A MIXED MONTE CARLO AND QUASI-MONTE CARLO SEQUENCE FOR MULTIDIMENSIONAL INTEGRAL ESTIMATION

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Abstract. In this paper, we propose a method for estimating an s-dimensional integral I. We define a new hybrid sequence that we call the H-mixed sequence. We obtain a probabilistic bound for the H-discrepancy of this sequence. We define a new estimator for a multidimensional integral using the H-mixed sequence. We prove a central limit theorem for this estimator. We show that by using our estimator, we obtain asymptotically a smaller variance than by using the crude Monte Carlo method. We also compare our method with the Monte Carlo and Quasi-Monte Carlo methods on a numerical example.

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1. Introduction

We consider the problem of estimating integrals of the form

$$I = \int_{[0,1]^s} f(x)dH(x), \tag{1}$$

where $f : [0,1]^s \to \mathbb{R}$ is the function we want to integrate and $H : \mathbb{R}^s \to [0,1]$ is a distribution function on $[0,1]^s$. In the continuous case, the integral I can be rewritten as

$$I = \int_{[0,1]^s} f(x)h(x)dx,$$

where h is the density function corresponding to the distribution function H.

In the Monte Carlo (MC) method (see [11]), the integral I is estimated by sums of the form

$$\hat{I}_N = \frac{1}{N}\sum_{k=1}^{N} f(x_k),$$

where $x_k = (x_k^{(1)}, \dots, x_k^{(s)})$, $k \geq 1$, are independent identically distributed random points on $[0,1]^s$, with the common density function h.

In the Quasi-Monte Carlo (QMC) method (see [11]), the integral I is approximated by sums of the form $\frac{1}{N}\sum_{k=1}^{N} f(x_k)$, where $(x_k)_{k\geq 1}$ is a H-distributed low-discrepancy sequence on $[0,1]^s$.

In [7], [8] and [9], Okten considered integrals of the form

$$I_1 = \int_{[0,1]^s} f(x)dx,$$

and proposed a method for estimating the integral I_1, using a so-called *mixed sequence* on $[0,1]^s$, which combines pseudorandom and low-discrepancy vectors. Each element of the s-mixed sequence $(x_k)_{k\geq 1}$ is obtained by concatenating two vectors q_k and X_k, i.e., $x_k = (q_k, X_k)$, $k \geq 1$, where $(q_k)_{k\geq 1}$ is a d-dimensional low-discrepancy sequence and X_k, $k \geq 1$, are independent uniformly distributed random vectors on $[0,1]^{s-d}$.

In this paper, we extend the results obtained by Okten to the case when the integral is of the form (1). First, we remember some basic notions and definitions that will be used in this paper. Next, we define a new hybrid sequence that we call the *H-mixed sequence* and obtain probabilistic bounds for the H-discrepancy of this sequence. Continuing, we define a new estimator for the integral I, using our H-mixed sequence, and prove a central limit theorem for this estimator. In the last paragraph, we consider a numerical example, in which we compare our estimator with the ones obtained by using the MC and QMC methods.

2. A probabilistic bound for the H-discrepancy of the H-mixed sequence

We start this paragraph by recalling some useful notions (see [10]).

Definition 1. (H-discrepancy) *Consider an s-dimensional continuous distribution on $[0,1]^s$, with distribution function H. Let λ_H be the probability measure induced by H. Let $P=(x_k)_{k\geq 1}$ be a sequence of points in $[0,1]^s$. The H-discrepancy of the first N terms of sequence P is defined as*

$$D_{N,H}(x_1,\dots,x_N)=\sup_{J\subseteq[0,1]^s}\left|\frac{1}{N}A_N(J,P)-\lambda_H(J)\right|,$$

where the supremum is calculated over all subintervals $J=\prod_{i=1}^s[a_i,b_i]\subseteq[0,1]^s$; A_N counts the number of elements of sequence P, falling into the interval J, i.e.,

$$A_N(J,P)=\sum_{k=1}^N 1_J(x_k),$$

1_J is the characteristic function of J. The sequence P is called H-distributed if $D_{N,H}(x_1,\dots,x_N)\to 0$ as $N\to\infty$. The H-distributed sequence P is said to be a low-discrepancy sequence if

$$D_{N,H}(x_1,\dots,x_N)=\mathcal{O}\big((\log N)^s/N\big)\quad\text{for all } N\geq 2.$$

Definition 2. *Consider an s-dimensional continuous distribution on $[0,1]^s$, with density function h and distribution function H. For a point $u=\big(u^{(1)},\dots,u^{(s)}\big)\in[0,1]^s$, the marginal density functions h_l, $l=1,\dots,s$, are defined by*

$$h_l\big(u^{(l)}\big)=\underbrace{\int\dots\int}_{[0,1]^{s-1}}h\big(t^{(1)},\dots,t^{(l-1)},u^{(l)},t^{(l+1)},\dots t^{(s)}\big)dt^{(1)}\dots dt^{(l-1)}dt^{(l+1)}\dots dt^{(s)},$$

and the marginal distribution functions H_l, $l=1,\dots,s$, are defined by

$$H_l\big(u^{(l)}\big)=\int_0^{u^{(l)}}h_l(t)dt.$$

In this paper, we consider s-dimensional continuous distributions on $[0,1]^s$, with independent marginals, i.e.,

$$H(u)=\prod_{l=1}^s H_l(u^{(l)}),\ \forall u=(u^{(1)},\dots,u^{(s)})\in[0,1]^s.$$

This can be expressed, using the marginal density functions, as follows:

$$h(u) = \prod_{l=1}^{s} h_l(u^{(l)}),\ \forall u = (u^{(1)}, \ldots, u^{(s)}) \in [0,1]^s.$$

Consider an integer $0 < d < s$. Using the marginal density functions, we construct the following density functions on $[0,1]^d$ and $[0,1]^{s-d}$, respectively:

$$h_q(u) = \prod_{l=1}^{d} h_l(u^{(l)}),\ \forall u = (u^{(1)}, \ldots, u^{(d)}) \in [0,1]^d,$$

and

$$h_X(u) = \prod_{l=d+1}^{s} h_l(u^{(l)}),\ \forall u = (u^{(d+1)}, \ldots, u^{(s)}) \in [0,1]^{s-d}.$$

The corresponding distribution functions are

$$H_q(u) = \int_0^{u^{(1)}} \cdots \int_0^{u^{(d)}} h_q\big(t^{(1)}, \ldots, t^{(d)}\big) dt^{(1)} \ldots dt^{(d)}, \tag{2}$$

where $u = (u^{(1)}, \ldots, u^{(d)}) \in [0,1]^d$, and

$$H_X(u) = \int_0^{u^{(d+1)}} \cdots \int_0^{u^{(s)}} h_X\big(t^{(d+1)}, \ldots, t^{(s)}\big) dt^{(d+1)} \ldots dt^{(s)}, \tag{3}$$

where $u = (u^{(d+1)}, \ldots, u^{(s)}) \in [0,1]^{s-d}$.

In the following definition, we introduce the new notion of a H-mixed sequence.

Definition 3. (H-mixed sequence) *Consider an s-dimensional continuous distribution on $[0,1]^s$, with distribution function H and independent marginals H_l, $l = 1, \ldots, s$. Let H_q and H_X be the distribution functions defined in (2) and (3), respectively. Let $(q_k)_{k\geq 1}$ be a H_q-distributed low-discrepancy sequence on $[0,1]^d$, with $q_k = (q_k^{(1)}, \ldots, q_k^{(d)})$, and X_k, $k \geq 1$, be independent and identically distributed random vectors on $[0,1]^{s-d}$, with distribution function H_X, where $X_k = (X_k^{(d+1)}, \ldots, X_k^{(s)})$. A sequence $(m_k)_{k\geq 1}$, with the general term given by*

$$m_k = (q_k, X_k), \quad k \geq 1, \tag{4}$$

is called a ***H-mixed sequence*** *on* $[0,1]^s$.

Remark 4. For an interval $J = \prod_{l=1}^{s}[a_l, b_l] \subseteq [0,1]^s$, we define the subintervals $J' = \prod_{l=1}^{d}[a_l, b_l] \subseteq [0,1]^d$ and $J'' = \prod_{l=d+1}^{s}[a_l, b_l] \subseteq [0,1]^{s-d}$ (i.e. $J = J' \times J''$).

Let $(m_k)_{k\geq 1}$ be a H-mixed sequence on $[0,1]^s$, with the general term given by (4). Based on definitions (1) and (3), the H-discrepancy of the set of points $(m_1, \ldots, m_N)$ can be expressed as

$$\begin{aligned}
D_{N,H}(m_1, \ldots, m_N) &= \sup_{J\subseteq[0,1]^s} \left| \frac{1}{N}\sum_{k=1}^{N} 1_J(m_k) - \lambda_H(J) \right| \\
&= \sup_{J\subseteq[0,1]^s} \left| \frac{1}{N}\sum_{k=1}^{N} 1_J(m_k) - \int_J dH(u) \right| \\
&= \sup_{J\subseteq[0,1]^s} \left| \frac{1}{N}\sum_{k=1}^{N} 1_J(m_k) - \prod_{l=1}^{s}[H_l(b_l) - H_l(a_l)] \right|,
\end{aligned}$$

and the H_q-discrepancy of the set of points $(q_1, \ldots, q_N)$ is given by

$$\begin{aligned}
D_{N,H_q}(q_1, \ldots, q_N) &= \sup_{J'\subseteq[0,1]^d} \left| \frac{1}{N}\sum_{k=1}^{N} 1_{J'}(q_k) - \lambda_{H_q}(J') \right| \\
&= \sup_{J'\subseteq[0,1]^d} \left| \frac{1}{N}\sum_{k=1}^{N} 1_{J'}(q_k) - \prod_{l=1}^{d}[H_l(b_l) - H_l(a_l)] \right|,
\end{aligned}$$

We consider the random variable $1_J(m_k)$ that is taking two values 1 and 0, with the probabilities deduced as follows:

$$\begin{aligned}
P(1_J(m_k) = 1) &= 1_{J'}(q_k)P(X_k \in J'') \\
&= 1_{J'}(q_k)P(X_k^{(d+1)} \in [a_{d+1}, b_{d+1}], \ldots, X_k^{(s)} \in [a_s, b_s]) \\
&= 1_{J'}(q_k)P(X_k^{(d+1)} \in [a_{d+1}, b_{d+1}]) \cdot \ldots \cdot P(X_k^{(s)} \in [a_s, b_s]) \\
&= 1_{J'}(q_k) \prod_{l=d+1}^{s} [H_l(b_l) - H_l(a_l)] \\
&= 1_{J'}(q_k)p,
\end{aligned}$$

where the product $\prod_{l=d+1}^{s}[H_l(b_l)-H_l(a_l)]$ was denoted by p. Thus, $P(1_J(m_k) = 0) = 1 - 1_{J'}(q_k)p$. Hence, the distribution of the random variable $1_J(m_k)$ is

$$1_J(m_k) : \begin{pmatrix} 1 & 0 \\ 1_{J'}(q_k) \cdot p & 1 - 1_{J'}(q_k) \cdot p \end{pmatrix}, \quad k \geq 1. \tag{5}$$

Lemma 5. *The random variable $1_J(m_k)$ has the expectation and the variance given by*

$$\begin{aligned} E(1_J(m_k)) &= 1_{J'}(q_k)p, && (6) \\ Var(1_J(m_k)) &= 1_{J'}(q_k)p(1-p). && (7) \end{aligned}$$

Furthermore,

$$Cov(1_J(m_i) \cdot 1_J(m_j)) = 0 \quad \text{for} \quad i,j \geq 1, \quad i \neq j. \tag{8}$$

Proof. We have

$$\begin{aligned} E(1_J(m_k)) &= 1 \cdot 1_{J'}(q_k)p + 0 \cdot (1 - 1_{J'}(q_k) \cdot p) \\ &= 1_{J'}(q_k)p. \end{aligned}$$

For the variance, we obtain

$$\begin{aligned} Var(1_J(m_k)) &= E(1_J(m_k)^2) - (E(1_J(m_k)))^2 \\ &= 1_{J'}(q_k)p - 1_{J'}(q_k)p^2 \\ &= 1_{J'}(q_k)p(1-p). \end{aligned}$$

The distribution of the product $1_J(m_i) \cdot 1_J(m_j)$ is

$$1_J(m_i) \cdot 1_J(m_j) : \begin{pmatrix} 1 & 0 \\ 1_{J'}(q_i)1_{J'}(q_j) \cdot p^2 & 1 - 1_{J'}(q_i)1_{J'}(q_j) \cdot p^2 \end{pmatrix}.$$

Hence, we get

$$E(1_J(m_i) \cdot 1_J(m_j)) = p^2 \cdot 1_{J'}(q_i)1_{J'}(q_j).$$

From this, the covariance is

$$Cov(1_J(m_i) \cdot 1_J(m_j)) \quad = \quad p^2 \cdot 1_{J'}(q_i) 1_{J'}(q_j) - 1_{J'}(q_i) p \cdot 1_{J'}(q_j) p = 0.$$

Corollary 6. *Let $1_J(m_k)$ be the random variable defined in (5). Then, we have*

$$E\Big(\frac{1}{N}\sum_{k=1}^{N} 1_J(m_k)\Big) = \frac{p}{N}\sum_{k=1}^{N} 1_{J'}(q_k), \tag{9}$$

$$Var\Big(\frac{1}{N}\sum_{k=1}^{N} 1_J(m_k)\Big) = \frac{p(1-p)}{N^2}\sum_{k=1}^{N} 1_{J'}(q_k), \tag{10}$$

$$Var\Big(\frac{1}{N}\sum_{k=1}^{N} 1_J(m_k) - \prod_{l=1}^{s}[H_l(b_l) - H_l(a_l)]\Big) \leq \frac{1}{4N}[D_{N,H_q}(q_k) + 1]. \tag{11}$$

Proof. The first two relations follow immediately from Lemma 5. For the last relation, we have

$$\begin{aligned} Var\Big(\frac{1}{N}\sum_{k=1}^{N} 1_J(m_k) - \prod_{l=1}^{s}[H_l(b_l) - H_l(a_l)]\Big) &= \frac{p(1-p)}{N^2}\sum_{k=1}^{N} 1_{J'}(q_k) \\ &= \frac{p(1-p)}{N}\sum_{k=1}^{N} \frac{1_{J'}(q_k)}{N} \\ &\leq \frac{1}{4N}[D_{N,H_q}(q_k) + \\ &\quad \prod_{l=1}^{d}[H_l(b_l) - H_l(a_l)]] \\ &\leq \frac{1}{4N}[D_{N,H_q}(q_k) + 1]. \end{aligned}$$

We introduce the following notations:

$$Z = \frac{1}{N}\sum_{k=1}^{N} 1_J(m_k), \tag{12}$$

and

$$u(J) = Z - \lambda_H(J), \tag{13}$$

where

$$\lambda_H(J) = \prod_{l=1}^{s} [H_l(b_l) - H_l(a_l)]. \tag{14}$$

We need the following lemma.

LEMMA 7. *If* $P(|u(J)| < a) \geq p$, *then* $P(sup_{J \subseteq [0,1]^s} |u(J)| < a) \geq p$.

Proof We have

$$|u(J)| < a, \ \forall J \subseteq [0,1]^s.$$

Hence, we get

$$sup_{J \subseteq [0,1]^s} |u(J)| < a.$$

For the corresponding events, we have the implication

$$\{|u(J)| < a\} \subseteq \{sup_{J \subseteq [0,1]^s} |u(J)| < a\}.$$

Finally, using a well-known inequality from probability theory, we obtain

$$p \leq P(|u(J)| < a) \leq P(sup_{J \subseteq [0,1]^s} |u(J)| < a).$$

Thus, we get the desired inequality

$$P(sup_{J \subseteq [0,1]^s} |u(J)| < a) \geq p.$$

Our main result, which gives a probabilistic error bound for the H-mixed sequences, is presented in the next theorem.

THEOREM 8. *If* $(m_k)_{k \geq 1} = (q_k, X_k)_{k \geq 1}$ *is a* H*-mixed sequence, then* $\forall \varepsilon > 0$ *we have*

$$P(D_{N,H}(m_k) \leq \varepsilon + D_{N,H_q}(q_k)) \geq 1 - \frac{1}{\varepsilon^2} \frac{1}{4N} \Big(D_{N,H_q}(q_k) + 1 \Big). \tag{15}$$

Proof. We apply the Chebyshev's Inequality for the random variable $u(J)$, defined in (13), and obtain

$$P(|u(J) - E(u(J)| < \varepsilon) \geq 1 - \frac{Var(u(J))}{\varepsilon^2}, \tag{16}$$

which is equivalent to

$$P(-\varepsilon+E(u(J))<u(J)<\varepsilon+E(u(J)))\geq 1-\frac{Var(u(J))}{\varepsilon^2}$$
$$\geq 1-\frac{1}{\varepsilon^2}\frac{1}{4N}\Big(D_{N,H_q}(q_k)+1\Big). \quad (17)$$

We have the following chain of implications:

$$-\varepsilon+E(u(J))<u(J)<\varepsilon+E(u(J))\Rightarrow$$

$$|u(J)|\leq \max(|-\varepsilon+E(u(J))|,|\varepsilon+E(u(J))|)\Rightarrow$$
$$|u(J)|\leq \sup_{J\subseteq[0,1]^s}\max(|-\varepsilon+E(u(J))|,|\varepsilon+E(u(J))|)\Rightarrow$$
$$|u(J)|\leq \max(\varepsilon+D_{N,H_q}(q_k),\varepsilon+D_{N,H_q}(q_k))\Rightarrow$$
$$|u(J)|\leq \varepsilon+D_{N,H_q}(q_k). \quad (18)$$

From relations (17) and (18), we obtain

$$P(|u(J)|\leq\varepsilon+D_{N,H_q}(q_k))\geq 1-\frac{1}{\varepsilon^2}\frac{1}{4N}\Big(D_{N,H_q}(q_k)+1\Big). \quad (19)$$

We apply Lemma 7 and get

$$P(sup_{J\subseteq[0,1]^s}|u(J)|\leq\varepsilon+D_{N,H_q}(q_k))\geq 1-\frac{1}{\varepsilon^2}\frac{1}{4N}\Big(D_{N,H_q}(q_k)+1\Big). \quad (20)$$

Hence, we obtain the inequality that we have to prove:

$$P(D_{N,H}(m_k)\leq\varepsilon+D_{N,H_q}(q_k))\geq 1-\frac{1}{\varepsilon^2}\frac{1}{4N}\Big(D_{N,H_q}(q_k)+1\Big). \quad (21)$$

Remark 9. We know that if $(q_k)_{k\geq 1}$ is a H_q-distributed low-discrepancy sequence on $[0,1]^d$, then

$$D_{N,H_q}(q_k)=O\Big(\frac{(logN)^d}{N}\Big).$$

Hence

$$D_{N,H_q}(q_k)\leq c_d\Big(\frac{(logN)^d}{N}\Big).$$

If we consider $\varepsilon = \frac{1}{\sqrt[4]{N}}$ in Theorem 8 then, as $N \to \infty$, we have

$$\frac{D_{N,H_q}(q_k)+1}{4\sqrt{N}} \leq \frac{c_d\left(\frac{(logN)^d}{N}\right)+1}{4\sqrt{N}} \longrightarrow 0.$$

In conclusion, when $N \to \infty$, we have

$$D_{N,H}(m_k) \longrightarrow 0, \tag{22}$$

with probability

$$p_1 = 1 - \frac{D_{N,H_q}(q_k)+1}{4\sqrt{N}} \longrightarrow 1. \tag{23}$$

3. An estimator for a multidimensional integral using H-mixed sequences

In order to estimate integrals of the form (1), we introduce the following estimator, which extends the one given by Okten (see [9]).

Definition 10. *Let $(m_k)_{k\geq 1} = (q_k, X_k)_{k\geq 1}$ be an s-dimensional H-mixed sequence, introduced by us in Definition 3, with $q_k = (q_k^{(1)}, \dots, q_k^{(d)})$ and $X_k = (X_k^{(d+1)}, \dots, X_k^{(s)})$. We define the following estimator for the integral I:*

$$\theta_m = \frac{1}{N}\sum_{k=1}^{N} f(m_k). \tag{24}$$

We consider the independent random variables:

$$Y_k = f(m_k) = f(q_k^{(1)}, \dots, q_k^{(d)}, X_k^{(d+1)}, \dots, X_k^{(s)}), \quad k \geq 1. \tag{25}$$

We denote the expectation of Y_k by

$$E(Y_k) = \mu_k, \tag{26}$$

and the variance of Y_k by

$$Var(Y_k) = \sigma_k^2. \tag{27}$$

We assume that

$$0 < \sigma_k^2 < \infty, \tag{28}$$

and we denote

$$0 < \sigma_{(N)}^2 = \sigma_1^2 + \ldots + \sigma_N^2 < \infty. \tag{29}$$

Remark 11. The estimator θ_m, defined in relation (24), is a biased estimator of the integral I, its convergence to I being asymptotic

$$E(\theta_m) \to I, \ as \ N \to \infty.$$

Proposition 12. *We assume that f is bounded on $[0,1]^s$ and that the functions*

$$f_1(x^{(1)}, \ldots, x^{(d)}) = \int_{[0,1]^{s-d}} (f(x^{(1)}, \ldots, x^{(s)}))^2 \prod_{l=d+1}^{s} h_l(x^{(l)}) dx^{(d+1)} \cdot \ldots \cdot dx^{(s)},$$

$$f_2(x^{(1)}, \ldots, x^{(d)}) = \Big[\int_{[0,1]^{s-d}} f(x^{(1)}, \ldots, x^{(s)}) \prod_{l=d+1}^{s} h_l(x^{(l)}) dx^{(d+1)} \cdot \ldots \cdot dx^{(s)} \Big]^2$$

are Riemann integrable. Then, we have

$$\frac{\sigma_{(N)}^2}{N} \longrightarrow L, \ as \ N \longrightarrow \infty,$$

where

$$L = \int_{[0,1]^s} (f(x^{(1)}, \ldots, x^{(s)}))^2 \prod_{l=1}^{s} h_l(x^{(l)}) dx^{(1)} \ldots dx^{(s)} -$$

$$- \int_{[0,1]^d} \Big[\int_{[0,1]^{s-d}} f(x^{(1)}, \ldots, x^{(s)}) \prod_{l=d+1}^{s} h_l(x^{(l)}) dx^{(d+1)} \cdot \ldots \cdot dx^{(s)} \Big]^2 \cdot$$

$$\cdot \prod_{l=1}^{d} h_l(x^{(l)}) dx^{(1)} \ldots dx^{(d)}.$$

Proof. From the fact that f_1 is Riemann integrable, it follows that

$$\begin{aligned}
&\frac{1}{N}\sum_{k=1}^{N} f_1(q_k^{(1)},\dots,q_k^{(d)}) \longrightarrow \int_{[0,1]^d} f_1(x^{(1)},\dots,x^{(d)})\prod_{l=1}^{d} h_l(x^{(l)})dx^{(1)}\dots dx^{(d)} = \\
= &\int_{[0,1]^d}\Big[\int_{[0,1]^{s-d}}(f(x^{(1)},\dots,x^{(s)}))^2\prod_{l=d+1}^{s} h_l(x^{(l)})dx^{(d+1)}\cdot\ldots\cdot dx^{(s)}\Big]\cdot \\
&\cdot\prod_{l=1}^{d} h_l(x^{(l)})dx^{(1)}\dots dx^{(d)}.
\end{aligned}$$

As f_2 is Riemann integrable, we obtain

$$\begin{aligned}
&\frac{1}{N}\sum_{k=1}^{N} f_2(q_k^{(1)},\dots,q_k^{(d)}) \longrightarrow \int_{[0,1]^d} f_2(x^{(1)},\dots,x^{(d)})\prod_{l=1}^{d} h_l(x^{(l)})dx^{(1)}\dots dx^{(d)} = \\
= &\int_{[0,1]^d}\Big[\int_{[0,1]^{s-d}}f(x^{(1)},\dots,x^{(s)})\prod_{l=d+1}^{s} h_l(x^{(l)})dx^{(d+1)}\cdot\ldots\cdot dx^{(s)}\Big]^2\cdot \\
&\prod_{l=1}^{d} h_l(x^{(l)})dx^{(1)}\dots dx^{(d)}.
\end{aligned}$$

We also know that

$$\begin{aligned}
&Var(Y_k) = \sigma_k^2 = Var(f(m_k)) = Var(f(q_k^{(1)},\dots,q_k^{(d)},X_k^{(d+1)},\dots,X_k^{(s)})) \\
= &\int_{[0,1]^{s-d}}(f(q_k^{(1)},\dots,q_k^{(d)},x^{(d+1)},\dots,x^{(s)}))^2\prod_{l=d+1}^{s} h_l(x^{(l)})dx^{(d+1)}\cdot\ldots\cdot dx^{(s)} - \\
- &\Big[\int_{[0,1]^{s-d}}f(q_k^{(1)},\dots,q_k^{(d)},x^{(d+1)},\dots,x^{(s)})\prod_{l=d+1}^{s} h_l(x^{(l)})dx^{(d+1)}\cdot\ldots\cdot dx^{(s)}\Big]^2 \\
= &f_1(q_k^{(1)},\dots,q_k^{(d)}) - f_2(q_k^{(1)},\dots,q_k^{(d)}).
\end{aligned}$$

Hence, we get

$$\begin{aligned}
\frac{\sigma_{(N)}^2}{N} &= \frac{\sigma_1^2+\ldots+\sigma_N^2}{N} \\
&= \frac{\sum_{k=1}^{N} f_1(q_k^{(1)},\dots,q_k^{(d)})}{N} - \frac{\sum_{k=1}^{N} f_2(q_k^{(1)},\dots,q_k^{(d)})}{N} \longrightarrow L,\ as\ N\longrightarrow\infty.
\end{aligned}$$

Now, we formulate and prove the main result of this paragraph.

THEOREM 13. *In the same hypothesis as in Proposition 12 and, in addition, assuming that $L \neq 0$, we have*

a)

$$Y_{(N)} = \frac{\sum_{k=1}^{N} Y_k - \sum_{k=1}^{N} \mu_k}{\sigma_{(N)}} \longrightarrow Y, \quad \textit{as } N \to \infty, \tag{30}$$

where the random variable Y has the standard normal distribution.

b) If we denote the crude Monte Carlo estimator for the integral (1) by θ_{MC}, then

$$Var(\theta_m) \leq Var(\theta_{MC}), \tag{31}$$

meaning that, by using our estimator, we obtain asymptotically a smaller variance than by using the classical Monte Carlo method.

Proof. a) As f is bounded, it follows that the random variables

$$Y_k = f(m_k), \ k \geq 1, \tag{32}$$

are also bounded.

From the facts that $L \neq 0$, $(\sigma^2_{(N)})_{N \geq 1}$ is a strictly increasing sequence, and

$$\lim_{N \to \infty} \frac{\sigma^2_{(N)}}{N} = L,$$

it follows that

$$\lim_{N \to \infty} \sigma^2_{(N)} = \infty. \tag{33}$$

Applying Corollary 5, page 267, from Ciucu [3] for the sequence of independent non-identically distributed random variables $(Y_k)_{k \geq 1}$, which are bounded and verify relation (33), it follows that

$$Y_{(N)} = \frac{\sum_{k=1}^{N} Y_k - \sum_{k=1}^{N} \mu_k}{\sigma_{(N)}} \longrightarrow Y \in \mathrm{N}(0,1) \quad \text{when } N \to \infty,$$

meaning that the sequence of independent non-identically distributed random variables $(Y_k)_{k \geq 1}$ satisfies the central limit theorem.

b) We know that

$$Var(\theta_{MC}) = \frac{Var(f(X))}{N}, \tag{34}$$

and

$$Var(\theta_m) = \frac{\sigma^2_{(N)}}{N^2}. \tag{35}$$

Hence, the relation (31), which we have to prove, is equivalent to

$$L = \frac{\sigma^2_{(N)}}{N} \leq Var(f(X)). \tag{36}$$

We apply the Cauchy-Buniakovski inequality $[E(X)]^2 \leq E(X^2)$ (see [2]) and we obtain

$$\begin{aligned}
&\int_{[0,1]^d} \Big[\int_{[0,1]^{s-d}} f(x^{(1)}, \ldots, x^{(s)}) \prod_{l=d+1}^{s} h_l(x^{(l)}) dx^{(d+1)} \cdot \ldots \cdot dx^{(s)} \Big]^2 \cdot \\
&\cdot \prod_{l=1}^{d} h_l(x^{(l)}) dx^{(1)} \ldots dx^{(d)} \geq \\
\geq \; & \Big[\int_{[0,1]^d} \Big[\int_{[0,1]^{s-d}} f(x^{(1)}, \ldots, x^{(s)}) \prod_{l=d+1}^{s} h_l(x^{(l)}) dx^{(d+1)} \cdot \ldots \cdot dx^{(s)} \Big] \cdot \\
&\cdot \prod_{l=1}^{d} h_l(x^{(l)}) dx^{(1)} \ldots dx^{(d)} \Big]^2 = \\
= \; & \Big[\int_{[0,1]^s} f(x^{(1)}, \ldots, x^{(s)}) \prod_{l=1}^{s} h_l(x^{(l)}) dx^{(1)} \ldots dx^{(s)} \Big]^2
\end{aligned}$$

Multiplying with (-1) the above inequality and adding $\int_{[0,1]^s} (f(x^{(1)}, \ldots, x^{(s)}))^2 \cdot$ $\cdot \prod_{l=1}^{s} h_l(x^{(l)}) dx^{(1)} \ldots dx^{(s)}$, we obtain

$$L \leq Var(f(X)).$$

So, the inequality (31) is proved.

4. Numerical example

In this paragraph, we compare our method with the MC and QMC methods on a numerical example. We want to estimate the following s-dimensional integral ($s \geq 1$):

$$I = \int_{[0,1]^s} 18x^{(1)}x^{(2)} \cdot \ldots \cdot x^{(s-1)}(x^{(s)})^2 e^{x^{(1)}x^{(s)}} dH(x), \tag{37}$$

where we consider the distribution function

$$H(x^{(1)}, x^{(2)}, \ldots, x^{(s)}) = (x^{(1)})^2(x^{(2)})^2 \ldots (x^{(s)})^2$$

on $[0,1]^s$. The corresponding density function is $h(x^{(1)}, x^{(2)}, \ldots, x^{(s)}) = 2^s x^{(1)}x^{(2)} \ldots x^{(s)}$. The marginal distribution functions are

$$H_1(x^{(1)}) = (x^{(1)})^2,\ H_2(x^{(2)}) = (x^{(2)})^2,\ \ldots\ , H_s(x^{(s)}) = (x^{(s)})^2.$$

and their inverses are $H_i^{-1}(x^{(i)}) = \sqrt{(x^{(i)})},\ i = 1, \ldots, s$.

The exact value of the integral I is

$$\frac{18(3e-8)2^s}{3^{s-2}}.$$

In the following, we compare numerically our estimator θ_m, with the estimators obtained using the MC and QMC methods. As a measure of comparison, we will use the relative errors produced by these three methods.

The MC estimate is defined as follows:

$$\theta_{MC} = \frac{1}{N}\sum_{k=1}^{N} f(x_k^{(1)}, \ldots, x_k^{(s)}), \tag{38}$$

where $x_k = (x_k^{(1)}, \ldots, x_k^{(s)})$, $k \geq 1$, are independent identically distributed random points on $[0,1]^s$, with the common distribution function H.

In order to generate such a point x_k, we proceed as follows. We first generate a random point $\omega_k = (\omega_k^{(1)}, \ldots, \omega_k^{(s)})$, where $\omega_k^{(i)}$ is a point uniformly distributed on $[0,1]$, for each $i = 1, \ldots, s$. Then, for each component $\omega_k^{(i)}$, $i = 1, \ldots, s$, we apply the inversion method and obtain that $H_i^{-1}(\omega_k^{(i)})$ is a point with the distribution function H_i. As the s-dimensional distribution with the distribution function H has independent marginals, it follows that $x_k = (H_1^{-1}(\omega_k^{(1)}), \ldots, H_s^{-1}(\omega_k^{(s)}))$ is a point on $[0,1]^s$, with the distribution function H.

The QMC estimate is defined as follows:

$$\theta_{QMC} = \frac{1}{N} \sum_{k=1}^{N} f(x_k^{(1)}, \ldots, x_k^{(s)}), \tag{39}$$

where $(x_k)_{k \geq 1} = (x_k^{(1)}, \ldots, x_k^{(s)})_{k \geq 1}$ is a H-distributed low-discrepancy sequence on $[0,1]^s$.

In order to generate such a sequence $(x_k)_{k \geq 1}$, we first consider a low-discrepancy sequence on $[0,1]^s$, $\omega = (\omega_k^{(1)}, \ldots, \omega_k^{(s)})_{k \geq 1}$. Then, by applying the inverses of the marginals on each dimension, we obtain that $(x_k)_{k \geq 1} = (H_1^{-1}(\omega_k^{(1)}), \ldots, H_s^{-1}(\omega_k^{(s)}))_{k \geq 1}$ is a H-distributed low-discrepancy sequence on $[0,1]^s$ (see [10]).

During our experiments, we employed as low-discrepancy sequences on $[0,1]^s$ the Halton sequences (see [6]).

The estimate proposed by us earlier is:

$$\theta_m = \frac{1}{N} \sum_{k=1}^{N} f(q_k^{(1)}, \ldots, q_k^{(d)}, X_k^{(d+1)}, \ldots, X_k^{(s)}). \tag{40}$$

where $(q_k, X_k)_{k \geq 1}$ is an s-dimensional H-*mixed sequence* on $[0,1]^s$.

In order to obtain such a H-mixed sequence, we first construct the H_q-distributed low-discrepancy sequence $(q_k)_{k \geq 1}$ on $[0,1]^d$ (the distribution function H_q was defined in (2)) and next, the independent and identically distributed random points x_k, $k \geq 1$ on $[0,1]^{s-d}$, with the common distribution function H_X (the distribution function H_X was defined in (3)). Finally, we concatenate q_k and x_k for each $k \geq 1$.

In our experiments, we used as low-discrepancy sequences on $[0,1]^d$ for the H-mixed sequences, the Halton sequences (see [6]).

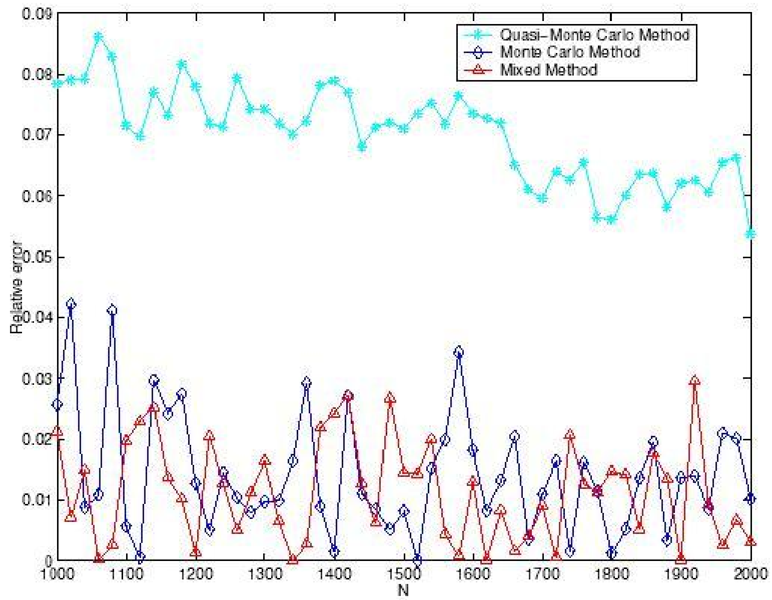

Figure 1: Simulation results for $s = 10$ and $d = 3$.

In our tests, we have considered the following dimensions of the integral I: $s = 8$, 10, 11, 12, 14, 15. We present the numerical results for a number of 51 samples, having sizes from $N = 1000$ to $N = 2000$, with a step size of 20. We have also changed the dimension of the deterministic part of the H-mixed sequence from $d = 3$ to $d = 8$, in order to determine an "optimal" dimension for the deterministic part of the H-mixed sequence.

The MC and mixed estimates are the mean values obtained in 10 independent runs, while the QMC estimate is the result of a single run. In our graphs the relative error produced using each of the three methods is plotted against the number of samples N.

We present the numerical results for $s = 10$ and $d = 3$ in Figure 1, and for $s = 15$ and $d = 5$ in Figure 2. We observe that the H-mixed sequences outperform their low-discrepancy versions, giving much better results.

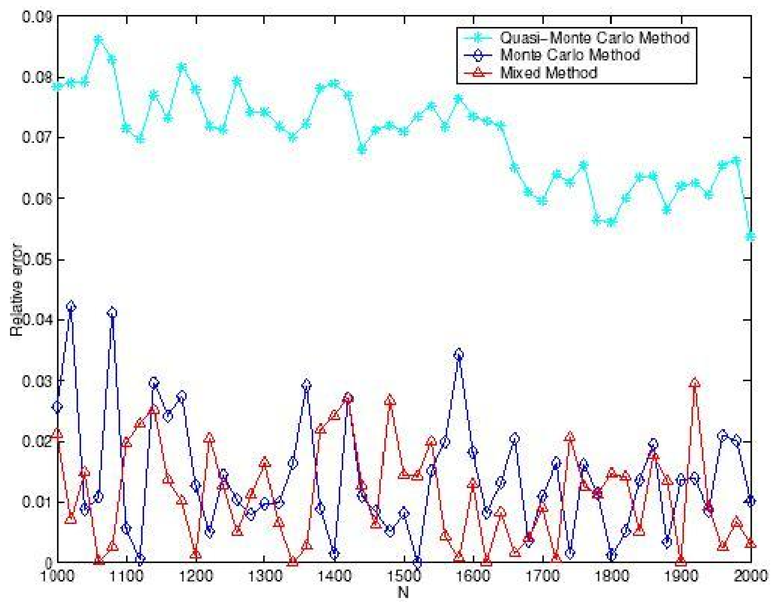

Figure 2: Simulation resu1ts for $s = 15$ and $d = 5$.

Also, the H-mixed sequences produced a better overall error reduction than the pseudorandom sequences, in both situations we present here. We also remark that, in order to achieve these improvements using H-mixed sequences, the dimension of the deterministic part should be around one third of the problem dimension. Considering a higher dimension of the deterministic component leads us to worse results compared with the MC method. However, the relative errors are still much smaller than the ones produced by using low-discrepancy sequences.

Next, we draw the following conclusions from all the tests we performed, considering the parameters (s, d) presented above:

1. the relative errors for all three estimates are very small, even for small sample sizes,

2. the behavior of the H-mixed sequence is superior to the one of the low-discrepancy sequence, regardless of the dimension of the problem,

3. the performance of the H-mixed sequence is better than the one of the pseudorandom sequence, for most of the sample sizes,

4. to achieve a good performance of our H-mixed estimator, we observe from our tests that the dimension of the deterministic part should be around one third of the dimension of the problem. However, this is not a general rule and is depending on the test problem or practical situation, where the H-mixed sequence is applied.

In conclusion, by properly chosen the dimension d of the deterministic part in the s-dimensional H-mixed sequence, we can achieve considerable improvements in error reduction, compared with the MC and QMC estimates, even for high dimensions and moderate sample sizes.

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