)^n}\binom{n+k-1}{k}\left(\frac{x}{1+x}\right)^k. \tag{3}$$

In both cases $x_{n,k} = k/n$, $k \in \mathbb{N}_0$.

In the second case, can be used so-called summation integral operators in Durrmeyer sense which are expressed by formulas of the type

$$(\Lambda_n^* f)(x) = \sum_{k=0}^{\infty} a_{n,k}(x) \int_0^{\infty} a_{n,k}(t) f(t) dt, \ n \in \mathbb{N}, \ x \in \mathbb{R}_+.$$

In other words, they are integral generalizations of the operators defined by (1). For example, the Durrmeyer generalization of Szász-Mirakjan operators has been obtained by S.M. Mazhar and V. Totik [7]. For similar generalization of Baskakov operators, see [8]. These kind of generalizations have their inspiration in the work of J.L. Durrmeyer [2].

Starting from these constructions, lately appeared mixed operators which use different basis functions. Using the notation introduced at (2) and (3) we indicate some examples.

(i) $$(L_n f)(x) = n \sum_{\nu=0}^{\infty} v_{n,\nu}(x) \int_0^{\infty} s_{n,\nu}(t) f(t) dt, \ n \in \mathbb{N}, \ x \in \mathbb{R}_+,$$

for $f \in L_p(\mathbb{R}_+)$, $p \geq 1$, see Gupta and Srivastava [6].

(ii) $$(L_n f)(x) = (n-1) \sum_{\nu=1}^{\infty} s_{n,\nu}(x) \int_0^{\infty} v_{n,\nu-1}(t) f(t) dt + e^{-nx} f(0), \tag{4}$$

$n \geq 2, \ x \in \mathbb{R}_+$, where $f \in C_\gamma(\mathbb{R}_+) = \{g \in C(\mathbb{R}_+) : |g(t)| \leq M e^{\gamma t} \text{ for some } M > 0\}$, $\gamma > 0$ fixed, see [4; *Eq. (1.1)*].

(iii) $$(L_n f)(x) = \sum_{\nu=1}^{\infty} \beta_{n,\nu}(x) \int_0^{\infty} s_{n,\nu-1}(t) f(t) dt + n(1+x)^{-n-1} f(0), \tag{5}$$

$n \in \mathbb{N}$, $x \in \mathbb{R}_+$, where $f \in C_\gamma(\mathbb{R}_+)$, see [3] and [5]. Here the weights $\beta_{n,\nu}$ are given by Beta functions as follows

$$\beta_{n,\nu}(x) = \frac{1}{B(n, \nu+1)} \frac{x^\nu}{(1+x)^{n+\nu+1}}, \ \nu \geq 1.$$

It can be observed that the last two classes of operators are discretely defined at the point zero.

Inspired by the previous constructions, mainly the work of Vijay Gupta, the goal of this paper is to introduce a general class of mixed integral type operators. The construction is indicated in Section 2. Our main result is presented in Section 3. It aimed at determining an upper bound for the error of approximation by using the smoothness modulus.

2. The operators

Let $(a_{n,k})_{k\geq 0}$, $(b_{n,k})_{k\geq 0}$ be two sequences of continuous and positive functions defined on $\mathbb{R}_+$ such that the following relations hold

$$\sum_{k=0}^{\infty} a_{n,k} = \mathbf{1}, \quad \sum_{k=0}^{\infty} b_{n,k} = \mathbf{1}, \quad \int_0^{\infty} b_{n,k}(t)dt := c_{n,k} < \infty, \tag{6}$$

where $\mathbf{1}$ denotes the constant function on $\mathbb{R}_+$ of constant value 1.

For each $n \in \mathbb{N}$, we define the operator

$$(V_n f)(x) = f(0)a_{n,0}(x) + \sum_{k=1}^{\infty} \frac{a_{n,k}(x)}{c_{n,k}} \int_0^{\infty} b_{n,k}(t)f(t)dt, \ x \in \mathbb{R}_+, \tag{7}$$

where $f \in \mathcal{F}(\mathbb{R}_+)$, this space consisting of all real valued functions f defined on $\mathbb{R}_+$ with the properties $b_{n,k}f$ belongs to the Lebesgue space $L_1(\mathbb{R}_+)$ for each $k \in \mathbb{N}$ and the series from the right hand side of relation (7) is convergent. Denoting by $C_B(\mathbb{R}_+)$ the space of all continuous and bounded functions defined on $\mathbb{R}_+$ it is evident that $C_B(\mathbb{R}_+) \subset \mathcal{F}(\mathbb{R}_+)$.

Remark. Examining relation (7), we deduce that V_n, $n \in \mathbb{N}$, are linear and positive operators. Moreover,

$$(V_n\mathbf{1})(x) = \sum_{k=0}^{\infty} a_{n,k}(x) = 1, \tag{8}$$

consequently the operators reproduce the constant functions.

Among the particular cases included by this construction, we mention the following.

For $n \geq 2$, choosing $a_{n,k} = s_{n,k}$ and $b_{n,k} = v_{n,k-1}$ (see (2) and (3)) we get $c_{n,k} = (n-1)^{-1}$ and we reobtain the operators defined by (4).

Choosing $a_{n,k} = \beta_{n,k}$ and $b_{n,k} = s_{n,k-1}$ we get $c_{n,k} = n^{-1}$ and we reobtain the operators defined by (5).

Set e_j, $j \in \mathbb{N}$, the monomial of degree j, i.e., $e_j(t) = t^j$, $t \geq 0$. Also, we can write $\mathbf{1} = e_0$.

As regards our operators V_n, $n \in \mathbb{N}$, we ask additional conditions to be fulfilled, more exactly, we impose that the polynomials of first and respectively second degree to be transformed into polynomials of first respectively second degree which vanish at the origin. This means

$$(V_n e_1)(x) = (1+\alpha_n)x \quad \text{and} \quad (V_n e_2)(x) = (1+\beta_n)x^2 + \gamma_n x, \ x \in \mathbb{R}_+. \tag{9}$$

In accordance with the well known Bohman-Korovkin criterion, if one has

$$\lim_n \alpha_n = \lim_n \beta_n = \lim_n \gamma_n = 0 \tag{10}$$

then, on any compact $K \subset \mathbb{R}_+$,

$$\lim_{n\to\infty} \|V_n f - f\|_K = 0 \tag{11}$$

takes place, where $\|h\|_K = \sup\limits_{x\in K} |h(x)|$, $h \in C(\mathbb{R}_+)$. In other words, conditions (9) and (10) guarantee that $(V_n)_{n\geq 1}$ becomes an approximation process on any compact $K \subset \mathbb{R}_+$. For more details [1; *Section 4.2*] can be consulted.

Our concern is to establish the error of approximation of this approximation sequence of linear positive operators.

Set $\varphi_x(t) = t - x$ for any $(t, x) \in \mathbb{R}_+ \times \mathbb{R}_+$. By a straightforward computation, the linearity of our operators and relations (8), (9) imply

$$0 \leq (V_n \varphi_x^2)(x) = (\beta_n - 2\alpha_n)x^2 + \gamma_n x. \tag{12}$$

At this moment we recall the notion of modulus of continuity associated to a function $f \in C(\mathbb{R}_+)$ on the compact interval $[0, a]$, where $a > 0$ is fixed. It is denoted by $\omega(f; \cdot)_{[0,a]}$ and is defined as follows

$$\omega(f;\delta)_{[0,a]} = \sup\{|f(t) - f(x)| : |t - x| \leq \delta,\ t, x \in [0, a]\},\ \delta \geq 0.$$

Among the properties of $\omega(f; \cdot)_{[0,a]}$, also called the first modulus of smoothness of f, we mention: it is a non-decreasing function and for every $\delta > 0$ one has

$$\omega(f; |t - x|)_{[0,a]} \leq \left(1 + \frac{(t - x)^2}{\delta^2}\right) \omega(f; \delta)_{[0,a]},\ (t, x) \in [0, a] \times [0, a]. \tag{13}$$

3. Main result

We consider the following space of functions

$$E_2(\mathbb{R}_+) = \{f \in C(\mathbb{R}_+) :\ |f(x)| \leq M(1 + x^2) \text{ for some } M > 0\}.$$

Lemma 1. *Let $\tau > 0$ be fixed. For every $f \in E_2(\mathbb{R}_+)$, $x \in [0, \tau]$ and $t \in \mathbb{R}_+$ one has*

$$|f(t) - f(x)| \leq M_\tau (t - x)^2 + \left(1 + \frac{(t - x)^2}{\delta^2}\right) \omega(f; \delta)_{[0,\tau+1]},\ \delta > 0, \tag{14}$$

where M_τ is a constant depending only on f and τ.

Proof. Let $x \in [0,\tau]$ arbitrarily fixed. Let $t \in \mathbb{R}_+$.

Case 1. $t \le \tau + 1$, consequently $|t-x| \le \tau+1$. By using relation (13) we can write

$$|f(t)-f(x)| \le \sup_{\substack{u,v\in[0,\tau+1]\\ |u-v|\le|t-x|}} |f(u)-f(v)| = \omega(f;|t-x|)_{[0,\tau+1]}$$
$$\le \left(1+\frac{(t-x)^2}{\delta^2}\right)\omega(f;\delta)_{[0,\tau+1]}.$$

Case 2. $t > \tau+1$, consequently $t-x>1$. Since $f \in E_2(\mathbb{R}_+)$ we get

$$\begin{aligned}|f(t)-f(x)| &\le M(2+x^2+t^2)\\ &= M((2+2x^2)+2x(t-x)+(t-x)^2)\\ &\le M((2+2x^2)(t-x)^2+2x(t-x)^2+(t-x)^2)\\ &\le M(\sup_{x\in[0,\tau]}(3+2x+2x^2))(t-x)^2\\ &= M_\tau(t-x)^2.\end{aligned}$$

Combining the above two cases, we obtain the desired result. □

The rate of convergence of the operators $V_n f$, $n \in \mathbb{N}$, to f for all $f \in E_2(\mathbb{R}_+)$ will be read as follows.

Theorem 1. *Let $\tau > 0$ be fixed. Let V_n, $n \in \mathbb{N}$, be the operators defined by (7) such that (9) takes place. For each $f \in E_2(\mathbb{R}_+)$ the following relation*

$$|(V_n f)(x) - f(x)| \le M_\tau \delta_n^2(x) + 2\omega(f;\delta_n(x))_{[0,\tau+1]},\ x\in[0,\tau], \tag{15}$$

holds, where

$$\delta_n(x) = \sqrt{(\beta_n - 2\alpha_n)x^2 + \gamma_n x} \tag{16}$$

and M_τ is a constant depending only on f and τ.

Proof. Let $n \in \mathbb{N}$, $x \in [0,\tau]$ arbitrarily fixed. Taking in view the following facts: $V_n e_0 = e_0$ (see (8)), V_n is linear and positive operator consequently it is monotone, inequality (14) and identity (12), we can write successively

$$\begin{aligned}|(V_n f)(x) - f(x)| &= |V_n(f - f(x);x)|\\ &\le V_n(|f-f(x)|;x)\\ &\le V_n\left(M_\tau\varphi_x^2 + \left(1+\frac{\varphi_x^2}{\delta^2}\right)\omega(f;\delta)_{[0,\tau+1]};x\right)\\ &= M_\tau((\beta_n-2\alpha_n)x^2+\gamma_n x)\\ &+\left(1+\frac{(\beta_n-2\alpha_n)x^2+\gamma_n x}{\delta^2}\right)\omega(f;\delta)_{[0,\tau+1]}.\end{aligned}$$

Choosing $\delta = \delta_n(x)$, see (16), we arrive at (15). $\square$

The pointwise error approximation given by (16) implies the following global rate of convergence

$$\|V_n f - f\|_{[0,\tau]} \le M_\tau \|\delta_n^2\|_{[0,\tau]} + 2\omega(f; \|\delta_n\|_{[0,\tau]})_{[0,\tau+1]}. \tag{17}$$

Under the assumptions (10), examining (17), clearly $\lim\limits_n \|\delta_n^j\|_{[0,\tau]} = 0$, $j \in \{1,2\}$, and we reobtain the identity (11) for $K = [0,\tau]$.

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