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THE HARDY-LITTLEWOOD-SOBOLEV INEQUALITY FOR GENERALIZED RIESZ POTENTIALS

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Abstract. In this study, the inequality of Hardy-Littlewood-Sobolev type is established for generalized Riesz potentials depending on the generalized $\lambda-$distance.

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1. Introduction

It is well known that the Hardy-Littlewood-Sobolev inequality for the classical Riesz potential [9]. Çınar studied this inequality for Riesz potential with the kernel depending on $\lambda-$distance [2]. On the other hand, Yıldırım gave the Hardy-Littlewood-Sobolev inequality for the generalized Riesz potential generated by the generalized shift operator [13]. Different problems for convolution type integrals with the kernel depending on $\lambda-$distance were studied in [1]-[4],[7],[10]-[12],[14],[16].

In this paper, we have defined the generalized Riesz potential generated by the $\lambda-$distance and the generalized shift operator, and we have studied the Hardy-Littlewood-Sobolev inequality for this potential.

Firstly we give some notations and definitions.

Let $\lambda_1, \lambda_2, ..., \lambda_n$ be positive numbers with $|\lambda| = \lambda_1 + \lambda_2 + ... + \lambda_n$ and for $R_n^+ = \{x : x = (x_1, x_2, ..., x_n), x_1 > 0, x_2 > 0, ..., x_n > 0\}$, $x, y \in R_n^+$

$$|x-y|_\lambda := (|x_1 - y_1|^{\frac{1}{\lambda_1}} + |x_2 - y_2|^{\frac{1}{\lambda_2}} + ... + |x_n - y_n|^{\frac{1}{\lambda_n}})^{\frac{|\lambda|}{n}}. \qquad (1)$$

The expression $|x-y|_\lambda$ is called the $\lambda-$distance between the points x and y. It can be seen that for $\lambda_i = \frac{1}{2}, i = 1, 2, ..., n$ the $\lambda-$distance become ordinary Euclidean distance $|x-y|$. For a positive number ρ and $x \in R_n^+$ define $\rho^\lambda x = (\rho^{\lambda_1} x_1, ..., \rho^{\lambda_n} x_n)$. Then we have

1. $|x|_\lambda = 0 \Longleftrightarrow x = \theta$
2. $\left|\rho^\lambda x\right| = \rho^{\frac{|\lambda|}{n}} |x|_\lambda$
3. $|x-y|_\lambda \leq C(|x|_\lambda + |y|_\lambda)$

where $C = 2^{\left(1+\frac{1}{\lambda_{\min}}\right)\frac{|\lambda|}{n}}$, $\lambda_{\min} = \min\{\lambda_1, \lambda_2, ..., \lambda_n\}$.

It is known the generalized shift operator the following equality

$$T^y_{x_1,...,x_n} f(x) := [\prod_{i=1}^n \frac{\Gamma(\nu_i+\frac{1}{2})}{\Gamma(\nu_i)\Gamma(\frac{1}{2})}] \int_0^\pi \dots \int_0^\pi f\left(\sqrt{x_1^2+y_1^2-2x_1y_1\cos\varphi_1}, ... \right.$$
$$\left. \dots, \sqrt{x_n^2+y_n^2-2x_ny_n\cos\varphi_n}\right) (\prod_{i=1}^n \sin^{2\nu_i-1}\varphi_i d\varphi_i)$$

as in [5], [8], [13], [15].

Now we define the generalized translation operator generated by the generalized $\lambda-$distance and the generalized shift operator as

$$\begin{aligned}(T^y_x)_\lambda |x|_\lambda := \quad & C_\nu \int_0^\pi ... \int_0^\pi \Big[(x_1^2+y_1^2-2x_1y_1\cos\varphi_1)^{\frac{1}{2\lambda_1}} + ... \\ & + (x_n^2+y_n^2-2x_ny_n\cos\varphi_n)^{\frac{1}{2\lambda_n}}\Big]^{\frac{|\lambda|}{n}} \left(\prod_{i=1}^n \sin^{\frac{\nu_i}{\lambda_i}-1}\varphi_i \, d\varphi_i\right)\end{aligned} \tag{2}$$

where $C_\nu = \pi^{-\frac{n}{2}}[\prod_{i=1}^n \frac{\Gamma(\frac{\nu_i+\lambda_i}{2\lambda_i})}{\Gamma(\frac{\nu_i}{2\lambda_i})}]$, $\nu_1 > 0, \nu_2 > 0, ..., \nu_n > 0$ and $|\nu| = \nu_1 + \nu_2 + ... + \nu_n$. In the equality (2) if we take $\lambda_i = \frac{1}{2}$, $i = 1, 2, ..., n$, we obtained the generalized shift operator which is given in [6],[13],[15].

$L_{p,\nu,\lambda} := L_{p,\nu,\lambda}(R_n^+)$ is defined with respect to the Lebesque- Stieljes measure $(\prod_{i=1}^n x_i^{\frac{\nu_i}{\lambda_i}})dx$ (It is clear the Lebesque- Stieljes mesure is no invariant in translation. But we never are using such as properties of measure) as follows [13],[15]:

$$L_{p,\nu,\lambda} = L_{p,\nu,\lambda}(R_n^+) = \left\{ f : \|f\|_{p,\nu,\lambda} = \left(\int_{R_n^+} |f(x)|^p (\prod_{i=1}^n x_i^{\frac{\nu_i}{\lambda_i}}) dx \right)^{\frac{1}{p}} < \infty \right\}, 1 \leq p < \infty.$$

We also define $B_{\nu,\lambda}-$ convolution operator as

$$(f * K)(x) := \int_{R_n^+} f(y) \, (T^y_x)_\lambda \, K(x) \left(\prod_{i=1}^n y^{\frac{\nu_i}{\lambda_i}}\right) \, dx.$$

Now we define the following $B_{\nu,\lambda}-$convolution type operator which is obtained by the $\lambda-$distance and the generalized shift operator:

$$\left(I_{\nu,\lambda}^{\alpha}\ f\right)(x) := \int\limits_{R_n^+} f(y)T_x^y(|x|^{\alpha-\frac{n}{|\lambda|}(|\lambda|+|\nu|)})\left(\prod_{i=1}^{n} y_i^{\frac{\nu_i}{\lambda_i}}\right)dy, \quad 0<\alpha<\tfrac{n(|\lambda|+|\nu|)}{|\lambda|}. \quad (3)$$

$I_{\nu,\lambda}^{\alpha}f$ is called a generalized Riesz Potentials generated by the $\lambda-$distance and the generalized shift operator. For $\lambda_i = \frac{1}{2}$, $i = 1,2,...,n$, we have the Riesz Potential generated by the generalize shift operator which is given in [5],[13],[15]. It can be seen that for $\lambda_i = \frac{1}{2}$ and $\varphi_i = 1$, $i = 1,2,...,n$ the generalized Riesz potential generated by the $\lambda-$distance and the generalized shift operator become the classical Riesz potential. We show that the generalized Riesz potential generated by the $\lambda-$distance and the generalized shift operator has a weak $(p,q)-$type for some p and q in the sense of [9]. It means, there exist a positive constant $C_{p,q,\nu,\lambda}$ independent on function f such that for any $\beta>0$ the inequality

$$mes\{x : |(I_{\nu,\lambda}^{\alpha}f)(x)| > \beta\} \leq \left(C_{p,q,\nu,\lambda}\frac{\|f\|_{p,\nu,\lambda}}{\beta}\right)^q \quad (4)$$

is hold. Here, $mesE := \int\limits_E \left(\prod_{i=1}^{n} x_i^{\frac{\nu_i}{\lambda_i}}\right)dx$, $E\subset R_n^+$.

In this study, we consider spherical coordinates by the following formulas:

$$x_1 = (\rho\cos\varphi_1)^{2\lambda_1}, ..., x_n = (\rho\sin\varphi_1\sin\varphi_2 ... \sin\varphi_{n-1})^{2\lambda_n}$$

we obtained that $|x|_\lambda = \rho^{\frac{2|\lambda|}{n}}$. It can be seen that the Jacobian $J(\rho,\varphi)$ of this transformation is $J(\rho,\varphi) = \rho^{2|\lambda|-1}\Omega(\varphi)$, where $\Omega(\varphi)$ is the bounded function, which depend only on angles $\varphi_1, \varphi_2, ..., \varphi_{n-1}$.

Lemma 1. *There are the following properties for the* $(T_x^y)_\lambda |x|_\lambda$,

i. $(T_x^y)_\lambda .1 = 1$

ii.$|(T_x^y)_\lambda |x|_\lambda|^p \leq (T_x^y)_\lambda |x|_\lambda^p$, $\frac{1}{p}+\frac{1}{p'}=1$, $1<p<\infty$.

Proof: **i.** From the definition of $(T_x^y)_\lambda$ and equality

$$\int\limits_0^\pi \sin^{\frac{\nu_i}{\lambda_i}-1}\varphi_i d\varphi_i = \frac{\Gamma(\frac{\nu_i}{2\lambda_i})\Gamma(\frac{1}{2})}{\Gamma(\frac{\nu_i}{2\lambda_i}+\frac{1}{2})}$$

we have

$$(T_x^y)_\lambda .1 = C_\nu \int\limits_0^\pi ... \int\limits_0^\pi \left(\prod_{i=1}^{n}\sin^{\frac{\nu_i}{\lambda_i}-1}\varphi_i\ d\varphi_i\right) = C_\nu.\frac{1}{C_\nu} = 1$$

ii. From Hölder's inequality and (**i**), we have

$$
\begin{aligned}
&\left|(T_x^y)_\lambda |x|_\lambda\right|^p = \left| C_\nu \int_0^\pi ... \int_0^\pi \Psi(x,y,\alpha) \left(\prod_{i=1}^n \sin^{\frac{\nu_i}{\lambda_i}-1} \varphi_i \, d\varphi_i \right) \right|^p \\
&\leq \left(C_\nu \int_0^\pi ... \int_0^\pi \Psi^p(x,y,\alpha) \left(\prod_{i=1}^n \sin^{\frac{\nu_i}{\lambda_i}-1} \varphi_i \, d\varphi_i \right) \right) \left(C_\nu \int_0^\pi ... \int_0^\pi \left(\prod_{i=1}^n \sin^{\frac{\nu_i}{\lambda_i}-1} \varphi_i \, d\varphi_i \right) \right)^{\frac{p}{p'}} \\
&\leq (T_x^y)_\lambda |x|_\lambda^p
\end{aligned}
$$

where $\Psi(x,y,\alpha) = \left[(x_1^2 + y_1^2 - 2x_1 y_1 \cos\varphi_1)^{\frac{1}{2\lambda_1}} + ... + (x_n^2 + y_n^2 - 2x_n y_n \cos\varphi_n)^{\frac{1}{2\lambda_n}} \right]^{\frac{|\lambda|}{n}}$.

Remark 1. Let $x_i, y_i \in R^+,\ i = 1, 2, ..., n$. In this case there is the following inequality for the generalized translation operator generated by the $\lambda-$distance and the generalized shift operator.

$$
\begin{aligned}
&(x_i - y_i)^2 \leq x_i^2 + y_i^2 - 2x_i y_i \cos\varphi_i \leq (x_i + y_i)^2 \\
&|x_i - y_i|^{\frac{1}{\lambda_i}} \leq (x_i^2 + y_i^2 - 2x_i y_i \cos\varphi_i)^{\frac{1}{2\lambda_i}} \leq (x_i + y_i)^{\frac{1}{\lambda_i}} \\
&|x - y|_\lambda \leq (T_x^y)_\lambda |x|_\lambda \leq |x + y|_\lambda
\end{aligned}
$$

where $\varphi_i \in [0, \pi]$.

Now, we prove the following Hardy-Littlewood-Sobolev type theorem for potential $I_{\nu,\lambda}^\alpha f$.

Theorem 1. *Let* $1 \leq p < q < \infty$, $2^{\left(1+\frac{1}{\lambda_{\min}}\right)\frac{|\lambda|}{n}+1} |x|_\lambda \leq |y|_\lambda$ *and* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha|\lambda|}{n(|\nu|+|\lambda|)}$.

a. *If* $f \in L_{p,\nu,\lambda}(R_n^+)$, *then* $I_{\nu,\lambda}^\alpha f$ *is absolutely convergent almost everywhere.*

b. *If* $p > 1$, *then*

$$\|I_{\nu,\lambda}^\alpha f\|_{q,\nu,\lambda} \leq C_\alpha(p,q,\nu,\lambda) \|f\|_{p,\nu,\lambda} \tag{5}$$

c. *If* $f \in L_{p,\nu,\lambda}(R_n^+)$, *then* $I_{\nu,\lambda}^\alpha f$ *has weak* $(1,q)-$*type, where* $q = 1 - \frac{\alpha|\lambda|}{n(|\nu|+|\lambda|)}$.

Proof **.** First we assume that $K(x) = |x|^{\alpha - \frac{n}{|\lambda|}(|\nu|+|\lambda|)}$. Let us decompose K as $K_1 + K_\infty$, where

$$K_1(x) = \begin{cases} K(x) & \text{if } |x|_\lambda \leq \mu \\ 0 & \text{if } |x|_\lambda > \mu \end{cases}, \quad K_\infty(x) = \begin{cases} K(x) & \text{if } |x|_\lambda > \mu \\ 0 & \text{if } |x|_\lambda \leq \mu \end{cases}$$

and μ is a fixed positive constant which need not be specified. It is obvious that

$$\begin{aligned}\left(I^{\alpha}_{\nu,\lambda}\, f\right)(x) &= \int\limits_{R_n^+} f(y)\,(T_x^y)_\lambda\, K(x)\left(\prod_{i=1}^{n} y_i^{\frac{\nu_i}{\lambda_i}}\right)dy \\ &= \int\limits_{R_n^+} f(y)\,(T_x^y)_\lambda\, K_1(x)\left(\prod_{i=1}^{n} y_i^{\frac{\nu_i}{\lambda_i}}\right)dy \\ &+ \int\limits_{R_n^+} f(y)\,(T_x^y)_\lambda\, K_\infty(x)\left(\prod_{i=1}^{n} y_i^{\frac{\nu_i}{\lambda_i}}\right)dy \\ &= I_1(x)+I_2(x).\end{aligned} \tag{6}$$

If we apply the Hölder inequality to $I_1(x)$ with $pp' = p + p'$, then we obtain the following inequality

$$\int\limits_{R_n^+} I_1^p(x)\left(\prod_{i=1}^{n} y_i^{\frac{\nu_i}{\lambda_i}}\right)dy \leq \|\,(T_x^y)_\lambda\, K_1\|_{1,\nu,\lambda}^{\frac{p+p'}{p}}\|f\|_{p,\nu,\lambda}^{p}. \tag{7}$$

However, we obtain the following inequality for $\|\,(T_x^y)_\lambda\, K_1\|_{1,\nu,\lambda}$ by the Remark 1 and $2^{(1+\frac{1}{\lambda_{\min}})\frac{|\lambda|}{n}+1}\,|x|_\lambda \leq |y|_\lambda$.

$$\begin{aligned}\|\,(T_x^y)_\lambda\, K_1\|_{1,\nu,\lambda} &= \int\limits_{|y|_\lambda\leq\mu} (T_x^y)_\lambda\, |x|_\lambda^{\alpha-\frac{n}{|\lambda|}(|\nu|+|\lambda|)}\left(\prod_{i=1}^{n} y_i^{\frac{\nu_i}{\lambda_i}}\right)dy \\ &\leq \int\limits_{|y|_\lambda\leq\mu} |x-y|_\lambda^{\alpha-\frac{n}{|\lambda|}(|\nu|+|\lambda|)}\left(\prod_{i=1}^{n} y_i^{\frac{\nu_i}{\lambda_i}}\right)dy \\ &\leq C_1\int\limits_{|y|_\lambda\leq\mu} |y|_\lambda^{\alpha-\frac{n}{|\lambda|}(|\nu|+|\lambda|)}\left(\prod_{i=1}^{n} y_i^{\frac{\nu_i}{\lambda_i}}\right)dy \leq C_2\mu^{2|\lambda|\frac{\alpha}{n}}\end{aligned} \tag{8}$$

where C_2 is a constant depending on the $\Omega(\varphi)$ with respect to angles coordinates, α and λ. Since $f \in L_{p,\nu,\lambda}(R_n^+)$ and $\|\,(T_x^y)_\lambda\, K_1\|_{1,\nu,\lambda} < \infty$ we have

$$\|I_1\|_{p,\nu,\lambda} \leq C_2\mu^{2|\lambda|\frac{\alpha}{n}}\|f\|_{p,\nu,\lambda} < \infty. \tag{9}$$

The integral I_2 may be direct calculated by the Hölder inequality. Then we have

$$|I_2| \leq \|\,(T_x^y)_\lambda\, K_\infty\|_{p',\nu,\lambda}\|f\|_{p,\nu,\lambda}. \tag{10}$$

where $p' = \frac{p}{p-1}$. Moreover, $(\alpha - \frac{n}{|\lambda|}(|\nu| + |\lambda|))p' < -\frac{n}{|\lambda|}(|\nu| + |\lambda|)$ is equivalent to $q < \infty$ by $\frac{1}{q} = \frac{1}{p} - \frac{\alpha|\lambda|}{n(|\nu|+|\lambda|)}$. Now we show that $\|\,(T_x^y)_\lambda\, K_\infty\|_{p',\nu,\lambda}$ is finite. We have the following inequality by Lemma and Remark 1

$$\begin{aligned}
\| (T_x^y)_\lambda K_\infty\|_{p',\nu,\lambda} &= \left(\int\limits_{|y|_\lambda>\mu} \left[(T_x^y)_\lambda |x|_\lambda^{\alpha-\frac{n}{|\lambda|}(|\nu|+|\lambda|)} \right]^{p'} \left(\prod_{i=1}^n y_i^{\frac{\nu_i}{\lambda_i}} \right) dy \right)^{\frac{1}{p'}} \\
&\le \left(\int\limits_{|y|_\lambda>\mu} (T_x^y)_\lambda |x|_\lambda^{(\alpha-\frac{n}{|\lambda|}(|\nu|+|\lambda|))p'} \left(\prod_{i=1}^n y_i^{\frac{\nu_i}{\lambda_i}} \right) dy \right)^{\frac{1}{p'}} \\
&\le \left(\int\limits_{|y|_\lambda>\mu} |x-y|_\lambda^{(\alpha-\frac{n}{|\lambda|}(|\nu|+|\lambda|))p'} \left(\prod_{i=1}^n y_i^{\frac{\nu_i}{\lambda_i}} \right) dy \right)^{\frac{1}{p'}} \\
&\le C_3 \left(\left[\rho^{2p' \frac{|\lambda|}{n}[\alpha-\frac{n(|\nu|+|\lambda|)}{|\lambda|p}]} \right] \Bigg|_\mu^\infty \right)^{\frac{1}{p'}}.
\end{aligned}$$

Thus we get $\| (T_x^y)_\lambda K_\infty\|_{p',\nu,\lambda} < \infty$ by hypothesis

$$\frac{n}{|\lambda|}(|\nu|+|\lambda|) \left(\frac{\alpha|\lambda|}{n(|\nu|+|\lambda|)} - \frac{1}{p} \right) < 0.$$

This means that I_2 is also finite. Note that the last inequality follows from $\frac{1}{q} = \frac{1}{p} - \frac{\alpha|\lambda|}{n(|\nu|+|\lambda|)}$. From (6), (9) and (10) it follows that $I_{\nu,\lambda}^\alpha f$ is finite almost everywhere. Thus the part **(a)** of theorem is proved.

Now we prove the part **(c)**. Obviously, it is sufficient to prove this fact in case $\|f\|_{p,\nu,\lambda} = 1$ and with 2β replace β in (4).

Since $\left(I_{\nu,\lambda}^\alpha f \right)(x) = I_1(x) + I_2(x)$ in view of (6) we have the inequality

$$mes\{x : |(I_{\nu,\lambda}^\alpha f)(x)| > 2\beta\} \le mes\{x : |I_1(x)| > \beta\} + mes\{x : |I_2(x)| > \beta\}. \quad (11)$$

Consider the right side of (11) inequality. Denoting $E_1 = \{x : \ |I_1(x)| > \beta\}$, then we see that

$$mes\{x : |I_1(x)| > \beta\} \quad \le \int\limits_{E_1} \left(\frac{|I_1(x)|}{\beta} \right)^p \left(\prod_{i=1}^n x_i^{\frac{\nu_i}{\lambda_i}} \right) dx. \quad (12)$$

Applying the generalized Minkowsky inequality and using the definition of the kernel $K_1(x)$ we obtain

$$\int\limits_{E_1} \left(\frac{|I_1(x)|}{\beta} \right)^p \left(\prod_{i=1}^n x_i^{\frac{\nu_i}{\lambda_i}} \right) dx \le C_4 \mu^{2\alpha\frac{|\lambda|}{n}p}$$

where C_4 is a constant depending on p, ν, λ, α. Using this inequality in (12) we have

$$mes\{x : |I_1(x)| > \beta\} \le C_4 \left(\frac{\mu^{2\alpha\frac{|\lambda|}{n}}}{\beta}\right)^p. \tag{13}$$

Consider the second term in (11). Let $E_2 = \{x : \ |I_2(x)| > \beta\}$. Applying the Hölder inequality we see that the inequality

$$|I_2(x)| \le \|K_\infty\|_{p',\nu,\lambda} \|f\|_{p,\nu,\lambda} = C_5 \mu^{-n\frac{|\lambda|+|\nu|}{q}}.$$

Therefore choosing $\mu = \left(C_5^{-1}\beta\right)^{-\frac{q}{n(|\lambda|+|\nu|)}}$, then for all $x \in R_n^+$ $|I_1(x)| \le \infty$ and so $mes\{x : |I_2(x)| > \beta\} = 0$. By (11) and (13), we have

$$mes\{x : |(I_{\nu,\lambda}^\alpha f)(x) > 2\beta|\} \le C_5 \left(\frac{\|f\|_{p,\nu,\lambda}}{\beta}\right)^q.$$

where C_5 is a constant depending on p, q, ν, λ and α. Consequently, under condition $1 \le p < q < \infty$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha|\lambda|}{n(|\nu|+|\lambda|)}$, $(I_{\nu,\lambda}^\alpha f)(x)$ has a weak $(p,q)-$type.

b. To prove this part we use the Marcinkiewicz interpolation theorem [1]. By part **(c)** the operator $I_{\nu,\lambda}^\alpha f$ is the weak type-(p,q) where $\frac{1}{q} = \frac{1}{p} - \frac{\alpha|\lambda|}{n(|\nu|+|\lambda|)}$. In special case $p = 1$ this operator is the weak type-$(1,q)$ where $\frac{1}{q} = 1 - \frac{\alpha|\lambda|}{n(|\nu|+|\lambda|)}$. Using the Marcinkiewicz interpolation theorem between (p_0, q_0) and (p_1, q_1) where

$$p_0 = 1, q_0 = \left(1 - \frac{\alpha|\lambda|}{n(|\nu|+|\lambda|)}\right)^{-1}, p_1 = p_1, q_1 = \left(\frac{1}{p_1} - \frac{n(|\nu|+|\lambda|)}{\alpha|\lambda|}\right)^{-1}.$$

We have that for potential $I_{\nu,\lambda}^\alpha f$ holds (5) and $\frac{1}{q} = \frac{1}{p} + \frac{\alpha|\lambda|}{n(|\nu|+|\lambda|)}$. The proof is completed.

Remark 2. The conditions $1 \le p < q < \infty$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha|\lambda|}{n(|\nu|+|\lambda|)}$ are also the necessary for (5). To prove this we assume that (5) holds for every function $f \in L_{p,\nu,\lambda}(R_n^+)$ and consider the dilation operator $\Im_{\rho^\lambda}$ defined by

$$\Im_{\rho^\lambda}(f)(x) := f(\rho^\lambda x), \ \rho > 0$$

where $\rho^\lambda x = (\rho^{\lambda_1}x_1, \rho^{\lambda_2}x_2, ..., \rho^{\lambda_n}x_n)$ and $x, y \in R_n^+$. Then simple calculation show that

$$\begin{aligned}
&I. \quad \Im_{\rho^{-\lambda}}\left[I_{\nu,\lambda}^\alpha \Im_{\rho^\lambda} f\right](x) &&= \rho^{-\alpha\frac{|\lambda|}{n}} I_{\nu,\lambda}^\alpha f(x)\\
&II. \quad \left\|\Im_{\rho^\lambda} f\right\|_{p,\nu,\lambda} &&= \rho^{-\frac{|\lambda|+|\nu|}{p}} \|f\|_{p,\nu,\lambda}\\
&III. \quad \left\|\Im_{\rho^{-\lambda}}^\alpha I_{\nu,\lambda}^\alpha f\right\|_{q,\nu,\lambda} &&= \rho^{\frac{|\lambda|+|\nu|}{q}} \left\|I_{\nu,\lambda}^\alpha f\right\|_{q,\nu,\lambda}.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\left\|\Im_{\rho^{-\lambda}}^{-\alpha\frac{|\lambda|}{n}} I_{\nu,\lambda}^{\alpha} f\right\|_{q,\nu,\lambda} &= \left\|\Im_{\rho^{-\lambda}}\left[I_{\nu,\lambda}^{\alpha}\Im_{\rho^{\lambda}} f\right]\right\|_{q,\nu,\lambda} && \text{from } I \\
&= \rho^{\frac{|\lambda|+|\nu|}{q}}\left\|I_{\nu,\lambda}^{\alpha}\Im_{\rho^{\lambda}} f\right\|_{q,\nu,\lambda} && \text{from } III \\
&\leq C_{\alpha}(p,q,\nu,\lambda)\rho^{\frac{|\lambda|+|\nu|}{q}}\left\|\Im_{\rho^{\lambda}} f\right\|_{q,\nu,\lambda} && \text{from}(5) \\
&\leq C_{\alpha}(p,q,\nu,\lambda)\rho^{\frac{|\lambda|+|\nu|}{q}}\rho^{-\frac{|\lambda|+|\nu|}{p}}\|f\|_{q,\nu,\lambda} && \text{from } II
\end{aligned}$$

and so

$$\left\|I_{\nu,\lambda}^{\alpha} f\right\|_{q,\nu,\lambda} \leq C_{\alpha}(p,q,\nu,\lambda)\rho^{\frac{\alpha|\lambda|}{n}+(|\lambda|+|\nu|)(\frac{1}{q}-\frac{1}{p})}\|f\|_{p,\nu,\lambda}. \tag{14}$$

The contradiction, which can be obtained from this inequality when $\rho \to 0\left(\text{if } \frac{1}{q} > \frac{1}{p} - \frac{\alpha|\lambda|}{n(|\lambda|+|\nu|)}\right)$ and when $\rho \to \infty \left(\text{if } \frac{1}{q} < \frac{1}{p} - \frac{\alpha|\lambda|}{n(|\lambda|+|\nu|)}\right)$.

Show that (5) holds only for if $\frac{1}{q} = \frac{1}{p} - \frac{\alpha|\lambda|}{n(|\lambda|+|\nu|)}$. Note that (5) does not hold for $p = q$. Really from the (14) it may be see that in the case $p = q$

$$\left\|I_{\nu,\lambda}^{\alpha} f\right\|_{q,\nu,\lambda} \leq C_{\alpha}(p,q,\nu,\lambda)\rho^{\frac{\alpha|\lambda|}{n}}\|f\|_{p,\nu,\lambda}.$$

But this is possible only when $\alpha = 0$. That is the potential $I_{\nu,\lambda}^{0}$ can not acting from $L_{p,\nu,\lambda}(R_n^+)$ to $L_{q,\nu,\lambda}(R_n^+)$.

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