Acta Universitatis Apulensis
ISSN: 1582-5329

No. 31/2012
pp. 129-140

SUBORDINATION RESULTS FOR SUBCLASSES OF ANALYTIC FUNCTIONS DEFINED BY CONVOLUTION

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Abstract. In this paper, we drive several interesting subordination results for subclasses of analytic functions defined by convolution. Also number of interesting applications of the subordination results are considered.

2000 *Mathematics Subject Classification*: 30C45.

1. Introduction

Let A denote the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\phi \in A$ be given by

$$\phi(z) = z + \sum_{k=2}^{\infty} c_k z^k. \tag{1.2}$$

Definition 1 *(Hadamard product or convolution). Given two functions f and ϕ in the class A, where $f(z)$ is given by* (1.1) *and $\phi(z)$ is given by* (1.2)*, the Hadamard product (or convolution) $f* \phi$ of f and ϕ is defined (as usual) by*

$$(f * \phi)(z) = z + \sum_{k=2}^{\infty} a_k c_k z^k = (\phi * f)(z). \tag{it1.3}$$

We also denote by K the class of functions $f(z) \in A$ that are convex in $\mathbb{U}$.

Following Goodman ([10] and [11]), Ronning ([18] and [19]) introduced and studied the following subclasses:
(i) A function $f(z)$ of the form (1.1) is said to be in the class $S_p(\alpha,\beta)$ of $\beta-$uniformly starlike functions if it satisfies the condition:

$$Re\left\{\frac{zf'(z)}{f(z)}-\alpha\right\}>\beta\left|\frac{zf'(z)}{f(z)}-1\right| \quad (z\in\mathbb{U}), \tag{1.4}$$

where $-1\leq\alpha<1$ and $\beta\geq0$.

(ii) A function $f(z)$ of the form (1.1) is said to be in the class $UCV(\alpha,\beta)$ of $\beta-$uniformly convex functions if it satisfies the condition:

$$Re\left\{1+\frac{zf''(z)}{f'(z)}-\alpha\right\}>\beta\left|\frac{zf''(z)}{f'(z)}\right| \quad (z\in\mathbb{U}), \tag{1.5}$$

where $-1\leq\alpha<1$ and $\beta\geq0$.
It follows from (1.4) and (1.5) that

$$f(z)\in UCV(\alpha,\beta)\Leftrightarrow zf'(z)\in S_p(\alpha,\beta). \qquad \text{(it1.6)}$$

For $-1\leq\alpha<1$, $0\leq\lambda\leq1$ and $\beta\geq0$, let $\mathrm{S}(g,\lambda;\alpha,\beta)$ be the subclass of A consisting of functions $f(z)$ of the form (1.1), functions $g(z)$ given by

$$g(z)=z+\sum_{k=2}^{\infty}b_kz^k \ \ (b_k>0), \tag{1.7}$$

and satisfying the analytic criterion:

$$Re\left\{\frac{z(f*g)'(z)}{(1-\lambda)(f*g)(z)+\lambda z(f*g)'(z)}-\alpha\right\}>\beta\left|\frac{z(f*g)'(z)}{(1-\lambda)(f*g)(z)+\lambda z(f*g)'(z)}-1\right|. \tag{1.8}$$

We note that:
(i) $\mathrm{S}(\frac{z}{(1-z)},0;\alpha,0)=\mathrm{S}^*(\alpha)$ and $\mathrm{S}(\frac{z}{(1-z)^2},0;\alpha,0)=C(\alpha)$ (see Robertson [17]);
(ii) $\mathrm{S}(\frac{z}{(1-z)},0;\alpha,1)=\mathrm{S}_p(\alpha)$ and $\mathrm{S}(\frac{z}{(1-z)^2},0;\alpha,1)=UCV(\alpha)$ (see Bharati et al. [4]);
(iii) $\mathrm{S}(\frac{z}{(1-z)},0;\alpha,\beta)=\mathrm{S}_p(\alpha,\beta)$ and $\mathrm{S}(\frac{z}{(1-z)^2},0;\alpha,\beta)=UCV(\alpha,\beta)$ (see Goodman [10], [11] and Ronning [18], [19]);
(iv) $\mathrm{S}(\frac{z}{(1-z)},\lambda;\alpha,\beta)=\mathrm{S}_p(\lambda,\alpha,\beta)$ and $\mathrm{S}(\frac{z}{(1-z)^2},\lambda;\alpha,\beta)=UCV(\lambda,\alpha,\beta)$ (see Murugusundaramoorthy and Magesh [16]);
(v) $\mathrm{S}(z+\sum\limits_{k=2}^{\infty}\frac{(a)_{k-1}}{(c)_{k-1}}z^k,0;\alpha,\beta)=\mathrm{S}(\alpha,\beta)$ $(c\neq0,-1,-2,...)$ (see Murugusundaramoorthy and Magesh [14] and [15]);

(vi) $\mathrm{S}(z+\sum\limits_{k=2}^{\infty}k^{n}z^{k},0;\alpha,\beta)=\mathrm{S}(n,\alpha,\beta)$ $(n\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$,where $\mathbb{N}=\{1,2,...\})$ (see Rosy and Murugusundaramoorthy [20]);
(vii) $\mathrm{S}(z+\sum\limits_{k=2}^{\infty}[1+\delta(k-1)]^{n}z^{k},0;\alpha,\beta)=\mathrm{S}_{\delta}(n,\alpha,\beta)$ $(\delta\geq 0,n\in\mathbb{N}_0)$ (see Aouf and Mostafa [2]).

Also we note that:
(i) $\mathrm{S}(g\ ,\lambda;\alpha,0)=\mathrm{S}(g,\lambda,\alpha)$

$$=\left\{f\in A: Re\left\{\frac{z(f*g)'(z)}{(1-\lambda)(f*g)(z)+\lambda z(f*g)'(z)}\right\}>\alpha\ \ (-1\leq\alpha<1,0\leq\lambda\leq 1,z\in\mathbb{U})\right\};$$

(ii) $\mathrm{S}(z+\sum\limits_{k=2}^{\infty}\Gamma_k(\alpha_1)z^{k},\lambda;\alpha,\beta)=\mathrm{S}_{q,s}(\alpha_i,\beta_j;\lambda,\alpha,\beta)$

$$=\{f\in A: Re\left\{\frac{z\left(H_{q,s}(\alpha_1,\beta_1)f(z)\right)'}{(1-\lambda)H_{q,s}(\alpha_1,\beta_1)f(z)+\lambda z\left(H_{q,s}(\alpha_1,\beta_1)f(z)\right)'}-\alpha\right\}$$
$$>\beta\left|\frac{z\left(H_{q,s}(\alpha_1,\beta_1)f(z)\right)'}{(1-\lambda)H_{q,s}(\alpha_1,\beta_1)f(z)+\lambda z\left(H_{q,s}(\alpha_1,\beta_1)f(z)\right)'}-1\right|\},$$

where $\Gamma_k(\alpha_1)$ is defined by

$$\Gamma_k(\alpha_1)=\frac{(\alpha_1)_{k-1}....(\alpha_q)_{k-1}}{(\beta_1)_{k-1}....(\beta_s)_{k-1}(1)_{k-1}} \tag{1.9}$$

$$(\alpha_i>0,i=1,..,q;\beta_j>0,j=1,..,s;q\leq s+1,q,s\in\mathbb{N}_0,\mathbb{N}_0=\mathbb{N}\cup\{0\},\mathbb{N}=\{1,2,..\}),$$

and the operator $H_{q,s}(\alpha_1,\beta_1)$ was introduced and studied by Dziok and Srivastava (see [7] and [8]), and contains many other operators;

(iii) $\mathrm{S}(z+\sum\limits_{k=2}^{\infty}\left[\frac{\ell+1+\mu(k-1)}{\ell+1}\right]^{m}z^{k},\lambda;\alpha,\beta)=\mathrm{S}(m,\mu,\ell;\alpha,\beta)$

$$=\left\{f\in A: Re\left\{\frac{z(I^{m}(\mu,\ell)f(z))'}{(1-\lambda)I^{m}(\mu,\ell)f(z)+\lambda z(I^{m}(\mu,\ell)f(z))'}-\alpha\right\}>\beta\left|\frac{z(I^{m}(\mu,\ell)f(z))'}{(1-\lambda)I^{m}(\mu,\ell)f(z)+\lambda z(I^{m}(\mu,\ell)f(z))'}-1\right|\right\},$$

where $m\in\mathbb{N}_0$, $\mu,\ell\geq 0$, $z\in\mathbb{U}$ and the operator $I^{m}(\mu,\ell)$ was defined by Cătaş et al. (see [6]), and contains many other operators;

(iv) $\mathrm{S}(z+\sum\limits_{k=2}^{\infty}C_k(b,\mu)z^{k},\lambda;\alpha,\beta)=\mathrm{S}_b^{\mu}(\lambda;\alpha,\beta)$

$$= \left\{ f \in A : Re \left\{ \frac{z\left(J_b^{\mu} f\ (z)\right)'}{(1-\lambda)J_b^{\mu} f\ (z)+\lambda z\left(J_b^{\mu} f\ (z)\right)'} - \alpha \right\} > \beta \left| \frac{z\left(J_b^{\mu} f\ (z)\right)'}{(1-\lambda)J_b^{\mu} f\ (z)+\lambda z\left(J_b^{\mu} f\ (z)\right)'} - 1 \right| \right\},$$

where $C_k(b,\mu)$ is defined by

$$C_k(b,\mu) = \left(\frac{1+b}{k+b}\right)^{\mu} (\mu \in \mathbb{C}, b \in \mathbb{C} \setminus \{\mathbb{Z}_0^-\};\ \mathbb{Z}_0^- = \mathbb{Z} \setminus \mathbb{N}), \tag{1.10}$$

and the operator J_b^{μ} was introduced by Srivastava and Attiya [23], and contains many other operators.

Remark 1. By taking $\lambda = 0$ in the class $\mathrm{S}_b^{\mu}(\lambda;\alpha,\beta)$, we get the class $\mathrm{S}_b^{\mu}(\alpha,\beta)$, which was defined by Murugusundaramoorthy [13].

Definition 2 *(Subordination Principle). For two functions f and ϕ, analytic in $\mathbb{U}$, we say that the function $f(z)$ is subordinate to $\phi(z)$ in $\mathbb{U}$,written $f(z) \prec \phi(z)$, if there exists a Schwarz function $w(z)$, which (by definition) is analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$, such that $f(z) = \phi\ (w(z))$. Indeed it is known that*

$$f(z) \prec \phi(z) \Rightarrow f(0) = \phi(0) \text{ and } f(\mathbb{U}) \subset \phi(\mathbb{U}\ ).$$

Furthermore, if the function ϕ is univalent in $\mathbb{U}$, then we have the following equivalence (see [5] *and* [12]*):*

$$f\ (z) \prec \phi\ (z) \Leftrightarrow f(0) = \phi(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}). \tag{it1.11}$$

Definition 3 *(Subordinating Factor Sequence)* [24]. *A sequence $\{c_k\}_{k=1}^{\infty}$ of complex numbers is said to be a subordinating factor sequence if, whenever f of the form* (1.1) *is analytic, univalent and convex in $\mathbb{U}$, we have*

$$\sum_{k=2}^{\infty} c_k a_k z^k \prec f(z) \quad (a_1 = 1; z \in \mathbb{U}\ ). \tag{it1.12}$$

2. Main Results

Unless otherwise mentioned, we shall assume in the reminder of this paper that, $-1 \le \alpha < 1$, $0 \le \lambda \le 1$, $\beta \ge 0$, $z \in \mathbb{U}$ and $g(z)$ is given by (1.7) with $b_{k+1} \ge b_k > 0$ $(k \ge 2)$.

To prove our main result we need the following lemmas.
Lemma 1 [24]. *The sequence* $\{c_k\}_{k=1}^{\infty}$*is a subordinating factor sequence if and only if*

$$Re\left\{1+2\sum_{k=1}^{\infty} d_k z^k\right\} > 0. \qquad \text{(it2.1)}$$

Now, we prove the following lemma which gives a sufficient condition for functions belonging to the class $\mathrm{S}(g,\lambda;\alpha,\beta)$.

Lemma 2. *A function* $f(z)$ *of the form* (1.1) *is said to be in the class* $S(g,\lambda;\alpha,\beta)$ *if*

$$\sum_{k=2}^{\infty}\left\{k\,(1+\beta)-(\alpha+\beta)\,[1+\lambda\,(k-1)]\right\} b_k\,|a_k| \leq 1-\alpha. \qquad \text{(it2.2)}$$

Proof. Assume that, the inequality (2.2) holds true. Then it suffices to show that

$$\beta\left|\frac{z\ (f{*}g)\ '(z)}{(1{-}\lambda)(f{*}g)(z){+}\lambda z\ (f{*}g)\ '(z)}-1\right| - Re\left\{\frac{z\ (f{*}g)\ '(z)}{(1{-}\lambda)(f{*}g)(z){+}\lambda z\ (f{*}g)\ '(z)}-1\right\} \leq 1-\alpha. \quad (2.3)$$

We have

$$\beta\left|\frac{z\ (f{*}g)\ '(z)}{(1{-}\lambda)(f{*}g)(z){+}\lambda z\ (f{*}g)\ '(z)}-1\right| - Re\left\{\frac{z\ (f{*}g)\ '(z)}{(1{-}\lambda)(f{*}g)(z){+}\lambda z\ (f{*}g)\ '(z)}-1\right\}$$

$$\leq (1+\beta)\left|\frac{z\ (f*g)\ '(z)}{(1-\lambda)\,(f*g)(z)+\lambda z\ (f*g)\ '(z)}-1\right|$$

$$\leq \frac{(1+\beta)\sum\limits_{k=2}^{\infty}(1-\lambda)\,(k-1)\,b_k\,|a_k|}{1-\sum\limits_{k=2}^{\infty}[1+\lambda\,(k-1)]\,b_k\,|a_k|} \leq 1-\alpha.$$

This completes the proof of Lemma 2.

Let $\mathrm{S}^*(g,\lambda;\alpha,\beta)$ denote the class of $f(z)\in A$ whose coefficients satisfy the condition (2.2). We note that $\mathrm{S}^*(g,\lambda;\alpha,\beta)\subseteq \mathrm{S}(g,\lambda;\alpha,\beta)$.
Employing the technique used earlier by Attiya [3] and Srivastava and Attiya [22], we prove:

Thereom 1. *Let* $f(z)\in S^*(g,\lambda;\alpha,\beta)$. *Then*

$$\frac{[2+\beta-\alpha-\lambda(\alpha+\beta)]\,b_2}{2\{1-\alpha+[2+\beta-\alpha-\lambda(\alpha+\beta)]\,b_2\}}(f*h)(z)\prec h(z), \tag{it2.4}$$

for every function $h\in K$, *and*

$$Re\{f(z)\}>-\frac{\{1-\alpha+[2+\beta-\alpha-\lambda(\alpha+\beta)]\,b_2\}}{[2+\beta-\alpha-\lambda(\alpha+\beta)]\,b_2}. \tag{it2.5}$$

The constant factor $\frac{[2+\beta-\alpha-\lambda(\alpha+\beta)]b_2}{2\{1-\alpha+[2+\beta-\alpha-\lambda(\alpha+\beta)]b_2\}}$ *in the subordination result* (2.4) *can not be replaced by a larger one.*

Proof. Let $f(z)\in S^*(g,\lambda;\alpha,\beta)$ and suppose that $h(z)=z+\sum\limits_{k=2}^{\infty}c_kz^k$, then

$$\frac{[2+\beta-\alpha-\lambda(\alpha+\beta)]\,b_2}{2\{1-\alpha+[2+\beta-\alpha-\lambda(\alpha+\beta)]\,b_2\}}(f*h)(z)$$

$$=\frac{[2+\beta-\alpha-\lambda(\alpha+\beta)]\,b_2}{2\{1-\alpha+[2+\beta-\alpha-\lambda(\alpha+\beta)]\,b_2\}}\left(z+\sum_{k=2}^{\infty}c_ka_kz^k\right). \tag{2.6}$$

Thus, by using Definition 3, the subordination result holds true if

$$\left\{\frac{[2+\beta-\alpha-\lambda(\alpha+\beta)]\,b_2}{2\{1-\alpha+[2+\beta-\alpha-\lambda(\alpha+\beta)]\,b_2\}}a_k\right\}_{k=1}^{\infty}$$

is a subordinating factor sequence, with $a_1=1$. In view of Lemma 1, this is equivalent to the following inequality:

$$Re\left\{1+\sum_{n=1}^{\infty}\frac{[2+\beta-\alpha-\lambda(\alpha+\beta)]\,b_2}{\{1-\alpha+[2+\beta-\alpha-\lambda(\alpha+\beta)]\,b_2\}}a_kz^k\right\}>0. \tag{2.7}$$

Now, since

$$\Psi(k)=\{k(1+\beta)-(\alpha+\beta)[1+\lambda(k-1)]\}\,b_k$$

is an increasing function of k $(k\geq 2)$, we have:

$$Re\left\{1+\frac{[2+\beta-\alpha-\lambda(\alpha+\beta)]\,b_2}{\{1-\alpha+[2+\beta-\alpha-\lambda(\alpha+\beta)]\,b_2\}}\sum_{k=1}^{\infty}a_kz^k\right\}$$

$$=Re\left\{1+\frac{[2+\beta-\alpha-\lambda(\alpha+\beta)]b_2}{\{1-\alpha+[2+\beta-\alpha-\lambda(\alpha+\beta)]b_2\}}z+\frac{\sum\limits_{k=2}^{\infty}[2+\beta-\alpha-\lambda(\alpha+\beta)]b_2a_kz^k}{\{1-\alpha+[2+\beta-\alpha-\lambda(\alpha+\beta)]b_2\}}\right\}$$

$$\geq 1-\frac{[2+\beta-\alpha-\lambda(\alpha+\beta)]\,b_2}{\{1-\alpha+[2+\beta-\alpha-\lambda(\alpha+\beta)]\,b_2\}}r - \frac{1}{\{1-\alpha+[2+\beta-\alpha-\lambda(\alpha+\beta)]\,b_2\}}\sum_{k=2}^{\infty}[2+\beta-\alpha-\lambda(\alpha+\beta)]\,b_k\,|a_k|\,r^k$$

$$\geq 1-\frac{[2+\beta-\alpha-\lambda(\alpha+\beta)]\,b_2}{\{1-\alpha+[2+\beta-\alpha-\lambda(\alpha+\beta)]\,b_2\}}r - \frac{1}{\{1-\alpha+[2+\beta-\alpha-\lambda(\alpha+\beta)]\,b_2\}}\sum_{k=2}^{\infty}\{k(1+\beta)-(\alpha+\beta)[1+\lambda(k-1)]\}\,b_k\,|a_k|\,r^k$$

$$\geq 1-\frac{[2+\beta-\alpha-\lambda(\alpha+\beta)]\,b_2}{\{1-\alpha+[2+\beta-\alpha-\lambda(\alpha+\beta)]\,b_2\}}r-\frac{1-\alpha}{\{1-\alpha+[2+\beta-\alpha-\lambda(\alpha+\beta)]\,b_2\}}r$$

$$\geq 1-\frac{[2+\beta-\alpha-\lambda(\alpha+\beta)]\,b_2}{\{1-\alpha+[2+\beta-\alpha-\lambda(\alpha+\beta)]\,b_2\}}-\frac{1-\alpha}{\{1-\alpha+[2+\beta-\alpha-\lambda(\alpha+\beta)]\,b_2\}} > 0 \quad (|z|=r<1),$$

where we have also made use of assertion (2.2) of Lemma 2. Thus (2.7) holds true in $\mathbb{U}$. This proves the inequality (2.4). The inequality (2.5) follows from (2.4) by taking the convex function

$$h(z)=\frac{z}{1-z}=z+\sum_{k=2}^{\infty}z^k\in K. \tag{2.8}$$

To prove the sharpness of the constant

$$\frac{[2+\beta-\alpha-\lambda(\alpha+\beta)]\,b_2}{2\{1-\alpha+[2+\beta-\alpha-\lambda(\alpha+\beta)]\,b_2\}},$$

we consider the function $f_0(z)\in \mathrm{S}^*(g,\lambda;\alpha,\beta)$ given by

$$f_0(z)=z-\frac{1-\alpha}{[2+\beta-\alpha-\lambda(\alpha+\beta)]\,b_2}z^2.$$

Thus from (2.4), we have

$$\frac{[2+\beta-\alpha-\lambda(\alpha+\beta)]\,b_2}{2\{1-\alpha+[2+\beta-\alpha-\lambda(\alpha+\beta)]\,b_2\}}f_0(z)\prec\frac{z}{1-z}.$$

It is easily verified that

$$\min_{|z|\leq r}\left\{Re\left(\frac{[2+\beta-\alpha-\lambda(\alpha+\beta)]\,b_2}{2\{1-\alpha+[2+\beta-\alpha-\lambda(\alpha+\beta)]\,b_2\}}f_0(z)\right)\right\}=-\frac{1}{2}. \tag{2.9}$$

This show that the constant $\frac{[2+\beta-\alpha-\lambda(\alpha+\beta)]b_2}{2\{1-\alpha+[2+\beta-\alpha-\lambda(\alpha+\beta)]b_2\}}$ is the best possible. This completes the proof of Theorem 1.

Remark 2. (i) Taking $\lambda=0$ and $g(z)=z+\sum\limits_{k=2}^{\infty}\frac{(a)_{k-1}}{(c)_{k-1}}z^k(c\neq 0,-1,-2,...)$ in Theorem 1, we obtain the result obtained by Frasin [9,Theorem 2.1];
(ii) Taking $g(z)=\frac{z}{(1-z)}$ and $g(z)=\frac{z}{(1-z)^2}$, respectively, in Theorem 1, we obtain the results obtained by Murugusundaramoorthy and Magesh [16, Theorem 2.1 and Theorem 2.3, respectively];
(iii) Taking $\lambda=0$, $g(z)=\frac{z}{(1-z)}$ and $g(z)=\frac{z}{(1-z)^2}$, respectively, in Theorem 1, we obtain the results obtained by Frasin [9, Corollary 2.2 and Corollary 2.5, respectively];
(iv) Taking $\beta=\lambda=0$, $g(z)=\frac{z}{(1-z)}$ and $g(z)=\frac{z}{(1-z)^2}$, respectively, in Theorem 1, we obtain the results obtained by Frasin [9, Corollary 2.3 and Corollary 2.6, respectively];
(v) Taking $\alpha=\beta=\lambda=0$ and $g(z)=\frac{z}{(1-z)}$ in Theorem 1, we obtain the result obtained by Singh [21, Corollary 2.2];
(vi) Taking $\alpha=\beta=\lambda=0$ and $g(z)=\frac{z}{(1-z)^2}$ in Theorem 1, we obtain the result obtained by Frasin [9, Corollary 2.7];
(vii) Taking $\lambda=0$, $\beta=1$, $g(z)=\frac{z}{(1-z)}$ and $g(z)=\frac{z}{(1-z)^2}$, respectively, in Theorem 1, we obtain the results obtained by Aouf et al. [1, Corollary 1 and Corollary 2, respectively];
$(viii)$ Taking $\lambda=0$, $g(z)=z+\sum\limits_{k=2}^{\infty}k^nz^k(n\in\mathbb{N}_0)$ and $g(z)=z+\sum\limits_{k=2}^{\infty}[1+\delta(k-1)]^n z^k$ $(\delta\geq 0,\ n\in\mathbb{N}_0)$, respectively, in Theorem 1, we obtain the results obtained by Aouf et al. [1, Corollary 4 and Corollary 6, respectively];

Also, we establish subordination results for the associated subclasses, $\mathrm{S}^*(g,\lambda,\alpha)$, $\mathrm{S}^*_{q,s}(\alpha_i,\beta_j;\lambda,\alpha,\beta)$, $\mathrm{S}^*(m,\mu,\ell;\alpha,\beta)$, $\mathrm{S}^{*\mu}_{b}(\lambda;\alpha,\beta)$ and $\mathrm{S}^{*\mu}_{b}(\alpha,\beta)$, whose coefficients satisfy the condition (2.2) in the special cases as mentioned in the introduction.

By taking $\beta=0$ in Lemma 2 and Theorem 1, we obtain the following corollary:
Corollary 1. *Let the function $f(z)$ defined by* (1.1) *be in the class $S^*(g,\lambda,\alpha)$ and*

satisfy the condition

$$\sum_{k=2}^{\infty}\{k-\alpha\left[1+\lambda\left(k-1\right)\right]\}b_k\left|a_k\right|\leq 1-\alpha. \qquad \text{(it2.10)}$$

Then for every function $h\in K$, we have

$$\frac{\left(2-\alpha-\lambda\alpha\right)b_2}{2\left[1-\alpha+\left(2-\alpha-\lambda\alpha\right)b_2\right]}\left(f*h\right)(z)\prec h(z), \qquad \text{(it2.11)}$$

and

$$Re\left\{f(z)\right\}>-\frac{\left[1-\alpha+\left(2-\alpha-\lambda\alpha\right)b_2\right]}{\left(2-\alpha-\lambda\alpha\right)b_2}. \qquad \text{(it2.12)}$$

The constant factor $\frac{(2-\alpha-\lambda\alpha)b_2}{2[1-\alpha+(2-\alpha-\lambda\alpha)b_2]}$ *in the subordination result* (2.11) *can not be replaced by a larger one.*

By taking $b_k=\Gamma_k(\alpha_1)$, where $\Gamma_k(\alpha_1)$ is defined by (1.9), in Lemma 2 and Theorem 1, we obtain the following corollary:

Corollary 2. *Let the function $f(z)$ defined by* (1.1) *be in the class $S^*_{q,s}(\alpha_i,\beta_j;\lambda,\alpha,\beta)$ and satisfy the condition*

$$\sum_{k=2}^{\infty}\{k\left(1+\beta\right)-\left(\alpha+\beta\right)\left[1+\lambda\left(k-1\right)\right]\}\Gamma_k(\alpha_1)\left|a_k\right|\leq 1-\alpha. \qquad \text{(it2.13)}$$

Then for every function $h\in K$, we have

$$\frac{\left[2+\beta-\alpha-\lambda\left(\alpha+\beta\right)\right]\Gamma_2(\alpha_1)}{2\left\{1-\alpha+\left[2+\beta-\alpha-\lambda\left(\alpha+\beta\right)\right]\Gamma_2(\alpha_1)\right\}}\left(f*h\right)(z)\prec h(z), \qquad \text{(it2.14)}$$

and

$$Re\left\{f(z)\right\}>-\frac{\left\{1-\alpha+\left[2+\beta-\alpha-\lambda\left(\alpha+\beta\right)\right]\Gamma_2(\alpha_1)\right\}}{\left[2+\beta-\alpha-\lambda\left(\alpha+\beta\right)\right]\Gamma_2(\alpha_1)}. \qquad \text{(it2.15)}$$

The constant factor $\frac{[2+\beta-\alpha-\lambda(\alpha+\beta)]\Gamma_2(\alpha_1)}{2\{1-\alpha+[2+\beta-\alpha-\lambda(\alpha+\beta)]\Gamma_2(\alpha_1)\}}$ *in the subordination result* (2.14) *can not be replaced by a larger one.*

By taking $b_k=\left[\frac{\ell+1+\mu(k-1)}{\ell+1}\right]^m$ $(m\in\mathbb{N}_0,\ \mu,\ell\geq 0)$ in Lemma 2 and Theorem 1, we obtain the following corollary:

Corollary 3. *Let the function $f(z)$ defined by* (1.1) *be in the class $S^*(m,\mu,\ell;\alpha,\beta)$ and satisfy the condition*

$$\sum_{k=2}^{\infty}\{k\left(1+\beta\right)-\left(\alpha+\beta\right)\left[1+\lambda\left(k-1\right)\right]\}\left[\frac{\ell+1+\mu(k-1)}{\ell+1}\right]^m\left|a_k\right|\leq 1-\alpha. \qquad \text{(it2.16)}$$

Then for every function $h \in K$, we have

$$\frac{[2+\beta-\alpha-\lambda(\alpha+\beta)](\ell+1+\mu)^m}{2\{(\ell+1)^m(1-\alpha)+[2+\beta-\alpha-\lambda(\alpha+\beta)](\ell+1+\mu)^m\}}(f*h)(z) \prec h(z), \tag{it2.17}$$

and

$$Re\{f(z)\} > -\frac{\{(\ell+1)^m(1-\alpha)+[2+\beta-\alpha-\lambda(\alpha+\beta)](\ell+1+\mu)^m\}}{[2+\beta-\alpha-\lambda(\alpha+\beta)](\ell+1+\mu)^m}. \tag{it2.18}$$

The constant factor $\frac{[2+\beta-\alpha-\lambda(\alpha+\beta)](\ell+1+\mu)^m}{2\{(\ell+1)^m(1-\alpha)+[2+\beta-\alpha-\lambda(\alpha+\beta)](\ell+1+\mu)^m\}}$ *in the subordination result* (2.17) *can not be replaced by a larger one.*

By taking $b_k = C_k(b,\mu)$, where $C_k(b,\mu)$ is defined by (1.10), in Lemma 2 and Theorem 1, we obtain the following corollary:

Corollary 4. *Let the function $f(z)$ defined by* (1.1) *be in the class* $\mathrm{S}^{*\mu}_b(\lambda;\alpha,\beta)$ *and satisfy the condition*

$$\sum_{k=2}^{\infty}\{k(1+\beta)-(\alpha+\beta)[1+\lambda(k-1)]\}|C_k(b,\mu)||a_k| \le 1-\alpha. \tag{it2.19}$$

Then for every function $h \in K$, we have

$$\frac{[2+\beta-\alpha-\lambda(\alpha+\beta)]|C_2(b,\mu)|}{2\{(1-\alpha)+[2+\beta-\alpha-\lambda(\alpha+\beta)]|C_2(b,\mu)|\}}(f*h)(z) \prec h(z), \tag{it2.20}$$

and

$$Re\{f(z)\} > -\frac{\{(1-\alpha)+[2+\beta-\alpha-\lambda(\alpha+\beta)]|C_2(b,\mu)|\}}{[2+\beta-\alpha-\lambda(\alpha+\beta)]|C_2(b,\mu)|}. \tag{it2.21}$$

The constant factor $\frac{[2+\beta-\alpha-\lambda(\alpha+\beta)]|C_2(b,\mu)|}{2\{(1-\alpha)+[2+\beta-\alpha-\lambda(\alpha+\beta)]|C_2(b,\mu)|\}}$ *in the subordination result* (2.20) *can not be replaced by a larger one.*

By taking $\lambda = 0$ in Corollary 4, we obtain the following corollary:

Corollary 5. *Let the function $f(z)$ defined by* (1.1) *be in the class* $S^{*\mu}_b(\alpha,\beta)$ *and satisfy the condition*

$$\sum_{k=2}^{\infty}[k(1+\beta)-(\alpha+\beta)]|C_k(b,\mu)||a_k| \le 1-\alpha. \tag{it2.22}$$

Then for every function $h \in K$, we have

$$\frac{(2+\beta-\alpha)|C_2(b,\mu)|}{2[1-\alpha+(2+\beta-\alpha)|C_2(b,\mu)|]}(f*h)(z) \prec h(z), \tag{it2.23}$$

and

$$Re\{f(z)\} > -\frac{[1-\alpha+(2+\beta-\alpha)\,|C_2(b,\mu)|]}{(2+\beta-\alpha)\,|C_2(b,\mu)|}. \qquad \text{(it2.24)}$$

The constant factor $\frac{(2+\beta-\alpha)|C_2(b,\mu)|}{2[1-\alpha+(2+\beta-\alpha)|C_2(b,\mu)|]}$ *in the subordination result* (2.23) *can not be replaced by a larger one.*

Remark 3. Corollary 5, corrects the result obtained by Murugusundaramoorthy [13, Theorem 2.1].

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