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PARTIAL SUMS OF GENERALIZED CLASS OF HARMONIC UNIVALENT FUNCTIONS INVOLVING A GAUSSIAN HYPERGEOMETRIC FUNCTION

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Abstract. The purpose of the present paper is to establish some new results giving the sharp bounds of the real parts of ratios of harmonic univalent functions to its sequences of partial sums by involving the Gaussian hypergeometric function. Relevant connections of the results presented here with various known results are briefly indicated. We also mention results which are associated with certain classical orthogonal polynomials (deduced from some of the main results) exhibiting the usefulness of the results presented in this paper.

Keywords and Phrases: Harmonic functions, Analytic univalent functions, partial sums, Gaussian hypergeometric function, Legendre polynomials, Jacobi polynomials, Laguerre polynomials.

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1. Introduction

A continuous complex-valued function $f = u+iv$ is said to be harmonic in a simply-connected domain D if both u and v are real harmonic in D. In any simply-connected domain we can write $f = h + \overline{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h'(z)| > |g'(z)|, \quad z \in D$; see Clunie and Sheil-Small [4].

Denote by S_H the class of functions $f = h+\overline{g}$ which are harmonic univalent and sense-preserving in the open unit disk $U = \{z : |z| < 1\}$ for which $f(0) = f_z(0)-1 = 0$. For $f = h + \overline{g} \in S_H$, we may express the analytic functions h and g as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = \sum_{k=1}^{\infty} b_k z^k, \quad |b_1| < 1. \tag{1}$$

For basic results on harmonic functions, one may refer to the book by Duren [10], see also Ahuja [2] and Ponnusamy and Rasila ([18], [19]).

Note that S_H reduces to the class S of normalized analytic univalent functions, if the co-analytic part of its member is zero, i.e. $g \equiv 0$, and for this class $f(z)$ may be expressed as

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{2}$$

Let $\phi(z) \in S_H$ be a fixed function of the form

$$\phi(z) = z + \sum_{k=2}^{\infty} c_k z^k + \overline{\sum_{k=1}^{\infty} d_k z^k}, \quad (d_k \geq c_k \geq c_2 > 0; k \geq 2 \text{ and } |d_1| < 1). \tag{3}$$

We now introduce a class $S_H(c_k, d_k, \delta)$ consisting of functions of the form 1 which satisfies the inequality that

$$\sum_{k=2}^{\infty} c_k |a_k| + \sum_{k=1}^{\infty} d_k |b_k| \leq \delta, \quad \text{where} \quad \delta > 0, \tag{4}$$

and we note that if $d_k = 0$, then the class $S_H(c_k, d_k, \delta)$ reduces to the class $S_\phi(c_k, \delta)$ which was studied by Frasin [13]. In this case condition 4 reduces to

$$\sum_{k=2}^{\infty} c_k |a_k| \leq \delta, \text{ where } \delta > 0. \tag{5}$$

It is easy to see that various subclasses of S_H consisting of functions $f(z)$ of the form:

$$h(z) = z - \sum_{k=2}^{\infty} |a_k| z^k, g(z) = \sum_{k=1}^{\infty} |b_k| z^k, |b_1| < 1, \tag{6}$$

can be represented by $S_H(c_k, d_k, \delta)$ for suitable choices of c_k, d_k and δ studied earlier by various researchers. We illustrate some known consequences by noting the following relationships:

1. $S_H(k, k, 1) \equiv S_H^*$ (Silverman [28]; Silverman and Silvia [29]).
2. $S_H(k - \alpha, k + \alpha, 1 - \alpha) \equiv S_H^*(\alpha)$ (Jahangiri [14], (see also [24])).
3. $S_H(2k - 1 - \alpha, 2k + 1 + \alpha, 1 - \alpha) \equiv G_H(\alpha)$ (Rosy *et al.* [25]).
4. $S_H(k(1 - \alpha\lambda) - \alpha(1 - \lambda), k(1 - \alpha\lambda) + \alpha(1 - \lambda), 1 - \alpha) \equiv S_H^*(\alpha, \lambda)$ (Öztürk *et al.* [17]).
5. $S_H(\lambda_k - \alpha, \mu_k + \alpha, 1 - \alpha) \equiv S_H(\Phi, \Psi, \alpha)$ (Frasin [11]).

6. $S_H(k^m - \alpha k^n, k^m - \alpha k^n, 1-\alpha) \equiv HS(m,n,\alpha)$ (Dixit and Porwal [6], (see also [9])).

7. $S_H(\lambda_k(1-\alpha\lambda) - \alpha(1-\lambda), \mu_k(1-\alpha\lambda) + \alpha(1-\lambda), 1-\alpha) \equiv S_H(\Phi, \Psi, \alpha, \lambda)$ (Dixit and Porwal [8]).

8. $S_H((k-\alpha)(\phi(k,\lambda)), (k+\alpha)(\phi(k,\lambda)), 1-\alpha) \equiv S^*_{H,\lambda}(\alpha)$ (Dixit and Porwal [7]).

9. $$S_H((k(\beta+1) - t(\beta+\gamma))\Gamma(\alpha_1,k), (k(\beta+1) + t(\beta+\gamma))\Gamma(\alpha_1,k)), (k-1)!(1-\gamma))$$
$$\equiv G_H(\alpha_1,\beta,\gamma,t) \text{ (Porwal \textit{et al.} [22])}$$

In 1985, Silvia [30] studied the partial sums of convex functions of order α. Later on, Silverman [27], Abubaker and Darus [1], Dixit and Porwal [5], Frasin ([12], [13]), Murugusundaramoorthy *et al.* [15], Orahan et al. [16], Raina and Bansal [23] and Rosy *et al.* [26] studied and generalized the results on partial sums for various classes of analytic functions. Very recently, Porwal [20], Porwal and Dixit [21] also studied the partial sums of harmonic univalent functions.

We denote the sequences of certain partial sums of functions of the form 1 (with $b_1 = 0$) as follows:

$$f_m(z) = z + \sum_{k=2}^{m} a_k z^k + \sum_{k=2}^{\infty} \overline{b_k z^k},$$

$$f_n(z) = z + \sum_{k=2}^{\infty} a_k z^k + \sum_{k=2}^{n} \overline{b_k z^k},$$

$$f_{m,n}(z) = z + \sum_{k=2}^{m} a_k z^k + \sum_{k=2}^{n} \overline{b_k z^k},$$

where the coefficients of f are sufficiently small to satisfy the condition 4.

The Gaussian hypergeometric function $F(a,b;c;z)$, $z \in U$, used in this paper is defined by the series:

$$F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_n} z^n,$$

which has vast applications in various fields such as conformal mappings, quasi conformal theory, continued fractions and in several other areas of applied mathematics and engineering. Here a, b, c are complex numbers such that $c \neq 0, -1, -2, -3...$; $(a)_0 = 1$ for $a \neq 0$, and for each positive integer n, $(a)_n = a(a+1)(a+2)...(a+n-1)$

is the familiar Pochhammer symbol. Results regarding $F(a,b;c;z)$ when $Re(c-a-b)$ is positive, zero or negative are abundant in literature. These cases for Gaussian hypergeometric function are discussed in [31].

In the present paper, we investigate results pertaining to the class $S_H(c_k,d_k,\delta)$ by using the Gaussian hypergeometric function and determine sharp lower bounds for $Re\left\{\frac{f(z)*H(a,b;c;z)}{f_m(z)*H(a,b;c;z)}\right\}$, $Re\left\{\frac{f_m(z)*H(a,b;c;z)}{f(z)*H(a,b;c;z)}\right\}$, $Re\left\{\frac{(f(z)*H(a,b;c;z))'}{(f_m(z)*H(a,b;c;z))'}\right\}$, $Re\left\{\frac{(f_m(z)*H(a,b;c;z))'}{(f(z)*H(a,b;c;z))'}\right\}$, $Re\left\{\frac{f(z)*H(a,b;c;z)}{f_n(z)*H(a,b;c;z)}\right\}$, $Re\left\{\frac{f_n(z)*H(a,b;c;z)}{f(z)*H(a,b;c;z)}\right\}$, $Re\left\{\frac{(f(z)*H(a,b;c;z))'}{(f_n(z)*H(a,b;c;z))'}\right\}$, $Re\left\{\frac{(f_n(z)*H(a,b;c;z))'}{(f(z)*H(a,b;c;z))'}\right\}$ $Re\left\{\frac{f(z)*H(a,b;c;z)}{f_{m,n}(z)*H(a,b;c;z)}\right\}$, $Re\left\{\frac{f_{m,n}(z)*H(a,b;c;z)}{f(z)*H(a,b;c;z)}\right\}$, $Re\left\{\frac{(f(z)*H(a,b;c;z))'}{(f_{m,n}(z)*H(a,b;c;z))'}\right\}$, $Re\left\{\frac{(f_{m,n}(z)*H(a,b;c;z))'}{(f(z)*H(a,b;c;z))'}\right\}$, where

$$H(a,b;c;z) = zF(a,b;c;z).$$

Several consequences of the main theorems are deduced and also as special cases, we consider the cases involving some classical orthogonal polynomials, like, the Legendre polynomials, the Jacobi polynomials and the Laguerre polynomials.

2. Main Results

We begin by determining the sharp lower bounds for $Re\left\{\frac{f(z)*H(a,b;c;z)}{f_m(z)*H(a,b;c;z)}\right\}$ which is contained in the following:

Theorem 2.1. *If f of the form 1 with $b_1 = 0$, satisfies the condition 4 with*

$$c_k \geq \begin{cases} \frac{\delta(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!} & if \quad k = 2,3,....,m \\ \frac{c_{m+1}}{\frac{(a)_m(b)_m}{(c)_m m!}} \frac{(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!} & if \quad k = m+1, m+2... \end{cases},$$

then

$$Re\left\{\frac{f(z)*H(a,b;c;z)}{f_m(z)*H(a,b;c;z)}\right\} \geq \frac{c_{m+1} - \frac{\delta(a)_m(b)_m}{(c)_m(m)!}}{c_{m+1}}, \quad \text{for all } z \in U. \tag{7}$$

The result 7 is sharp with the function given by

$$f(z) = z + \frac{\delta}{c_{m+1}} z^{m+1}. \tag{8}$$

Proof.
To obtain sharp lower bound given by 7, let us put

$$\frac{1+\omega(z)}{1-\omega(z)} = \frac{c_{m+1}}{\frac{\delta(a)_m(b)_m}{(c)_m(m)!}} \left[\frac{f(z)*H(a,b;c;z)}{f_m(z)*H(a,b;c;z)} - \frac{c_{m+1} - \frac{\delta(a)_m(b)_m}{(c)_m(m)!}}{c_{m+1}}\right]$$

$$= \frac{1+\sum_{k=2}^{m}\frac{(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!}a_k r^{k-1}e^{i(k-1)\theta}+\sum_{k=2}^{\infty}\frac{(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!}\overline{b_k}r^{k-1}e^{-i(k+1)\theta} + \frac{c_{m+1}}{\frac{\delta(a)_m(b)_m}{(c)_m(m)!}}\left[\sum_{k=m+1}^{\infty}\frac{(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!}a_k r^{k-1}e^{i(k-1)\theta}\right]}{1+\sum_{k=2}^{m}\frac{(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!}a_k r^{k-1}e^{i(k-1)\theta}+\sum_{k=2}^{\infty}\frac{(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!}\overline{b_k}r^{k-1}e^{-i(k+1)\theta}},$$

then

$$\omega(z) = \frac{\frac{c_{m+1}}{\frac{\delta(a)_m(b)_m}{(c)_m(m)!}}\left[\sum_{k=m+1}^{\infty}\frac{(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!}a_k r^{k-1}e^{i(k-1)\theta}\right]}{2+2\left(\sum_{k=2}^{m}\frac{(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!}a_k r^{k-1}e^{i(k-1)\theta}+\sum_{k=2}^{\infty}\frac{(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!}\bar{b}_k r^{k-1}e^{-i(k+1)\theta}\right) + \frac{c_{m+1}}{\frac{\delta(a)_m(b)_m}{(c)_m(m)!}}\left(\sum_{k=m+1}^{\infty}\frac{(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!}a_k r^{k-1}e^{i(k-1)\theta}\right)}.$$

Thus

$$|\omega(z)| \le \frac{\frac{c_{m+1}}{\frac{\delta(a)_m(b)_m}{(c)_m(m)!}}\left[\sum_{k=m+1}^{\infty}\frac{(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!}|a_k|\right]}{2-2\left(\sum_{k=2}^{m}\frac{(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!}|a_k|+\sum_{k=2}^{\infty}\frac{(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!}|b_k|\right) - \frac{c_{m+1}}{\frac{\delta(a)_m(b)_m}{(c)_m(m)!}}\left(\sum_{k=m+1}^{\infty}\frac{(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!}|a_k|\right)}.$$

This last expression is bounded above by 1, if and only if

$$\sum_{k=2}^{m}\frac{\delta(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!}|a_k|+\sum_{k=2}^{\infty}\frac{\delta(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!}|b_k| + \frac{c_{m+1}}{\frac{\delta(a)_m(b)_m}{(c)_m(m)!}}\sum_{k=m+1}^{\infty}\frac{\delta(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!}|a_k| \le 1.$$

In view of 4, it suffices to show that L.H.S. of 9 is bounded above by

$\sum_{k=2}^{\infty} \frac{c_k}{\delta}|a_k| + \sum_{k=2}^{\infty} \frac{d_k}{\delta}|b_k|,$ which is equivalent to

$$\sum_{k=2}^{m}\left(\frac{c_k}{\delta} - \frac{(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!}\right)|a_k| + \sum_{k=2}^{\infty}\left(\frac{d_k}{\delta} - \frac{(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!}\right)|b_k|$$
$$+ \sum_{k=m+1}^{\infty}\left(\frac{c_k}{\delta} - \frac{c_{m+1}}{\frac{\delta(a)_m(b)_m}{(c)_m m!}}\right)|a_k| \geq 0.$$

To see that $f(z) = z + \frac{\delta}{c_{m+1}} z^{m+1}$ gives the sharp result, we observe that for $z = re^{i\pi/m}$ that

$$\begin{aligned}\frac{f(z) * H(a,b;c;z)}{f_m(z) * H(a,b;c;z)} &= \frac{z + \frac{\delta(a)_m(b)_m}{(c)_m(m)!c_{m+1}} z^{m+1}}{z} \\ &= 1 - \frac{\delta(a)_m(b)_m}{(c)_m(m)!c_{m+1}} \\ &= \frac{c_{m+1} - \frac{\delta(a)_m(b)_m}{(c)_m(m)!}}{c_{m+1}},\end{aligned}$$

when $r \to 1^-$.

Adopting the same procedure as in Theorem 2.1, and performing simple calculations, we can obtain the sharp lower bounds for the real parts of the following ratios:
$Re\left\{\frac{f_m(z)*H(a,b;c;z)}{f(z)*H(a,b;c;z)}\right\}$, $Re\left\{\frac{(f(z)*H(a,b;c;z))'}{(f_m(z)*H(a,b;c;z))'}\right\}$, $Re\left\{\frac{(f_m(z)*H(a,b;c;z))'}{(f(z)*H(a,b;c;z))'}\right\}$;
$Re\left\{\frac{f(z)*H(a,b;c;z)}{f_n(z)*H(a,b;c;z)}\right\}$, $Re\left\{\frac{f_n(z)*H(a,b;c;z)}{f(z)*H(a,b;c;z)}\right\}$; $Re\left\{\frac{(f(z)*H(a,b;c;z))'}{(f_n(z)*H(a,b;c;z))'}\right\}$
and $Re\left\{\frac{(f_n(z)*H(a,b;c;z))'}{(f(z)*H(a,b;c;z))'}\right\}$.

The results corresponding to real parts of these ratios are contained in the following Theorems 2.2, 2.3, 2.4, 2.5, 2.6, 2.7 and 2.8.

Theorem 2.2. *If f of the form 1 with $b_1 = 0$, satisfies the condition 4 with*

$$c_k \geq \begin{cases} \frac{\delta(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!} & if \quad k = 2,3,....,m \\ \frac{c_{m+1}}{\frac{(a)_m(b)_m}{(c)_m m!}} \frac{(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!} & if \quad k = m+1, m+2... \end{cases},$$

then

$$Re\left\{\frac{f_m(z) * H(a,b;c;z)}{f(z) * H(a,b;c;z)}\right\} \geq \frac{c_{m+1}}{c_{m+1} + \frac{\delta(a)_m(b)_m}{(c)_m(m)!}}, \qquad \text{for all } z \in U. \tag{9}$$

The result 9 is sharp with the function given by 8.

Theorem 2.3. *If f of the form 1 with $b_1 = 0$, satisfies the condition 4 with*

$$c_k \geq \begin{cases} \frac{\delta k(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!} & \text{if} \quad k = 2, 3, ...m. \\ \frac{kc_{m+1}}{m+1\frac{(a)_m(b)_m}{(c)_m(m)!}} \frac{(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!} & \text{if} \quad k = m+1, m+2, ... \end{cases},$$

then

$$Re\left\{\frac{(f(z) * H(a,b;c;z))'}{(f_m(z) * H(a,b;c;z))'}\right\} \geq \frac{c_{m+1} - \frac{(m+1)\delta(a)_m(b)_m}{(c)_m(m)!}}{c_{m+1}}, \text{ for all } z \in U. \tag{10}$$

The result 10 is sharp with the function given by 8.

Theorem 2.4. *If f of the form 1 with $b_1 = 0$, satisfies the condition 4 with*

$$c_k \geq \begin{cases} k\frac{\delta(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!} & if \quad k = 2, 3, ..., m \\ \frac{kc_{m+1}}{(m+1)\frac{(a)_m(b)_m}{(c)_m(m)!}} \frac{(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!} & if \quad k = m+1, m+2... \end{cases},$$

then

$$Re\left\{\frac{(f_m(z) * H(a,b;c;z))'}{(f(z) * H(a,b;c;z))'}\right\} \geq \frac{c_{m+1}}{c_{m+1} + (m+1)\frac{\delta(a)_m(b)_m}{(c)_m(m)!}}, \quad (z \in U). \tag{11}$$

The result 11 is sharp with the function given by 8.

Theorem 2.5. *If f of the form 1 with $b_1 = 0$ satisfies the condition 4 with*

$$d_k \geq \begin{cases} \frac{\delta(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!} & if \quad k = 2, 3, ..., n \\ \frac{d_{n+1}}{\frac{(a)_n(b)_n}{(c)_n(n)!}} \frac{(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!} & if \quad k = n+1, n+2... \end{cases},$$

then

$$Re\left\{\frac{f(z) * H(a,b;c;z)}{f_n(z) * H(a,b;c;z)}\right\} \geq \frac{d_{n+1} - \frac{\delta(a)_n(b)_n}{(c)_n(n)!}}{d_{n+1}}, \quad (z \in U). \tag{12}$$

The result 12 is sharp with the function

$$f(z) = z + \frac{\delta}{d_{n+1}}\bar{z}^{n+1}. \tag{13}$$

Theorem 2.6. *If f of the form 1 with $b_1 = 0$, satisfies the condition 4 with*

$$d_k \geq \begin{cases} \frac{\delta(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!} & if \quad k = 2, 3, ..., n \\ \frac{d_{n+1}}{\frac{(a)_n(b)_n}{(c)_n(n)!}} \frac{(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!} & if \quad k = n+1, n+2... \end{cases},$$

then

$$Re\left\{\frac{f_n(z) * H(a,b;c;z)}{f(z) * H(a,b;c;z)}\right\} \geq \frac{d_{n+1}}{d_{n+1} + \frac{\delta(a)_n(b)_n}{(c)_n(n)!}}, \quad (z \in U). \tag{14}$$

The result 14 is sharp with the function given by 13.

Theorem 2.7. *If f of the form 1 with $b_1 = 0$, satisfies the condition 4 with*

$$d_k \geq \begin{cases} \frac{\delta k(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!} & \text{if} \quad k = 2, 3, ...n. \\ \frac{k d_{n+1}}{(n+1)\frac{(a)_n(b)_n}{(c)_n(n)!}} \frac{(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!} & \text{if} \quad k = n+1, n+2, ... \end{cases},$$

then

$$Re\left\{\frac{(f(z) * H(a,b;c;z))'}{(f_n(z) * H(a,b;c;z))'}\right\} \geq \frac{d_{n+1} - \frac{(n+1)\delta(a)_n(b)_n}{(c)_n n!}}{d_{n+1}}, \quad (z \in U). \tag{15}$$

The result 15 is sharp with the function given by 13.

Theorem 2.8. *If f of the form 1 with $b_1 = 0$, satisfies the condition 4 with*

$$d_k \geq \begin{cases} \frac{\delta k(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!} & \text{if} \quad k = 2, 3, ...n. \\ \frac{k d_{n+1}}{(n+1)\frac{(a)_n(b)_n}{(c)_n(n)!}} \frac{(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!} & \text{if} \quad k = n+1, n+2, ... \end{cases},$$

then

$$Re\left\{\frac{(f_n(z) * H(a,b;c;z))'}{(f(z) * H(a,b;c;z))'}\right\} \geq \frac{d_{n+1}}{d_{n+1} + \frac{(n+1)\delta(a)_n(b)_n}{(c)_n n!}}, \quad z \in U. \tag{16}$$

The result 16 is sharp with the function given by 13.

We next determine bound for $Re\left\{\frac{f(z)*H(a,b;c;z)}{f_{m,n}(z)*H(a,b;c;z)}\right\}$.

Theorem 2.9. *If f of the form 1 with $b_1 = 0$, satisfies the condition 4 with*

$$c_k \geq \begin{cases} \frac{\delta(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!} & if \quad k = 2, 3, ..., m \\ \frac{c_{m+1}}{\frac{(a)_m(b)_m}{(c)_m(m)!}} \frac{(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!} & if \quad k = m+1, m+2... \end{cases},$$

and

$$d_k \geq \begin{cases} \frac{\delta(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!} & if \quad k = 2,3,....,n \\ \frac{c_{m+1}}{\frac{(a)_n(b)_n}{(c)_n(n)!}} \frac{(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!} & if \quad k = n+1, n+2... \end{cases},$$

then

$$(i) \quad Re\left\{\frac{f(z) * H(a,b;c;z)}{f_{m,n}(z) * H(a,b;c;z)}\right\} \geq \frac{c_{m+1} - \frac{\delta(a)_m(b)_m}{(c)_m(m)!}}{c_{m+1}}, \quad (z \in U). \tag{17}$$

and if the condition

$$c_k \geq \begin{cases} \frac{\delta(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!} & if \quad k = 2,3,....,m \\ \frac{d_{n+1}}{\frac{(a)_m(b)_m}{(c)_m(m)!}} \frac{(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!} & if \quad k = m+1, m+2... \end{cases},$$

and

$$d_k \geq \begin{cases} \frac{\delta(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!} & if \quad k = 2,3,....,n \\ \frac{d_{n+1}}{\frac{(a)_n(b)_n}{(c)_n(n)!}} \frac{(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!} & if \quad k = n+1, n+2... \end{cases},$$

$$(ii) \quad Re\left\{\frac{f(z) * H(a,b;c;z)}{f_{m,n}(z) * H(a,b;c;z)}\right\} \geq \frac{d_{n+1} - \frac{\delta(a)_n(b)_n}{(c)_n(n)!}}{d_{n+1}}, \quad (z \in U). \tag{18}$$

The results 17 and 18 are sharp with the functions given by 8 and 13, respectively.

Proof. To obtain sharp lower bound given by 17, we put

$$\frac{1+\omega(z)}{1-\omega(z)} = \frac{c_{m+1}}{\frac{\delta(a)_m(b)_m}{(c)_m(m)!}}\left[\frac{f(z) * H(a,b;c;z)}{f_{m,n}(z) * H(a,b;c;z)} - \frac{c_{m+1} - \frac{\delta(a)_m(b)_m}{(c)_m(m)!}}{c_{m+1}}\right]$$

$$= \frac{1 + \sum_{k=2}^{m} \frac{(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!} a_k r^{k-1} e^{i(k-1)\theta} + \sum_{k=2}^{n} \frac{(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!} \overline{b_k} r^{k-1} e^{-i(k+1)\theta} + \frac{c_{m+1}}{\frac{\delta(a)_m(b)_m}{(c)_m(m)!}} \left[\sum_{k=m+1}^{\infty} \frac{(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!} a_k r^{k-1} e^{i(k-1)\theta} + \sum_{k=n+1}^{\infty} \frac{(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!} \overline{b_k} r^{k-1} e^{-i(k+1)\theta}\right]}{1 + \sum_{k=2}^{m} \frac{(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!} a_k r^{k-1} e^{i(k-1)\theta} + \sum_{k=2}^{n} \frac{(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!} \overline{b_k} r^{k-1} e^{-i(k+1)\theta}}.$$

So that

$$\omega(z) = \frac{\frac{c_{m+1}}{\frac{\delta(a)_m(b)_m}{(c)_m(m)!}}\left[\sum_{k=m+1}^{\infty}\frac{(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!}a_k r^{k-1}e^{i(k-1)\theta}+\sum_{k=n+1}^{\infty}\frac{(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!}\bar{b}_k r^{k-1}e^{-i(k+1)\theta}\right]}{\begin{array}{c}2+2\left(\sum_{k=2}^{m}\frac{(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!}a_k r^{k-1}e^{i(k-1)\theta}+\sum_{k=2}^{n}\frac{(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!}\bar{b}_k r^{k-1}e^{-i(k+1)\theta}\right)\\ +\frac{c_{m+1}}{\frac{\delta(a)_m(b)_m}{(c)_m(m)!}}\left(\sum_{k=m+1}^{\infty}\frac{(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!}a_k r^{k-1}e^{i(k-1)\theta}+\sum_{k=n+1}^{\infty}\frac{(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!}\bar{b}_k r^{k-1}e^{-i(k+1)\theta}\right)\end{array}}.$$

Then

$$|\omega(z)| \leq \frac{\frac{c_{m+1}}{\frac{\delta(a)_m(b)_m}{(c)_m(m)!}}\left[\sum_{k=m+1}^{\infty}\frac{(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!}|a_k|+\sum_{k=n+1}^{\infty}\frac{(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!}|b_k|\right]}{\begin{array}{c}2-2\left(\sum_{k=2}^{m}\frac{(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!}|a_k|+\sum_{k=2}^{n}\frac{(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!}|b_k|\right)\\ -\frac{c_{m+1}}{\frac{\delta(a)_m(b)_m}{(c)_m(m)!}}\left(\sum_{k=m+1}^{\infty}\frac{(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!}|a_k|+\sum_{k=n+1}^{\infty}\frac{(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!}|b_k|\right)\end{array}}.$$

The last expression is bounded above by 1 if and only if

$$\begin{array}{c}\sum_{k=2}^{m}\frac{(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!}|a_k|+\sum_{k=2}^{n}\frac{(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!}|b_k|+\frac{c_{m+1}}{\frac{\delta(a)_m(b)_m}{(c)_m(m)!}}\\ \left(\sum_{k=m+1}^{\infty}\frac{(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!}|a_k|+\sum_{k=n+1}^{\infty}\frac{(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!}|b_k|\right)\leq 1.\end{array} \tag{19}$$

It suffices to show that L.H.S. of 19 is bounded above by $\sum_{k=2}^{\infty}\frac{c_k}{\delta}|a_k|+\sum_{k=2}^{\infty}\frac{d_k}{\delta}|b_k|$, which is equivalent to

$$\begin{aligned}&\sum_{k=2}^{m}\left(\frac{c_k}{\delta}-\frac{(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!}\right)|a_k|+\sum_{k=2}^{n}\left(\frac{d_k}{\delta}-\frac{(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!}\right)|b_k|\\ +&\sum_{k=m+1}^{\infty}\left(\frac{c_k}{\delta}-\frac{c_{m+1}}{\frac{\delta(a)_m(b)_m}{(c)_m(m)!}}\frac{(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!}\right)|a_k|\\ +&\sum_{n+1}^{\infty}\left(\frac{d_k}{\delta}-\frac{c_{m+1}}{\frac{\delta(a)_m(b)_m}{(c)_m(m)!}}\frac{(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!}\right)|b_k|\geq 0.\end{aligned}$$

To see that $f(z) = z + \dfrac{\delta}{c_{m+1}} z^{m+1}$ gives the sharp result, we observe that for $z = re^{i\pi/m}$ that

$$\begin{aligned}\frac{f(z) * H(a,b;c;z)}{f_{m,n}(z) * H(a,b;c;z)} &= \frac{z+\frac{\delta(a)_m(b)_m}{c_{m+1}(c)_m(m)!}z^{m+1}}{z}\\ &= 1 + \frac{\delta(a)_m(b)_m}{(c)_m(m)!c_{m+1}} z^m\\ &= \frac{c_{m+1} - \frac{\delta(a)_m(b)_m}{(c)_m(m)!}}{c_{m+1}}\end{aligned}$$

when $r \to 1^-$.

For finding sharp lower bound given by 18, we follow the same steps as just mentioned above to arrive at the desired bound.

By applying the same technique as used in proving Theorem 2.9, we can derive the sharp lower bounds of real parts of the following ratios:
$Re\left\{\frac{f_{m,n}(z)*H(a,b;c;z)}{f(z)*H(a,b;c;z)}\right\}$, $Re\left\{\frac{(f(z)*H(a,b;c;z))'}{(f_{m,n}(z)*H(a,b;c;z))'}\right\}$ and $Re\left\{\frac{(f_{m,n}(z)*H(a,b;c;z))'}{(f(z)*H(a,b;c;z))'}\right\}$. The results obtained are given by the following Theorems 2.10, 2.11 and 2.12.

Theorem 2.10. *If f of the form 1 with $b_1 = 0$, satisfies the condition 4 with*

$$c_k \geq \begin{cases} \frac{\delta(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!} & if \quad k = 2,3,....,m \\ \frac{c_{m+1}}{\frac{(a)_m(b)_m}{(c)_m(m)!}} \frac{(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!} & if \quad k = m+1, m+2... \end{cases},$$

and

$$d_k \geq \begin{cases} \frac{\delta(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!} & if \quad k = 2,3,....,n \\ \frac{c_{m+1}}{\frac{(a)_n(b)_n}{(c)_n(n)!}} \frac{(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!} & if \quad k = n+1, n+2... \end{cases},$$

then

$$(i) \quad Re\left\{\frac{f_{m,n}(z) * H(a,b;c;z)}{f(z) * H(a,b;c;z)}\right\} \geq \frac{c_{m+1}}{c_{m+1} + \frac{\delta(a)_m(b)_m}{(c)_m(m)!}}, \quad (z \in U), \tag{20}$$

and if the condition

$$c_k \geq \begin{cases} \frac{\delta(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!} & if \quad k = 2,3,....,m \\ \frac{d_{n+1}}{\frac{(a)_m(b)_m}{(c)_m(m)!}} \frac{(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!} & if \quad k = m+1, m+2... \end{cases},$$

and

$$d_k \geq \begin{cases} \frac{\delta(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!} & if \quad k = 2,3,....,n \\ \frac{d_{n+1}}{\frac{(a)_n(b)_n}{(c)_n(n)!}} \frac{(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!} & if \quad k = n+1, n+2... \end{cases},$$

$$(ii)\quad Re\left\{\frac{f_{m,n}(z)*H(a,b;c;z)}{f(z)*H(a,b;c;z)}\right\}\geq\frac{d_{n+1}}{d_{n+1}+\frac{\delta(a)_n(b)_n}{(c)_n(n)!}},\quad (z\in U). \tag{21}$$

The results 20 and 21 are sharp with the functions given by 8 and 13, respectively.

Theorem 2.11. *If f of the form 1 with $b_1=0$, satisfies the condition 4 with*

$$c_k\geq\begin{cases}\frac{\delta k(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!} & \text{if}\quad k=2,3,...m.\\ \frac{kc_{m+1}}{m+1\frac{(a)_m(b)_m}{(c)_m(m)!}}\frac{(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!} & \text{if}\quad k=m+1,m+2,...\end{cases},$$

and

$$d_k\geq\begin{cases}\frac{\delta k(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!} & \text{if}\quad k=2,3,...n.\\ \frac{kc_{m+1}}{m+1\frac{(a)_m(b)_m}{(c)_m(m)!}}\frac{(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!} & \text{if}\quad \mathrm{k=n+1,n+2,...}\end{cases}.$$

Then

$$Re\left\{\frac{(f(z)*H(a,b;c;z))'}{(f_{m,n}(z)*H(a,b;c;z))'}\right\}\geq\frac{c_{m+1}-\frac{(m+1)\delta(a)_m(b)_m}{(c)_m(m)!}}{c_{m+1}},\ \text{forall } z\in U. \tag{22}$$

The result 22 is sharp with the function given by 8.

Theorem 2.12. *If f of the form 1 with $b_1=0$, satisfies the condition 4 with*

$$c_k\geq\begin{cases}\frac{\delta k(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!} & \text{if}\quad k=2,3,...m.\\ \frac{kc_{m+1}}{m+1\frac{(a)_m(b)_m}{(c)_m(m)!}}\frac{(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!} & \text{if}\quad k=m+1,m+2,...\end{cases}$$

and

$$d_k\geq\begin{cases}\frac{\delta k(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!} & \text{if}\quad k=2,3,...n.\\ \frac{kc_{m+1}}{m+1\frac{(a)_m(b)_m}{(c)_m(m)!}}\frac{(a)_{k-1}(b)_{k-1}}{(c)_{k-1}(k-1)!} & \text{if}\quad k=n+1,n+2,...\end{cases},$$

then

$$Re\left\{\frac{(f_{m,n}(z)*H(a,b;c;z))'}{(f(z)*H(a,b;c;z))'}\right\}\geq\frac{c_{m+1}}{c_{m+1}+\frac{(m+1)\delta(a)_m(b)_m}{(c)_m(m)!}},\ \text{for all } z\in U. \tag{23}$$

The result 23 is sharp with the function given by 8.

3. Some Consequences and Concluding Remarks

In this section we specifically point out the relevances of some of our main results with those results which have appeared recently in literature.

If we put $a = c,\ b = 1$ in Theorems 2.1 - 2.12, then we obtain the corresponding results of Porwal [20].

Next, if we put $a = c,\ b = 1,\ c_k = k - \alpha,\ d_k = k + \alpha,\ \delta = 1 - \alpha$ in Theorems 2.1 - 2.12, then we obtain the results of Porwal and Dixit [21].

Lastly, if we put $g \equiv 0,\ a = c,\ b = 1$, then we obtain the result of Frasin [13].

We mention below some corollaries giving sharp bounds of the real parts of the ratio of univalent functions to its sequences of partial sums.

By putting $a = c, b = 1$ in Theorem 2.1 for the function f of the form 2 with $c_k = k - \alpha$ and $\delta = 1 - \alpha$, then we obtain the following result of Silverman ([27], Theorem 1).

Corollary 3.1. *If f of the form 2 satisfies the condition 5 with $c_k = k - \alpha$, then*

$$Re\left\{\frac{f(z)}{f_m(z)}\right\} \geq \frac{m}{m+1-\alpha},\ z \in U.$$

The result is sharp for every m, with the extremal function given by

$$f(z) = z + \frac{1-\alpha}{m+1-\alpha} z^{m+1}. \tag{24}$$

On the other hand, if we put $a = c, b = 1$ in Theorem 2.3 for the function f of the form 2 with $c_k = k - \alpha$ and $\delta = 1 - \alpha$, then we obtain the following result of Silverman ([27], Theorem 4(i)).

Corollary 3.2. *If f of the form 2 satisfies the condition 5 with $c_k = k - \alpha$, then for $z \in U$:*

$$Re\left\{\frac{f'(z)}{f'_m(z)}\right\} \geq \frac{\alpha m}{m+1-\alpha}.$$

The result is sharp for every m, with the extremal function given by 24.

Also, if we put $a = c, b = 1$ in Theorem 2.1 for the function f of the form 2 belonging to the class $S_\phi(c_k, \delta)$, then we obtain the following result of Frasin [13].

Corollary 3.3. *If $f \in S_\phi(c_k, \delta)$, then*

$$Re\left\{\frac{f\left(z\right)}{f_m\left(z\right)}\right\} \geq \frac{c_{m+1} - \delta}{c_{m+1}}, \qquad (z \in U), \tag{25}$$

where $c_k \geq \begin{cases} \delta & if \quad k = 2, 3, ..., n \\ c_{m+1} & if \quad k = m+1, m+2, ... \end{cases}$.

The result is sharp for every m, with the extremal function given by

$$f(z) = z + \frac{\delta}{c_{m+1}} z^{m+1}. \tag{26}$$

Again, if we set $a = c, b = 1$ in Theorem 2.1, then we obtain the following result of Porwal [20].

Corollary 3.4. *If f of the form 1 with* $b_1 = 0$*, satisfies the condition 4 with*

$$c_k \geq \begin{cases} \delta & if \quad k = 2, 3, ..., m \\ c_{m+1} & if \quad k = m+1, m+2... \end{cases},$$

then

$$Re\left\{\frac{f(z)}{f_m(z)}\right\} \geq \frac{c_{m+1} - \delta}{c_{m+1}}, \quad \text{for all } z \in U. \tag{27}$$

The result 27 is sharp with the function given by 26.

We conclude this paper by considering some specific values for the parameters a, b, c involved in $H(a, b; c; z)$, so that it reduces to a familiar special function. By making use of these values in Theorems 2.1 - 2.12, we can obtain various interesting results. To illustrate, we consider some of the cases of our theorems as follows.

By choosing $a = -n, b = n+1, c = 1$ in $H(a, b; c; z)$, so that

$$H(a, b; c; z) = zF(-n, n+1; 1; z) = zP_n(1 - 2z),$$

where $P_n(z)$ is Legendre polynomial (see [3]).

Using this specialization, Theorem 2.1 yields the following result.

Corollary 3.5. *If f of the form 1 with* $b_1 = 0$*, satisfies the condition 4 with*

$$c_k \geq \begin{cases} \frac{\delta(-n)_{k-1}(n+1)_{k-1}}{(1)_{k-1}(k-1)!} & if \quad k = 2, 3, ..., m \\ \frac{c_{m+1}}{\frac{(-n)_m(n+1)_m}{(1)_m m!}} \frac{(-n)_{k-1}(n+1)_{k-1}}{(1)_{k-1}(k-1)!} & if \quad k = m+1, m+2, ..., n \end{cases},$$

then

$$Re\left\{\frac{f(z) * zP_n(1-2z)}{f_m(z) * zP_n(1-2z)}\right\} \geq \frac{c_{m+1} - \frac{\delta(-n)_m(n+1)_m}{(1)_m(m)!}}{c_{m+1}}, \quad \text{for all } z \in U, \tag{28}$$

provided that $n > m$.

The result 28 is sharp with the function given by

$$f(z) = z + \frac{\delta}{c_{m+1}} z^{m+1}. \tag{29}$$

Again, if we put $a = -n, b = 1 + \alpha + \beta + n; c = 1 + \alpha$, then we have

$$H(a, b; c; z) = zF(-n, 1+\alpha+\beta+n; 1+\alpha; z) = z\frac{n!}{(1+\alpha)_n} P_n^{(\alpha,\beta)}(1-2z) = JP(\alpha, \beta, n; z),$$

where $P_n^{(\alpha,\beta)}(z)$ is the well known Jacobi polynomial.

Theorem 2.1 with these above specializations gives the following result.

Corollary 3.6. *If f of the form 1 with $b_1 = 0$, satisfies the condition 4 with*

$$c_k \geq \begin{cases} \frac{\delta(-n)_{k-1}(1+\alpha+\beta+n)_{k-1}}{(1+\alpha)_{k-1}(k-1)!} & if \quad k = 2, 3, ..., m \\ \frac{c_{m+1}}{\frac{(-n)_m(1+\alpha+\beta+n)_m}{(1+\alpha)_m m!}} \frac{(-n)_{k-1}(1+\alpha+\beta+n)_{k-1}}{(1+\alpha)_{k-1}(k-1)!} & if \quad k = m+1, m+2, ..., n \end{cases},$$

then

$$Re\left\{\frac{f(z) * JP(\alpha, \beta, n; z)}{f_m(z) * JP(\alpha, \beta, n; z)}\right\} \geq \frac{c_{m+1} - \frac{\delta(-n)_m(1+\alpha+\beta+n)_m}{(1+\alpha)_m(m)!}}{c_{m+1}}, \quad \text{for all } z \in U, \tag{30}$$

provided that $n > m$.

The result 30 is sharp with the function given by 29.

Similarly, if $a = -n, b = 1, c = 1$, then

$$H(a, b; c; z) = zF(-n, 1; 1; z) = zL_n(z),$$

where $L_n(z)$ is a Laguerre polynomial. Based on these substitutions, Theorem 2.1 yields the following result.

Corollary 3.7. *If f of the form 1 with $b_1 = 0$, satisfies the condition 4 with*

$$c_k \geq \begin{cases} \frac{\delta(-n)_{k-1}}{(k-1)!} & if \quad k = 2, 3, ..., m \\ \frac{c_{m+1}}{\frac{(-n)_m}{m!}} \frac{(-n)_{k-1}}{(k-1)!} & if \quad k = m+1, m+2, ..., n \end{cases},$$

then

$$Re\left\{\frac{f(z) * zL_n(z)}{f_m(z) * zL_n(z))}\right\} \geq \frac{c_{m+1} - \frac{\delta(-n)_m}{m!}}{c_{m+1}}, \quad \text{for all } z \in U, \tag{31}$$

provided that $n > m$.

The result 31 is sharp with the function given by 29.

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References

[1] Afaf A. Ali Abubaker and M. Darus, *Partial sums of analytic functions involving generalized Cho-Kwon-Srivastava operator*, Int. J. Open Prob. Compl. Anal., **2** (3) (2010), 181-188.

[2] O.P. Ahuja, *Planar harmonic univalent and related mappings*, J. Inequal. Pure Appl. Math., **6** (4) (2005), Art. 122, 1-18.

[3] G.E. Andrews, R. Askey and R. Roy, *Special Functions*, Cambridge University Press, Cambridge, (1999).

[4] J. Clunie and T. Sheil-Small, *Harmonic univalent functions*, Ann. Acad. Sci. Fen. Series AI Math., **9** (1984), 3-25.

[5] K.K. Dixit and Saurabh Porwal, *A convolution approach on partial sums of certain analytic and univalent functions*, J. Inequal. Pure Appl. Math., **10** (4) (2009), Art.101, 1-17.

[6] K.K. Dixit and Saurabh Porwal, *On a subclass of harmonic univalent functions*, J. Inequal. Pure Appl. Math., 10 (1) (2009), Art. 27, 1-18.

[7] K.K. Dixit and Saurabh Porwal, *An application of fractional calculus to harmonic univalent functions*, Bull. Cal. Math. Soc., **102** (4), (2010), 343-352.

[8] K.K. Dixit and Saurabh Porwal, *Some properties of harmonic functions defined by convolution*, Kyungpook Math. J., **49** (4) (2009), 751-761.

[9] K.K. Dixit and Saurabh Porwal, *Salagean-type harmonic univalent functions with negative coefficients*, J. Raj. Acad. Phy. Sci., **8** (2), (2009), 147-158.

[10] P. Duren, *Harmonic mappings in the plane*, Cambridge University Press, Cambridge, (2004).

[11] B.A. Frasin, *Comprehensive family of harmonic univalent functions*, SUT J. Math., **42** (1) (2006), 145-155.

[12] B.A. Frasin, *Partial sums of certain analytic and univalent functions*, Acta Math. Acad. Paed. Nyir., **21** (2005), 135-145.

[13] B.A. Frasin, *Generalization of partial sums of certain analytic and univalent functions*, Appl. Math. Lett., **21** (7) (2008), 735-741.

[14] J.M. Jahangiri, *Harmonic functions starlike in the unit disk*, J. Math. Anal. Appl., **235** (1999), 470-477.

[15] G. Murugusundaramoorthy, K. Uma and M. Darus, *Partial sums of generalized class of analytic functions involving Hurwitz-Lerch zeta function*, Abstract Appl. Anal., (2011), Article ID 849250, 1-9.

[16] H. Orhan, E. Gunes and A. Kilicman, *Neighborhoods and partial sums of analytic functions based on Gaussian hypergeometic functions*, in S.S. Dragomir and A. Sofo, Advances in Inequalities for series (RGMIA Book series), 109-218.

[17] M. Öztürk, S. Yalcin and M. Yamankaradeniz, *Convex subclass of harmonic starlike functions*, Appl. Math. Comput., **154** (2004), 449-459.

[18] S. Ponnusamy and A. Rasila, *Planar harmonic mappings*, RMS Mathematics Newsletters, **17** (2) (2007), 40-57.

[19] S. Ponnusamy and A. Rasila, *Planar harmonic and quasi-conformal mappings*, RMS Mathematics Newsletters, **17** (3) (2007), 85-101.

[20] Saurabh Porwal, *Partial sums of certain harmonic univalent functions*, Lobachevskii J. Math., **32** (4) (2011), 366-375.

[21] Saurabh Porwal and K.K. Dixit, *Partial sums of starlike harmonic univalent functions*, Kyungpook Math. J., **50** (3) (2010), 433-445.

[22] Saurabh Porwal, K.K. Dixit, Vinod Kumar, A.L. Pathak and P. Dixit, *A new subclass of harmonic univalent functions defined by Dziok-Srivastava Operator*, Advan. Theor. Appl. Math., **5** (1), (2010), 109-119.

[23] R.K. Raina and Deepak Bansal, *Some properties of a new class of analytic functions defined in terms of a Hadamard product*, J. Inequl. Pure Appl. Math., **9** (1) (2008), Art. 22, 1-20.

[24] R.K. Raina and P. Sharma, *Harmonic univalent functions associated with Wright's generalized hypergeometric functions*, Int. Trans. Spec. Funct., **22** (8) (2011), 561-572.

[25] T. Rosy, B.A. Stephen, K.G. Subramanian and J.M. Jahangiri, *Goodman-Rønning-type harmonic univalent functions*, Kyungpook Math. J., **41**(1) (2001), 45-54.

[26] T. Rosy, K.G. Subramanian and G. Murugusundaramoorthy, *Neighbourhoods and partial sums of starlike based on Ruscheweyh derivatives*, J. Inequl. Pure Appl. Math., **4** (4) (2003), Art. 64, 1-8.

[27] H. Silverman, *Partial sums of starlike and convex functions*, J. Math. Anal. Appl., **209** (1997), 221-227.

[28] H. Silverman, *Harmonic univalent functions with negative coefficients*, J. Math. Anal. Appl., **220** (1998), 283-289.

[29] H. Silverman and E.M. Silvia, *Subclasses of harmonic univalent functions*, New Zealand J. Math., **28** (1999), 275-284.

[30] E.M. Silvia, *On partial Sums of convex functions of order* α, Houston J. Math., **11** (3) (1985), 397-404.

[31] A. Swaminathan, *Certain sufficiency conditions on Gaussian hypergeometric function*, J. Inequal. Pure Appl. Math., **5** (4) (2004), Art. 83, 1-10.

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