Acta Universitatis Apulensis No. 31/2012
ISSN: 1582-5329 pp. 47-52

SOME SUBCLASSES OF ANALYTIC FUNCTIONS INVOLVING λ-SPIRALLIKENESS OF ORDER α

Oana Crişan and Andreea Elena Tudor

Abstract. In the present paper, using the Sălăgean integral operator, we introduce two subclasses of analytic functions involving λ-spirallikeness of order α and study some inclusion properties.

2000 *Mathematics Subject Classification*: 30C45.

1. Introduction

Let $\mathcal{H}(U)$ denote the class of analytic functions in the unit disc $U = \{z \in \mathbb{C} \ : \ |z| < 1\}$ and $\mathcal{A}$ the class of functions from $\mathcal{H}(U)$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k.$$

For $0 \leq \alpha < 1$ and $|\lambda| < \pi/2$, a function $f \in \mathcal{A}$ is said to be λ-spirallike of order α in U if

$$\operatorname{Re}\left[e^{i\lambda}\frac{zf'(z)}{f(z)}\right] > \alpha\cos\lambda\,, \quad z \in U. \tag{1}$$

We denote by $S^{\lambda}(\alpha)$ the class consisting of functions from $\mathcal{A}$ which are λ-spirallike of order α in U, and $F^{\lambda}(\alpha)$ the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ for which

$$\operatorname{Re}\left[e^{i\lambda}\left(1+\frac{zf''(z)}{f'(z)}\right)\right] > \alpha\cos\lambda\,, \quad z \in U. \tag{2}$$

It follows from (1) and (2) that $f(z) \in F^{\lambda}(\alpha)$ if and only if $zf'(z) \in S^{\lambda}(\alpha)$.

The class $S^{\lambda}(\alpha)$ was considered by Libera in [4] and the class $F^{\lambda}(\alpha)$ was studied by Chichra in [3].

We note that $S^0(\alpha) = S^*(\alpha)$ and $F^0(\alpha) = K(\alpha)$, where $S^*(\alpha)$ and $K(\alpha)$ denote the classes of starlike functions of order α and convex functions of order α, introduced by Robertson in [6] and [7] . Also, $S^\lambda(0) = \hat{S}_\lambda$, where $\hat{S}_\lambda$ represents the class of spirallike functions of type λ, which was introduced by Špaček in [8].

We will next consider the integral operators I^n, defined by Sălăgean in [9] and L_c, introduced by Bernardi in [2].

For $f(z) \in \mathcal{A}$ and $n \in \mathbb{N} = \{0, 1, 2, \dots\}$, let

$$I^0 f(z) = f(z),$$

$$I^1 f(z) = I f(z) = \int_0^z f(t)t^{-1}dt,$$

and

$$I^n f(z) = I(I^{n-1} f(z)). \tag{3}$$

If $f \in \mathcal{A}$, $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, it is easy to show that

$$I^n f(z) = z + \sum_{k=2}^{\infty} \frac{a_k}{k^n} z^k, \quad z \in U,\ n \in \mathbb{N}_0 \tag{4}$$

and

$$z(I^n f(z))' = I^{n-1} f(z), \quad z \in U. \tag{5}$$

For $f(z) \in \mathcal{A}$ and $c \in \mathbb{N}$, we define the integral operator $L_c f(z)$ of the form

$$L_c f(z) = \frac{c+1}{z^c} \int_0^z f(t)t^{c-1}\mathrm{d}t \tag{6}$$

We will now introduce and study some properties of two new classes of analytic function, defined using the operator I^n:

$$S_n^\lambda(\alpha) = \{f \in \mathcal{A} \,|\, I^n f(z) \in S^\lambda(\alpha)\},$$

$$F_n^\lambda(\alpha) = \{f \in \mathcal{A} \,|\, I^n f(z) \in F^\lambda(\alpha)\}.$$

It is easy to see that $f(z) \in F_n^\lambda(\alpha)$ if and only if $zf'(z) \in S_n^\lambda(\alpha)$. Also, $S_0^\lambda(\alpha) = S^\lambda(\alpha)$ and $F_0^\lambda(\alpha) = F^\lambda(\alpha)$.

In order to prove our main theorems, we also need the following notions and results.

If $f(z)$ and $g(z)$ are in $\mathcal{H}(U)$, the function f is said to be subordinate to g if there exists a function w analytic in U, with $w(0) = 0$ and $|w(z)| < 1,\ z \in U$, such that $f(z) = g(w(z)),\ z \in U$. In this case, we write $f \prec g$ or $f(z) \prec g(z),\ z \in U$. If the function g is univalent in U, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subseteq g(U)$.

Lemma 1 ([5])**.** *Let h be a convex function in U with $\mathrm{Re}[\beta h(z)+\delta]>0$, $z\in U$. If q is an analytic function in U such that $q(0)=h(0)$ and $q(z)+\dfrac{zq'(z)}{\beta q(z)+\delta}\prec h(z)$, then $q(z)\prec h(z)$.*

2. Main Results

Theorem 1. *For all $n\in\mathbb{N}$, $S_n^\lambda(\alpha)\subset S_{n+1}^\lambda(\alpha)$, where $\lambda\in(-\pi/2,\pi/2)$ and $\alpha\in[0,1)$.*

Proof. Let $f\in S_n^\lambda(\alpha)$ and let

$$p(z)=\frac{z(I^{n+1}f(z))'}{I^{n+1}f(z)}=\frac{I^nf(z)}{I^{n+1}f(z)}\,. \tag{7}$$

Then $p(z)=1+a_1z+a_2z^2+\ldots$ is analytic in U and $p(z)\neq 0$, for all $z\in U$. Differentiating (13) logarithmically we have

$$\frac{z(I^nf(z))'}{I^nf(z)}-\frac{z(I^{n+1}f(z))'}{I^{n+1}f(z)}=\frac{zp'(z)}{p(z)}\,,$$

or, equivalently,

$$\frac{z(I^nf(z))'}{I^nf(z)}=p(z)+\frac{zp'(z)}{p(z)}\,. \tag{8}$$

Multiplying the last equation by $e^{i\lambda}$ and denoting $e^{i\lambda}p(z)$ by $q(z)$, (8) becomes

$$e^{i\lambda}\frac{z(I^nf(z))'}{I^nf(z)}=q(z)+\frac{zq'(z)}{e^{-i\lambda}q(z)}\,. \tag{9}$$

Because $f\in S_n^\lambda(\alpha)$, from (9) we obtain

$$\mathrm{Re}\left[e^{i\lambda}\frac{z(I^nf(z))'}{I^nf(z)}\right]=\mathrm{Re}\left[q(z)+\frac{zq'(z)}{e^{-i\lambda}q(z)}\right]>\alpha\cos\lambda,\quad z\in U. \tag{10}$$

Let h be a convex function such that $h(0)=q(0)=e^{i\lambda}$ and which transforms the unit disc U into the domain $\{z\in\mathbb{C}\,|\,\mathrm{Re}\,z>\alpha\cos\lambda\}$. Then (10) can be written as

$$q(z)+\frac{zq'(z)}{e^{-i\lambda}q(z)}\prec h(z),\quad z\in U. \tag{11}$$

Because $\lambda\in(-\pi/2,\pi/2)$, we have that $\mathrm{Re}[e^{-i\lambda}h(z)]>0$ for $z\in U$ and so applying Lemma 1 we obtain $q(z)\prec h(z)$. Therefore

$$\mathrm{Re}\left[e^{i\lambda}\frac{z(I^{n+1}f(z))'}{I^{n+1}f(z)}\right]>\alpha\cos\lambda,\quad z\in U. \tag{12}$$

which shows that $f\in S_{n+1}^\lambda(\alpha)$.

□

Theorem 2. *For all $n \in \mathbb{N}$, $F_n^\lambda(\alpha) \subset F_{n+1}^\lambda(\alpha)$, where $\lambda \in (-\pi/2, \pi/2)$ and $\alpha \in [0,1)$.*

Proof. If $f(z) \in F_n^\lambda(\alpha)$, then $I^n f(z) \in F^\lambda(\alpha)$, which is equivalent to $z(I^n f(z))' \in S^\lambda(\alpha)$. But this implies that $I^n(zf'(z)) \in S^\lambda(\alpha)$, or, equivalently, $zf'(z) \in S_n^\lambda(\alpha)$. From Theorem 1, we obtain that $zf'(z) \in S_{n+1}^\lambda(\alpha)$, which shows that $I^{n+1}(zf'(z)) \in S_\lambda(\alpha)$ and so $z(I^{n+1}f(z))' \in S^\lambda(\alpha)$. From this we have $I^{n+1}f(z) \in F^\lambda(\alpha)$ and so $f(z) \in F_{n+1}^\lambda(\alpha)$. □

Theorem 3. *Let $c \in \mathbb{N}$, $\lambda \in (-\pi/2, \pi/2)$, $\alpha \in [0,1)$. If $f(z) \in S_n^\lambda(\alpha)$ then $L_c f(z) \in S_n^\lambda(\alpha)$, for all $z \in U$.*

Proof. Let $f(z) \in S_n^\lambda(\alpha)$ and

$$p(z) = \frac{z\left(I^n L_c f(z)\right)'}{I^n L_c f(z)}\,. \tag{13}$$

Then $p(z) = 1 + a_1 z + a_2 z^2 + \ldots$ is analytic in U and $p(z) \neq 0$, for all $z \in U$. From (6), we obtain the following recurrence:

$$z\left(I^n L_c f(z)\right)' = (c+1) I^n f(z) - c I^n L_c f(z)\,. \tag{14}$$

Using (14) and (6), we get

$$p(z) + c = (c+1)\frac{I^n f(z)}{I^n L_c f(z)}\,. \tag{15}$$

Now, differentiating (15) logarithmically, we obtain

$$\frac{z\left(I^n f(z)\right)'}{I^n f(z)} = p(z) + \frac{zp'(z)}{p(z)+c}\,. \tag{16}$$

Multiplying (16) by $e^{i\lambda}$ and choosing $q(z) = e^{i\lambda} p(z)$, we have

$$e^{i\lambda}\frac{z\left(I^n f(z)\right)'}{I^n f(z)} = q(z) + \frac{zq'(z)}{e^{-i\lambda}q(z)+c}\,. \tag{17}$$

Let $h(z)$ be a convex function with $h(0) = e^{i\lambda}$ which maps the unit disk into the domain $\{z \in \mathbb{C} \,|\, \operatorname{Re} z > \alpha \cos\lambda\}$. In these conditions, from (17), we have

$$q(z) + \frac{zq'(z)}{e^{-i\lambda}q(z)+c} \prec h(z)\,. \tag{18}$$

We know that $\operatorname{Re} c \geq 0$ and $\lambda \in (-\pi/2, \pi/2)$, therefore $\operatorname{Re}\left[e^{-i\lambda}h(z)+c\right] > 0$. From Lemma 1, where $\beta = e^{-i\lambda}$ and $\delta = c$, follows that $q(z) \prec h(z)$, which means that

$$\operatorname{Re}\left[e^{i\lambda}\frac{z\left(I^n L_c f(z)\right)'}{I^n L_c f(z)}\right] > \alpha\cos\lambda\,. \tag{19}$$

$\square$

Theorem 4. *Let $c \in \mathbb{N}$, $\lambda \in (-\pi/2, \pi/2)$, $\alpha \in [0,1)$. If $f(z) \in F_n^\lambda(\alpha)$ then $L_c f(z) \in F_n^\lambda(\alpha)$, for all $z \in U$.*

Proof. If $f(z) \in F_n^\lambda(\alpha)$, then $zf'(z) \in S_n^\lambda(\alpha)$. From Theorem 3, we obtain that $L_c(zf'(z)) \in S_n^\lambda(\alpha)$, which shows that $z(L_c f(z))' \in S_n^\lambda(\alpha)$ and so $L_c f(z) \in F_n^\lambda(\alpha)$. $\square$

Acknowledgment. This work was possible with the financial support of the Sectoral Operation Programme for Human Resources Development 2007-2013, co-financed by the European Social Fund, under the project number POSDRU/107/1.5/ S/76841 with the title ”Modern Doctoral Studies: Internationalization and Interdisciplinarity”.

References

[1] M.K. Aouf, *The Sălăgean integral operator and strongly starlike functions*, Stud. Univ. Babeş-Bolyai Math., 56(2011), 109-115.

[2] S.D. Bernardi, *Convex and starlike univalent functions*, Trans. Amer. Math. Soc., 135(1969), 429-446.

[3] P.N. Chichra, *Regular functions $f(z)$ for which $zf'(z)$ is α-spiral-like*, Proc. Amer. Math. Soc., 49(1975), no. 1, 151-160.

[4] R.J. Libera, *Univalent λ-spiral functions*, Canad. J. Math, 19(1967), 449-456.

[5] S.S. Miller, P.T. Mocanu, *Differential Subordinations: Theory and Applications*, Series on Monographs and Textbooks in Pure and Applied Mathematics, vol. 225, Marcel Dekker, New York, 2000.

[6] M.S. Robertson, *On the theory of univalent functions*, Ann. Math., 37(1936), 374-408.

[7] M.S. Robertson, *Univalent functions $f(z)$ for which $zf'(z)$ is spirallike*, Michigan Math. J., 16(1969), 97-101.

[8] L. Špaček, *Contribution à la théorie des fonctions univalentes*, Časopis Pěst. Mat. , 62(1932), 12-19 (in Czech).

[9] G.Ş. Sălăgean, *Subclasses of univalent functions*, Lecture Notes in Math., Springer Verlang, Berlin, 1013(1983), 362-372.

Oana Crişan
Department of Mathematics
University of Babeş-Bolyai
1 Kogălniceanu Street
400084 Cluj-Napoca, Romania
email:*oana.crisan@ubbcluj.ro*

Andreea Elena Tudor
Department of Mathematics
University of Babeş-Bolyai
1 Kogălniceanu Street
400084 Cluj-Napoca, Romania
email:*andreea.tudor@ubbcluj.ro*