Acta Universitatis Apulensis No. 32/2012
ISSN: 1582-5329 pp. 85-101

EXISTENCE OF SOLUTIONS FOR TWO POINT BOUNDARY VALUE PROBLEMS FOR FRACTIONAL DIFFERENTIAL EQUATIONS WITH P-LAPLACIAN

Nemat Nyamoradi

Abstract. In this paper, we study the existence of positive solutions to boundary value problem for fractional differential equation

$$\begin{cases} {}^cD^{\sigma}_{0+}(\phi_p(u''(t))) - g(t)f(u(t)) = 0, \quad t \in (0,1), \\ \phi_p(u''(1)) = 0, \ \ \phi_p(u''(0)) = 0, \\ \alpha u(0) - \beta u'(0) = 0, \\ \gamma u(1) + \delta u'(1) = 0, \end{cases}$$

where ${}^cD^{\alpha}_{0+}$ is the Caputo's fractional derivative of order $1 < \sigma \leq 2$, $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $\alpha, \beta, \gamma, \delta \geq 0$ and $f \in C([0,\infty); [0,\infty))$, $g \in C((0,1); (0,\infty))$.

2000 *Mathematics Subject Classification*: 47H10, 26A33, 34A08.

1. Introduction

The existence of solutions for two point boundary value problems for fractional differential equations of the form

$$\begin{cases} {}^cD^{\sigma}_{0+}(\phi_p(u''(t))) - g(t)f(u(t)) = 0, \quad t \in (0,1), \\ \phi_p(u''(1)) = 0, \ \ \phi_p(u''(0)) = 0, \\ \alpha u(0) - \beta u'(0) = 0, \\ \gamma u(1) + \delta u'(1) = 0, \end{cases} \tag{1}$$

where ${}^cD^{\alpha}_{0+}$ is the Caputo's fractional derivative of order $1 < \sigma \leq 2$, $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $(\phi_p)^{-1} = \phi_q$, $\frac{1}{p} + \frac{1}{q} = 1$ and we assume that
(H1) $f : [0,\infty) \to [0,\infty)$ is continuous and

$g \in C((0,1);[0,+\infty))$ and

$$0 < \int_0^1 g(r)dr < \infty,$$

Moreover, $g(t)$ does not vanish identically on any subinterval of $[0,1]$.
(H1*) f is a nonnegative, lower semi-continuous function defined on $[0,+\infty)$, i.e. $\exists I \subset [0,+\infty)$; $\forall x_n \in I$, $x_n \to x_0$ $(n \to \infty)$, one has $f(x_0) \leq \underline{\lim}_{n\to\infty} f(x_n)$. Moreover, f has only a finite number of discontinuity points in each compact subinterval of $[0,+\infty)$.
(A1) $\rho = \gamma\beta + \alpha\gamma + \alpha\delta$, $0 < \eta := \min\left\{\frac{4\delta+\gamma}{4(\delta+\gamma)}, \frac{\alpha+4\beta}{4(\alpha+\beta)}\right\} < 1$.

Fractional differential equations have been of great interest recently. This is because of both the intensive development of the theory of fractional calculus itself and the applications of such constructions in various scientific fields such as physics, mechanics, chemistry, engineering, etc. For details, see [1-3] and the references therein. In [4], Liu, and Jia investigated the existence of multiple solutions for problem:

$$\begin{cases} {}^cD_{0+}^{\sigma}(p(t)u'(t)) + q(t)f(t,u(t)) = 0, \quad t > 0, \;\; 0 < \sigma < 1, \\ p(0)u'(0) = 0, \\ \lim_{t\to\infty} u(t) = \int_0^{+\infty} g(t)u(t)dt, \end{cases}$$

where ${}^cD_{0+}^{\sigma}$ is the standard Caputo derivative of order σ. Some existence results were given for the problem (1) with $\sigma = 2$ by Yanga et al. [5] and Zhao et al. [6].

The solution of differential equations of fractional order is much involved. Some analytical methods are presented, such as the popular Laplace transform method [7,8], the Fourier transform method [9], the iteration method [10] and Green function method [11,12]. Numerical schemes for solving fractional differential equations are introduced, for example, in [13,14,15]. Recently, a great deal of effort has been expended over the last years in attempting to find robust and stable numerical as well as analytical methods for solving fractional differential equations of physical interest. The Adomian decomposition method [16], homotopy perturbation method [17], homotopy analysis method [18], differential transform method [19] and variational method [20] are relatively new approaches to provide an analytical approximate solution to linear and nonlinear fractional differential equations.
The existence of solutions of initial value problems for fractional order differential equations have been studied in the literature [7,10,21,22] and the references therein.

In this work we will consider the existence of positive solutions to problem (1). we shall first give a new form of the solution, and then determine the properties of the Green's function for associated fractional boundary value problems; finally, by employing the Krasnoselskii's fixed point theorems, some sufficient conditions guaranteeing the existence of positive solution.

The rest of the article is organized as follows: in Section 2, we present some preliminaries that will be used in Section 3. The main results and proofs will be given in Section 3. Finally, in Section 4, an example are given to demonstrate the application of our main result.

2. Preliminaries

In this section, we present some notation and preliminary lemmas that will be used in the proofs of the main results.

We work in $C^1([0,1])$ with respect to the norm $||u|| = \max_{0\leq t\leq 1} u(t)$.

Definition 1. *Let X be a real Banach space. A non-empty closed set $P \subset X$ is called a cone of X if it satisfies the following conditions:*
(1) $x \in P, \mu \geq 0$ implies $\mu x \in P$,
(2) $x \in P, -x \in P$ implies $x = 0$.

Definition 2. *The Riemann-Liouville fractional integral operator of order $\alpha > 0$, of function $f \in L^1(R^+)$ is defined as*

$$I_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds,$$

where $\Gamma(\cdot)$ is the Euler gamma function.

Definition 3. *The Riemann-Liouville fractional derivative of order $\alpha > 0$, $n-1 < \alpha < n$, $n \in N$ is defined as*

$$D_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \Big(\frac{d}{dt}\Big)^n \int_0^t (t-s)^{n-\alpha-1} f(s)ds,$$

where the function $f(t)$ have absolutely continuous derivatives up to order $(n-1)$.

Lemma 1. ([23]) *The equality $D_{0^+}^{\gamma} I_{0^+}^{\gamma} f(t) = f(t)$, $\gamma > 0$ holds for $f \in L(0,1)$.*

Definition 4. ([7,23]) *The fractional derivative of f in the Caputo sense is defined as*

$${}^{c}D_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds, \ \ n-1 < \alpha < n,$$

where $n = [\alpha] + 1$.

Lemma 2. ([23-25]) *Let $\alpha > 0$. Then the differential equation*

$$^{c}D_{0+}^{\alpha}u(t) = 0$$

has a unique solution $u(t) = c_0 + c_1 t + \cdots + c_{n-1}t^{n-1}$, $c_i \in R$, $i = 1, \ldots, n$, there $n - 1 < \alpha \leq n$.

Lemma 3. ([23-25]) *Assume that $h \in C(0,1) \cap L(0,1)$ with a derivative of order $\alpha > 0$ that belongs to $C(0,1) \cap L(0,1)$. Then*

$$I_{0+}^{\alpha}\, {}^{c}D_{0+}^{\alpha}h(t) = h(t) + c_0 + c_1 t + \cdots + c_{n-1}t^{n-1},$$

for some $c_i \in R$, $i = 1, \ldots, n-1$, where $n - 1 < \alpha \leq n$.

In the following, we present the Green function of fractional differential equation boundary value problem.

Lemma 4. *Let $h(t) \in C([0,1])$ be a given function. Then the boundary value problem*

$$\begin{cases} {}^{c}D_{0+}^{\sigma}(\phi_p(u''(t))) - h(t) = 0, & t \in (0,1), \\ \phi_p(u''(1)) = 0, \ \ \phi_p(u''(0)) = 0, & \\ \alpha u(0) - \beta u'(0) = 0, & \\ \gamma u(1) + \delta u'(1) = 0, & \end{cases} \tag{2}$$

has a unique solution

$$u(t) = \int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)h(\tau)d\tau\Big)ds, \tag{3}$$

where

$$G(t,s) = \begin{cases} \frac{1}{\rho}(\gamma + \delta - \gamma t)(\beta + \alpha s), & 0 \leq s \leq t \leq 1, \\ \frac{1}{\rho}(\beta + \alpha t)(\gamma + \delta - \gamma s), & 0 \leq t \leq s \leq 1, \end{cases} \tag{4}$$

and

$$H(t,s) = \begin{cases} \frac{(1-s)^{\sigma-1} - (t-s)^{\sigma-1}}{\Gamma(\sigma)}, & 0 \leq s \leq t \leq 1, \\ \frac{(1-s)^{\sigma-1}}{\Gamma(\sigma)}, & 0 \leq t \leq s \leq 1. \end{cases} \tag{5}$$

proof. According to Lemma 3, we can obtain that

$$\phi_p(u''(t)) = I_{0+}^{\sigma} h(t) - c_1 - c_2 t = \frac{1}{\Gamma(\sigma)} \int_0^t (t-s)^{\sigma-1} h(s) ds - c_1 - c_2 t.$$

By the boundary conditions $\phi_p(u''(1)) = 0$ and $(\phi_p(u''(0)))' = 0$, we can calculate out that $c_2 = 0$ and $c_1 = I_{0+}^{\alpha} h(1)$. Consequently, the solution of problem (2) is

$$\phi_p(u''(t)) = I_{0+}^{\sigma} h(t) - I_{0+}^{\sigma} h(1).$$

Thus, the unique solution $\phi_p(u''(t))$ of problem (2) is

$$\begin{aligned} \phi_p(u''(t)) &= \frac{1}{\Gamma(\sigma)} \int_0^t (t-s)^{\sigma-1} h(s) ds - \frac{1}{\Gamma(\sigma)} \int_0^1 (1-s)^{\sigma-1} h(s) ds \\ &= -\int_0^t \frac{(1-s)^{\sigma-1} - (t-s)^{\sigma-1}}{\Gamma(\sigma)} h(s) ds - \int_t^1 \frac{(1-s)^{\sigma-1}}{\Gamma(\sigma)} h(s) ds \\ &= -\int_0^1 H(t,s) h(s) ds. \end{aligned}$$

Then, we get

$$u''(t) = -\phi_q\Big(\int_0^1 H(t,s) h(s) ds\Big).$$

Also, by calculation, it is easy to prove that Lemma 4 holds. So we omit its proof here.

Lemma 5. (See [26]). *Let $G(t,s)$ be given as in (4), then we have the following results:*

$$\begin{cases} \frac{G(t,s)}{G(s,s)} \le 1, & \text{for } t \in [0,1] \text{ and } s \in [0,1], \\ \frac{G(t,s)}{G(s,s)} \ge \eta, & \text{for } t \in \left[\frac{1}{4}, \frac{3}{4}\right] \text{ and } s \in [0,1]. \end{cases} \tag{6}$$

Proposition 1. *For $t, s \in [0,1]$, we have*

$$0 \le H(t,s) \le H(s,s) \le \frac{1}{\Gamma(\sigma)}.$$

Proposition 2. *Let $\theta \in (0, \frac{1}{2})$, then for all $s \in [0,1]$, we have*

$$\min_{\theta \le t \le 1-\theta} H(t,s) \ge \Big[1-(1-\theta)^{\sigma-1}\Big] H(s,s).$$

proof. For $\theta \in (0, \frac{1}{2})$, we have

$$\begin{aligned}
\min_{\theta \le t \le 1-\theta} H(t,s) &= \begin{cases} \frac{(1-s)^{\sigma-1}-(1-\theta-s)^{\sigma-1}}{\Gamma(\sigma)}, & s \in [0,\theta],\\ \min\{\frac{(1-s)^{\sigma-1}-(1-\theta-s)^{\sigma-1}}{\Gamma(\sigma)}, \frac{(1-s)^{\sigma-1}}{\Gamma(\sigma)}\} & \\ \quad = \frac{(1-s)^{\sigma-1}-(1-\theta-s)^{\sigma-1}}{\Gamma(\sigma)}, & s \in [\theta, 1-\theta],\\ \frac{(1-s)^{\sigma-1}}{\Gamma(\sigma)}, & s \in [1-\theta, 1]. \end{cases}\\
&= \begin{cases} \frac{(1-s)^{\sigma-1}-(1-\theta-s)^{\sigma-1}}{\Gamma(\sigma)}, & s \in [0,1-\theta],\\ \frac{(1-s)^{\sigma-1}}{\Gamma(\sigma)}, & s \in [1-\theta,1], \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
(1-s)^{\sigma-1}-(1-\theta-s)^{\sigma-1} &= (1-s)^{\sigma-1}-(1-\theta)^{\sigma-1}(1-s)^{\sigma-1}\\
&\ge [1-(1-\theta)^{\sigma-1}](1-s)^{\sigma-1}, \quad \text{for } s \in [0,1-\theta],\\
(1-s)^{\sigma-1} &\ge [1-(1-\theta)^{\sigma-1}](1-s)^{\sigma-1}, \quad \text{for } s \in [1-\theta,1].
\end{aligned}$$

Therefore, there has
It follows from Proposition 1 that

$$\min_{\theta \le t \le 1-\theta} H(t,s) \ge [1-(1-\theta)^{\sigma-1}]\frac{(1-s)^{\sigma-1}}{\Gamma(\sigma)} = [1-(1-\theta)^{\sigma-1}]H(s,s) \quad \text{for } s \in [0,1].$$

Thus, we complete the proof.

Remark 1. *Let $\theta = \frac{1}{4}$, then by Proposition 2, we have*

$$\min_{\frac{1}{4} \le t \le \frac{3}{4}} H(t,s) \ge \Big[1-(\frac{3}{4})^{\sigma-1}\Big] H(s,s) \quad \textit{for } s \in [0,1].$$

Lemma 6. *Let (H1) and (A1) hold. If $h(t) \in C([0,1])$ and $h \ge 0$, then the unique solution u of the problem (2) satisfies*

(i) $u(t) \geq 0$, *for* $t \in [0,1]$,
and
(ii) $\min_{\frac{1}{4} \leq t \leq \frac{3}{4}} u(t) \geq \Gamma ||u||$,
where $\Gamma := \eta\Big(1 - (\frac{3}{4})^{\sigma-1}\Big)^{q-1}$

proof. (i) By Lemma 5, Proposition 1 and the property of function ϕ_q it is obvious that we have

$$G(t,s) \geq 0, \quad H(t,s) \geq 0, \quad \phi_q\Big(\int_0^1 H(s,\tau)h(\tau)d\tau\Big) \geq 0,$$

so we get $u(t) \geq 0$.

(ii) From Lemma 5, Remark 1, for $t \in \Big[\frac{1}{4}, \frac{3}{4}\Big]$, we have

$$\begin{aligned} u(t) &= \int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)h(\tau)d\tau\Big)ds \\ &\geq \eta\Big(1 - (\frac{3}{4})^{\sigma-1}\Big)^{q-1}\int_0^1 G(s,s)\phi_q\Big(\int_0^1 H(s,s)h(\tau)d\tau\Big)ds \\ &\geq \eta\Big(1 - (\frac{3}{4})^{\sigma-1}\Big)^{q-1}||u||. \end{aligned}$$

Therefore, we get $\min_{\frac{1}{4} \leq t \leq \frac{3}{4}} u(t) \geq \Gamma ||u||$.

Then, choose a cone K is $C^1([0,1])$, by

$$K = \{u \in C[0,1] | u(t) \geq 0, \min_{\frac{1}{4} \leq t \leq \frac{3}{4}} u(t) \geq \Gamma \|u\|\},$$

and define an operator T by

$$(Tu)(t) = \int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)g(\tau)f(u(\tau))d\tau\Big)ds. \tag{7}$$

It is clear that the existence of a positive solution for the system (1) is equivalent to the existence of nontrivial fixed point of T in K.

Lemma 7. *Suppose that the conditions (H1) and (A1) hold, then* $T(K) \subseteq K$ *and* $T : K \to K$ *is completely continuous.*

proof. For any $u \in K$, by (7), we obtain $(Tu)(t) \geq 0$ and, for $t \in [0,1]$,

$$\begin{aligned}(Tu)(t) &= \int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)g(\tau)f(u(\tau))d\tau\Big)ds\\ &\le \int_0^1 G(s,s)\phi_q\Big(\int_0^1 H(s,s)g(\tau)f(u(\tau))d\tau\Big)ds.\end{aligned}$$

Thus, $||Tu|| \le \int_0^1 G(s,s)\phi_q\Big(\int_0^1 H(s,s)g(\tau)f(u(\tau))d\tau\Big)ds$.

On the other hand, for $t \in \Big[\frac{1}{4}, \frac{3}{4}\Big]$, we have

$$\begin{aligned}(Tu)(t) &= \int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)h(\tau)d\tau\Big)ds\\ &\ge \eta\Big(1-(\frac{3}{4})^{\sigma-1}\Big)^{q-1}\int_0^1 G(s,s)\phi_q\Big(\int_0^1 H(s,s)h(\tau)d\tau\Big)ds\\ &\ge \eta\Big(1-(\frac{3}{4})^{\sigma-1}\Big)^{q-1}||Tu|| = \Gamma||Tu||.\end{aligned}$$

Therefore, we get $TK \subseteq K$

By conventional arguments and Ascoli-Arzela theorem, one can prove $T : K \to K$ is completely continuous, so we omit it here.

Our approach is based on the following Guo-Krasnoselskii fixed point theorem of cone expansion-compression type [27].

Theorem 1. *Let E be a Banach space and $K \subseteq E$ a cone in E. Assume Ω_1 and Ω_2 are open subsets of E with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Let $T : K \bigcap(\overline{\Omega}_2\backslash\Omega_1) \to K$ be a completely continuous operator. In addition suppose either*
(A) $\|Tu\| \le \|u\|$, $\forall u \in K \cap \partial\Omega_1$ and $\|Tu\| \ge \|u\|$, $\forall u \in K \cap \partial\Omega_2$ or
(B) $\|Tu\| \ge \|u\|$, $\forall u \in K \cap \partial\Omega_1$ and $\|Tu\| \le \|u\|$, $\forall u \in K \cap \partial\Omega_2$
holds. Then T has a fixed point in $K \cap (\overline{\Omega}_2\backslash\Omega_1)$.

3. Main results

We define $\Omega_l = \{u \in K : ||u|| < l\}$, $\partial\Omega_l = \{u \in K : ||u|| = l\}$, where $l > 0$.
If $u \in \partial\Omega_l$, for $t \in [\frac{1}{4}, \frac{3}{4}]$, we have $\Gamma l \le u \le l$.
For convenience, we introduce the following notations. Let

$$\begin{aligned}f_l &= \inf\Big\{\frac{f(u)}{\phi_p(l)}\Big|u \in [\Gamma l, l]\Big\}, \qquad f^l = \sup\Big\{\frac{f(u)}{\phi_p(l)}\Big|u \in [0, l]\Big\},\\ f_\varrho &= \liminf_{u\to\varrho}\frac{f(u)}{\phi_p(u)}, \quad (\varrho := 0^+ \text{ or } +\infty),\end{aligned}$$

$$
\begin{aligned}
f^{\varrho} &= \limsup_{u\to\varrho} \frac{f(u)}{\phi_p(u)}, \quad (\varrho := 0^+ \text{ or } +\infty),\\
\frac{1}{\omega} &= \Big(\frac{1}{\Gamma(\sigma)}\Big)^{q-1}\Big(\int_0^1 G(s,s)ds\Big)\phi_q\Big(\int_0^1 g(\tau)d\tau\Big),\\
\frac{1}{M} &= \eta\Big(1-(\frac{3}{4})^{\sigma-1}\Big)^{q-1}\left(\frac{\Big(1-(\frac{3}{4})^{\sigma-1}\Big)^{\sigma-1}}{\Gamma(\sigma)}\right)^{q-1}\Big(\int_0^1 G(s,s)ds\Big)\phi_q\Big(\int_0^1 g(\tau)d\tau\Big).
\end{aligned}
$$

We always assume that (H1) hold in the following theorems.

Theorem 2. *Suppose that there exist constants $r, R > 0$ with $r < \Gamma R$ for $r < R$, such that the following two conditions*

(H2) $f^r \le \phi_p(\omega)$,

and

(H3) $f_R \ge \phi_p(M)$,

hold. Then the problem (1) has at least one positive solution $u \in K$ such that

$$0 < r \le ||u|| \le R.$$

proof. Case 1. We shall prove that the result holds when (H1) is satisfied. Without loss of generality, we suppose that $r < \Gamma R$ for $r < R$.

By (H2), (7), Proposition 1 and Lemma 5, for $t \in [0,1]$ and $u \in \Omega_r$, we have

$$
\begin{aligned}
(Tu)(t) &= \int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)g(\tau)f(u(\tau))d\tau\Big)ds\\
&\le \Big(\frac{1}{\Gamma(\sigma)}\Big)^{q-1} r\omega\Big(\int_0^1 G(s,s)ds\Big)\phi_q\Big(\int_0^1 g(\tau)d\tau\Big)\\
&= r = ||u||.
\end{aligned}
$$

This implies that $||Tu|| \le ||u||$ for $u \in \Omega_r$.

Also, by (H3), (7), Remark 1 and Lemma 5, for $t \in [0,1]$ and $u \in \Omega_R$, we have

$$
\begin{aligned}
(Tu)(t) &= \int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)g(\tau)f(u(\tau))d\tau\Big)ds\\
&\ge \int_{\frac{1}{4}}^{\frac{3}{4}} G(t,s)\phi_q\Big(\int_{\frac{1}{4}}^{\frac{3}{4}} H(s,\tau)g(\tau)f(u(\tau))d\tau\Big)ds
\end{aligned}
$$

$$\begin{aligned}
&\geq \eta\Big(1-(\frac{3}{4})^{\sigma-1}\Big)^{q-1}\left(\frac{\Big(1-(\frac{3}{4})^{\sigma-1}\Big)^{\sigma-1}}{\Gamma(\sigma)}\right)^{q-1} MR\Big(\int_{\frac{1}{4}}^{\frac{3}{4}} G(s,s)ds\Big)\phi_q\Big(\int_{\frac{1}{4}}^{\frac{3}{4}} g(\tau)d\tau\Big)\\
&= R = ||u||.
\end{aligned}$$

This implies that $||Tu|| \geq ||u||$ for $u \in \Omega_R$.

Therefore, by Theorem 1, it follows that T has a fixed-point u in $K \cap (\overline{\Omega_R} \setminus \Omega_r)$. This means that the problem (1) has at least one positive solution $u \in K$ such that $0 < r \leq ||u|| \leq R$.

Case 2. When (H1*) holds, by applying the linear approaching method on the domain of discontinuous points of f we can establish sequence $\{f_j\}_{j=1}^{\infty}$ satisfying the following two conditions

(i) $f_j \in C[0,\infty)$ and $0 \leq f_j \leq f_{j+1}$ on $[0,\infty)$,
and
(ii) $\lim_{j\to\infty} f_j = f$, $j = 1, 2, \ldots$, is pointwisely convergent on $[0,\infty)$.

By virtue of proof of Case 1, we know that when $f = f_j$, the problem (1) has a positive solution $u_j(t)$ where

$$u_j(t) = \int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)g(\tau)f_j(u_j(\tau))d\tau\Big)ds,$$

for all $t \in [0,1]$ and $r \leq ||u_j|| \leq R$, r, R are independent of j.

By uniform continuity of $G(t,s)$ on $[0,1]\times[0,1]$, for any $\epsilon > 0$ (adequate small), there exists $\vartheta > 0$ such that for $t_1, t_2 \in [0,1]$ and $|t_1 - t_2| < \vartheta$, one has $|G(t_1,s) - G(t_2,s)| < \epsilon$. Thus, for $t_1, t_2 \in [0,1]$ and $|t_1 - t_2| < \vartheta$, one has

$$\begin{aligned}
|u_j(t_1) - u_j(t_2)| &\leq \int_0^1 |G(t_1,s) - G(t_2,s)| \cdot \phi_q\Big(\int_0^1 H(s,\tau)g(\tau)f_j(u_j(\tau))d\tau\Big)ds\\
&\leq \Big(\frac{1}{\Gamma(\sigma)}\Big)^{q-1} \cdot \max_{||u_j||\leq R} f_j(u_j) \cdot \phi_q\Big(\int_0^1 g(\tau)d\tau\Big) \cdot \epsilon.
\end{aligned}$$

So we get that $\{u_j\}_{j=1}^{\infty}$ are equicontinuous on $[0,1]$. Thus, by Arzela-Asoli theorem, we know that there exists a convergent subsequence of $\{u_j\}_{j=1}^{\infty}$. For convenience, we denote this convergent subsequence with $\{u_j\}_{j=1}^{\infty}$. Without loss of generality, we suppose $\lim_{j\to\infty} u_j(t) = u(t)$, $\forall t \in [0,1]$, and $r \leq ||u|| \leq R$. By Fatou's Lemma and Lebesgue dominated convergence theorem, we have

$$\lim_{j\to\infty} u_j(t) \geq \int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)g(\tau)\lim_{j\to\infty} f_j(u_j(\tau))d\tau\Big)ds,$$

i.e.

$$u(t) \geq \int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)g(\tau)f(u(\tau))d\tau\Big)ds. \tag{8}$$

On the other hand, by the conditions (i) and (ii), we have

$$u_j(t) \leq \int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)g(\tau)f(u_j(\tau))d\tau\Big)ds.$$

By the lower semi-continuity of f, taking limits in above inequality as $j \to \infty$, we have

$$u(t) \leq \int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)g(\tau)f(u(\tau))d\tau\Big)ds. \tag{9}$$

By (8) and (9), we have

$$u(t) = \int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)g(\tau)f(u(\tau))d\tau\Big)ds.$$

Therefore $u(t)$ is a positive solution of the problem (1). This completes the proof of Theorem 2.

Similarly, we can obtain the following conclusion.

Theorem 3. *Suppose that there exist constants* $r, R > 0$ *with* $r < \Gamma R$ *for* $r < R$, *such that the following two conditions*

(H2)* $f^r < \phi_p(\omega)$,

and

(H3)* $f_R > \phi_p(M)$,

hold. Then the problem (1) has at least one positive solution $u \in K$ *such that*

$$0 < r < ||u|| < R.$$

Theorem 4. *Assume that one of the following two conditions*

(H4) $f^0 \leq \phi_p(\omega)$, $\qquad f_\infty \geq \phi_p(\frac{M}{\gamma})$,

and

(H5) $f_0 \geq \phi_p(\frac{M}{\gamma})$, $\qquad f^\infty \leq \phi_p(\omega)$
is satisfied. Then the problem (1) has at least one positive solution.

proof. We show that (H4) implies (H2) and (H3). Suppose that (H4) holds, then there exist r and R with $0 < r < \gamma R$, such that

$$\frac{f(u)}{\phi_p(u)} \leq \phi_p(\omega), \quad 0 < u \leq r$$

and

$$\frac{f(u)}{\phi_p(u)} \geq \phi_p(\frac{M}{\gamma}), \quad u \geq \gamma R.$$

Hence, we obtain

$$f(u) \leq \phi_p(\omega)\phi_p(u) \leq \phi_p(\omega)\phi_p(r) = \phi_p(r\omega), \quad 0 < u \leq r$$

and

$$f(u) \geq \phi_p(\frac{M}{\gamma})\phi_p(u) \geq \phi_p(\frac{M}{\gamma})\phi_p(\gamma R) = \phi_p(MR), \quad u \geq \gamma R.$$

Thus, (H2) and (H3) holds.

Therefore, by Theorem 2, the problem (1) has at least one positive solution.

Now suppose that (H5) holds, then there exist $0 < r < R$ with $Mr < \omega R$ such that

$$\frac{f(u)}{\phi_p(u)} \geq \phi_p(\frac{M}{\gamma}), \quad 0 < u \leq r. \tag{10}$$

and

$$\frac{f(u)}{\phi_p(u)} \leq \phi_p(\omega), \quad u \geq R. \tag{11}$$

By (10), it follows that

$$f(u) \geq \phi_p(\frac{M}{\gamma})\phi_p(u) \geq \phi_p(\frac{M}{\gamma})\phi_p(\gamma r) = \phi_p(Mr), \quad \gamma r \leq u \leq r.$$

So, the condition (H3) holds for r.

For (11), we consider two cases.

(i) If $f(u)$ is bounded, there exists a constant $D > 0$ such that $f(u) \leq D$, for $0 \leq u < \infty$. By (11), there exists a constant $\lambda \geq R$ with $Mr < \omega R \leq \lambda\omega$ satisfying $\phi_p(\lambda) \geq \max\{\phi_p(R), \frac{D}{\phi_p(\omega)}\}$ such that $f(u) \leq D \leq \phi_p(\lambda\omega)$ for $0 \leq u \leq \lambda$. This means that the condition (H2) holds for λ.

(ii) If $f(u)$ is unbounded, there exist $\lambda_1 \geq R$ with $Mr < \omega R \leq \lambda_1\omega$ such that $f(u) \leq f(\lambda_1)$ for $0 \leq u \leq \lambda_1$. This yields $f(u) \leq f(\lambda_1) \leq \phi_p(\lambda_1\omega)$ for $0 \leq u \leq \lambda_1$. Thus, condition (H2) holds for λ_1.

Therefore, by Theorem 2, the problem (1) has at least one positive solution. Theorem 4 is proved.

Remark 2. *It is obvious that Theorem 4 holds if f satisfies conditions $f^0 = 0$, $f_\infty = +\infty$ or $f_0 = +\infty$, $f^\infty = 0$.*

In this section, we give some conclusions about the existence of multiple positive solutions. We always suppose that (H1), (H1*) and (A1) hold in the following theorems.

Theorem 5. *Assume that one of the following two conditions*

(H6) $f^r < \phi_p(\omega)$,

and

(H7) $f_0 \geq \phi_p(\frac{M}{\gamma})$, $\qquad f_\infty \geq \phi_p(\frac{M}{\gamma})$

are satisfied. Then the problem (1) has at least two positive solutions such that

$$0 < ||u_1|| < r < ||u_2||.$$

proof. By the proof of Theorem 4, we can take $0 < r_1 < r < \gamma r_2$ such that $f(u) \geq \phi_p(r_1 M)$ for $\gamma r_1 \leq u \leq r_1$ and $f(u) \geq \phi_p(r_2 M)$ for $\gamma r_2 \leq u \leq r_2$. Therefore, by Theorems 3 and 4, it follows that problem (1) has at least two positive solutions such that $0 < ||u_1|| < r < ||u_2||$.

Theorem 6. *Assume that one of the following two conditions*

(H8) $f_R > \phi_p(M)$,

and

(H9) $f^0 \leq \phi_p(\omega)$, $\qquad f^\infty \leq \phi_p(\omega)$,

are satisfied. Then the problem (1) has at least two positive solutions such that

$$0 < ||u_1|| < R < ||u_2||.$$

Theorem 7. *Assume (H4) (or (H5)) holds, and there exist constants* $r_1, r_2 > 0$ *with* $r_1 M < r_2 \omega$ *(or* $r_1 < \gamma r_2$*) such that (H6) holds for* $r = r_2$ *(or* $r = r_1$*) and (H8) holds for* $R = r_1$ *(or* $R = r_2$*). Then the problem (1) has at least three positive solutions such that*

$$0 < ||u_1|| < r_1 < ||u_2|| < r_2 < ||u_3||.$$

The proofs of Theorems 6 and 7 are similar to that of Theorem 5, so we omit it here.

Theorem 8. *Let* $n = 2k + 1$, $k \in N$. *Assume (H4) (or (H5)) holds. If there exist constants* $r_1, r_2, \ldots, r_{n-1} > 0$ *with* $r_{2i} < \gamma r_{2i+1}$*, for* $1 \le i \le k - 1$ *and* $r_{2i-1} M < r_{2i}\omega$ *for* $1 \le i \le k$ *(or with* $r_{2i-1} < \gamma r_{2i}$*, for* $1 \le i \le k$ *and* $r_{2i} M < r_{2i+1}\omega$ *for* $1 \le i \le k-1$*) such that (H8) (or (H6)) holds for* r_{2i-1}, $1 \le i \le k$ *and (H6) (or (H8)) holds for* r_{2i}, $1 \le i \le k$. *Then the problem (1) has at least n positive solutions* $u_1, \ldots, u_n$ *such that*

$$0 < ||u_1|| < r_1 < ||u_2|| < r_2 < \cdots < ||u_{n-1}|| < r_{n-1} < ||u_n||.$$

4. Application

Example 1. Consider the following singular boundary value problems with a p-Laplacian operator

$$\begin{cases} {}^c D_{0+}^{\frac{3}{2}} (\phi_p(u''(t))) - t^{-\frac{1}{2}} f(u(t)) = 0, \quad t \in (0,1), \\ \phi_p(u''(1)) = 0, \ \ \phi_p(u''(0)) = 0, \\ u(0) - u'(0) = 0, \\ u(1) + u'(1) = 0, \end{cases} \tag{12}$$

where $p = \frac{3}{2}$,

$$f(u) = \begin{cases} e^{-u}, & 0 \le u \le 1, \\ (n+1)e^{-u}, & n < u \le n+1, \quad n = 1, 2, \ldots, 10, \\ e^{\sqrt{u}}, & u > 11. \end{cases}$$

We note that

$$\alpha=\beta=\gamma=\delta=1, \quad \rho=3, \quad \eta=\frac{5}{8}<1, \quad g(t)=t^{-\frac{1}{2}}, \quad \Gamma=\frac{10-5\sqrt{3}}{16}$$
$$f_0=+\infty, \quad f_\infty=+\infty, \quad \omega=\frac{9\pi}{104}, \quad M=\frac{72\pi}{65(2-\sqrt{3})^2}.$$

So, $f_\infty > \phi_p(\frac{M}{\Gamma})$ and $f_0 > \phi_p(\frac{M}{\Gamma})$. We choose $r=4$, then

$$f^r=\sup\Big\{\frac{f(u)}{\phi_p(r)}\Big|u\in[0,r]\Big\}=0.125<0.141=\phi_p(\omega).$$

Thus, (H5) and (H6) hold. Obviously, (H1), (H1*) and (A1) hold. By Theorem 5, the problem (12) has at least two positive solutions $u_1, u_2 \in K$ such that $0 < ||u_1|| < 4 < ||u_2||$.

Acknowledgments: The author would like to thank the referees for several comments and valuable suggestions.

References

[1] A.M.A. El-Sayed, *Nonlinear functional differential equations of arbitrary orders*, Nonlinear Anal., 33, (1998), 181-186.

[2] A.A. Kilbas, J.J. Trujillo, *Differential equations of fractional order: Methods, results and problems I*, Appl. Anal., 78, (2001), 153-192.

[3] A.A. Kilbas, J.J. Trujillo, *Differential equations of fractional order: Methods, results and problems II*, Appl. Anal., 81, (2002), 435-493.

[4] X. Liu, M. Jia, *Multiple solutions of nonlocal boundary value problems for fractional differential equations on the half-line*, Electron. J. Qualit. Diff. Equat., 56, (2011), 114.

[5] J. Yang, Z. Wei, *Existence of positive solutions for fourth-order m-point boundary value problems with a one-dimensional p-Laplacian operator*, Nonlinear Anal., 71, (2009), 2985-2996.

[6] J. Zhaoa, L. Wang, W. Ge, *Necessary and sufficient conditions for the existence of positive solutions of fourth order multi-point boundary value problems*, Nonlinear Anal., 72, (2010), 822-835.

[7] I. Podlubny, *Fractional Differential Equations*, Academic Press, New York, 1999.

[8] I. Podlubny, *The Laplace Transform Method for Linear Differential Equations of Fractional Order*, Slovac Academy of Science, Slovak Republic, 1994.

[9] K.S. Miller, B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley, New York, 1993.

[10] G. Samko, A. Kilbas, O. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach, Amsterdam, 1993.

[11] W. Schneider, W. Wyss, *Fractional diffusion and wave equations*, J. Math. Phys., 30, (1989), 134-144.

[12] F. Mainardi, Y. Luchko, G. Pagnini, *The fundamental solution of the space-time fractional diffusion equation*, Fract. Calc. Appl. Anal., 4, (2001), 153-192.

[13] K. Diethelm, N. Ford, A. Freed, *A predictor-corrector approach for the numerical solution of fractional differential equations*, Nonlinear Dynam., 29, (2002), 3-22.

[14] K. Diethelm, N. Ford, A. Freed, *Detailed error analysis for a fractional Adams Method*, Numer. Algorithms, 36, (2004), 31-52.

[15] Z. Odibat, S. Momani, *An algorithm for the numerical solution of differential equations of fractional order*, J. Appl. Math. Inform., 26(1-2), (2008), 15-27.

[16] Z. Odibat, S. Momani, *Numerical methods for nonlinear partial differential equations of fractional order*, Appl. Math. Modelling., 32(12), (2008), 28-39.

[17] Z. Odibat, S. Momani, *Modified homotopy perturbation method: Application to quadratic Riccati differential equation of fractional order*, Chaos Solitons Fractals, 36(1), (2008), 167-174.

[18] J. Cang, Y. Tan, H. Xu, S.J. Liao, *Series solutions of non-linear Riccati differential equations with fractional order*, Chaos Solitons Fractals, 40(1), (2009), 1-9.

[19] S. Momani, Z. Odibat, *A novel method for nonlinear fractional partial differential equations: Combination of DTM and generalized Taylor's formula*, J. Comput. Appl. Math., 220(1-2), (2008), 85-95.

[20] J. Feng, Z. Yong, *Existence of solutions for a class of fractional boundary value problems via critical point theory*, Comput. Math. Appl., 62, (2011), 1181-1199.

[21] R.P. Agarwal, M. Benchohra, S. Hamani, *A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions*, Acta Appl. Math., 109, (2010), 973-1033.

[22] V. Lakshmikantham, A.S. Vatsala, *Basic theory of fractional differential equations*, Nonlinear Anal. TMA, 69(8), (2008), 2677-2682.

[23] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2006.

[24] V. Lakshmikantham, S. Leela, JV. Devi, *Theory of Fractional Dynamic Systems*, Cambridge: Cambridge Scientific Publishers 2009.

[25] C. Bai, *Positive solutions for nonlinear fractional differential equations with*

coefficient that changes sign, Nonlinear Anal., 64(4), (2006), 677-685.

[26] W.C. Lian, F.H. Wong, C.C. Yeh, *On the existence of positive solutions of nonlinear second order differential equations*, Proc. Amer. Math. Soc., 124, (1996), 1117-1126.

[27] M.A. Krasnoselskii, *Positive solutions of operator equations*, Noordhoff, Groningen, Netherlands, 1964.

Nemat Nyamoradi
Department of Mathematics, Faculty of Sciences
Razi University, 67149 Kermanshah, Iran.
email:*neamat80@yahoo.com; nyamoradi@razi.ac.ir*