ACTA UNIVERSITATIS APULENSIS No 10/2005

Proceedings of the *International Conference on Theory and Application of Mathematics and Informatics ICTAMI 2005* - Alba Iulia, Romania

AN CONTINUATION RESULTS FOR FREDHOLM INTEGRAL EQUATIONS ON LOCALLY CONVEX SPACES

Adela Chis

Abstract. The continuation method is used to investigate the existence of solutions to Fredholm integral equations in locally convex spaces.

2000 *Mathematics Subject Classification*:45G,45N05,47H10,45B05.

1. Introduction

In this article we study the problem of the existence of solutions for the Fredholm integral equation

$$x(t) = \int_0^1 K(t,s,x(s))ds, \qquad t \in [0,1]. \tag{1}$$

where the functions x, K have values in a locally convex space.

In paper [8] the above equations are studied using fixed point theorems for self-maps. Our approach is based on the continuation method.

The results presented in this paper extend and complement those in [8]-[9].

We finish this section by stating the main result from [1] which will be used in the next section.

For a map $H : D \times [0,1] \to X$, where $D \subset X$, we will use the following notations:

$$\begin{array}{l} \Sigma = \{(x,\lambda) \in D \times [0,1] : H(x,\lambda) = x\}, \\ S = \{x \in D : H(x,\lambda) = x \text{ for some } \lambda \in [0,1]\}, \\ \Lambda = \{\lambda \in [0,1] : H(x,\lambda) = x \text{ for some } x \in D\}. \end{array} \tag{2}$$

THEOREM 1.*Let X be a set endowed with the separating gauge structures $\mathcal{P} = \{p_\alpha\}_{\alpha\in A}$ and $\mathcal{Q}^\lambda = \{q^\lambda_\beta\}_{\beta\in B}$ for $\lambda \in [0,1]$. Let $D \subset X$ be $\mathcal{P}$-sequentially closed, $H : D \times [0,1] \to X$ a map, and assume that the following conditions are satisfies:*

(i) for each $\lambda \in [0,1]$, there exists a function $\varphi_\lambda : B \to B$ and $a^\lambda \in [0,1)^B$, $a^\lambda = \{a^\lambda_\beta\}_{\beta\in B}$ such that

$$q^\lambda_\beta(H(x,\lambda), H(y,\lambda)) \le a^\lambda_\beta q^\lambda_{\varphi_\lambda(\beta)}(x,y), \tag{3}$$

$$\sum_{n=1}^{\infty} a^\lambda_\beta a^\lambda_{\varphi_\lambda(\beta)} a^\lambda_{\varphi^2_\lambda(\beta)} \ldots a^\lambda_{\varphi^{n-1}_\lambda(\beta)} q^\lambda_{\varphi^n_\lambda(\beta)}(x,y) < \infty \tag{4}$$

for every $\beta \in B$ and $x, y \in D$;

(ii) there exists $\rho > 0$ such that for each $(x,\lambda) \in \Sigma$, there is a $\beta \in B$ with

$$\inf\{q^\lambda_\beta(x,y) : y \in X\backslash D\} > \rho; \tag{5}$$

(iii) for each $\lambda \in [0,1]$, there is a function $\psi : A \to B$ and $c \in (0,\infty)^A$, $c = \{c_\alpha\}_{\alpha\in A}$ such that

$$p_\alpha(x,y) \le c_\alpha q^\lambda_{\psi(\alpha)}(x,y) \qquad \text{for all } \alpha \in A \text{ and } x, y \in X; \tag{6}$$

(iv) $(X,\mathcal{P})$ is a sequentially complete gauge space;

(v) if $\lambda \in [0,1]$, $x_0 \in D$, $x_n = H(x_{n-1},\lambda)$ for $n = 1,2,\ldots$, and $\mathcal{P}\text{-}\lim_{n\to\infty} x_n = x$, then $H(x,\lambda) = x$;

(vi) for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ with

$$q^\lambda_{\varphi^n_\lambda(\beta)}(x, H(x,\lambda)) \le (1 - a^\lambda_{\varphi^n_\lambda(\beta)})\varepsilon$$

for $(x,\mu) \in \Sigma$, $|\lambda - \mu| \le \delta$, all $\beta \in B$, and $n \in \mathbb{N}$.

In addition, assume that $H_0 := H(.,0)$ has a fixed point. Then, for each $\lambda \in [0,1]$, the map $H_\lambda := H(.,\lambda)$ has a unique fixed point.

REMARK 2.*Notice that, by condition (ii) we have: for each $(x,\lambda) \in \Sigma$, there is a $\beta \in B$ such that the set*

$$B(x,\lambda,\beta) = \{y \in X : q^\lambda_{\varphi^n_\lambda(\beta)}(x,y) \le \rho,\ \forall n \in \mathbb{N}\} \subset D. \tag{7}$$

The proof of Theorem 1, in [1], shows that the contraction condition (3) given on D, can be asked only on sets of the form (7), more exactely for $(x, \lambda) \in \Sigma$ *and* $y \in B(x, \lambda, \beta)$.

2.Existence Results

This section contains existence results for the equation (1).

Theorem 3.*Let E be a locally convex space, Hausdorff separated, complete by sequences, with the topology defined by the saturated and sufficient set of semi-norms* $\{|.|_\alpha, \alpha \in A\}$ *and let* $\delta > 0$ *be a fixed number. Assume that the following conditions are satisfied:*

(1) $K : [0,1]^2 \times E \to E$ *is continuous;*

(2) there exists $r = \{r_\alpha\}_{\alpha \in A}$ *such that, any solution x of the equation*

$$x(t) = \lambda \int_0^1 K(t, s, x(s))ds, \qquad t \in [0,1], \tag{8}$$

for some $\lambda \in [0,1]$ *satisfies* $|x(t)|_\alpha \leq r_\alpha$, *for all* $t \in [0,1]$ *and* $\alpha \in A$;

(3) there exists $\{L_\alpha\}_{\alpha \in A} \in [0,1)^A$ *such that*

$$|K(t,s,x) - K(t,s,y)|_\alpha \leq L_\alpha |x-y|_{f(\alpha)} \tag{9}$$

whenever $\alpha \in A$,*for all* $t, s \in [0,1]$ *and* $x, y \in E_r$ *where* $E_r = \{x \in E :$ *there exists* $\alpha \in A$ *such that* $|x|_\alpha \leq r_\alpha + \delta\}$;

(4)

$$\sum_{n=0}^{\infty} L_\alpha L_{f(\alpha)} ... L_{f^n(\alpha)} < \infty \tag{10}$$

for every $\alpha \in A$;

(5) for every $\alpha \in A$ *and for each continuous function* $g : [0,1] \to E$ *one has*

$$\sup\{|g(t)|_{f^n(\alpha)} : t \in [0,1],\, n = 0,1,2,...\} < \infty;$$

(6) there exists C with $0 < C \leq \dfrac{1 - L_{f^n(\alpha)}}{M_{f^n(\alpha)}}$ *for all* $\alpha \in A$ *and* $n \in \mathbb{N}$, *where* $M_\alpha := \sup\limits_{\substack{t,s \in [0,1], \\ |x|_{f(\alpha)} \leq r_{f(\alpha)}}} |K(t,s,x)|_\alpha$.

Then problem (1) has a solution.

Notice that $M_\alpha < \infty$. Indeed, from (9) we have

$$\begin{aligned} |K(t,s,x)|_\alpha &\le |K(t,s,x) - K(t,s,0)|_\alpha + |K(t,s,0)|_\alpha \\ &\le L_\alpha r_{f(\alpha)} + \max_{t,s\in[0,1]} |K(t,s,0)|_\alpha < \infty \end{aligned}$$

for all $t, s \in [0,1]$ and $x \in E$ with $|x|_{f(\alpha)} \le r_{f(\alpha)}$.

Proof. We shall apply Theorem 1. Let $X = C([0,1], X)$. For each $\alpha \in A$ we define the map $d_\alpha : X \times X \to \mathbb{R}_+$, by

$$d_\alpha(x,y) = \max_{t\in[0,1]} |x(t) - y(t)|_\alpha .$$

It is easy to show that d_α is a pseudo-metric on X and the family $\{d_\alpha\}_{\alpha\in A}$ defines on X a gauge structure, separated and complete by sequences.

Here $\mathcal{P} = \mathcal{Q}^\lambda = \{d_\alpha\}_{\alpha\in A}$ for every $\lambda \in [0,1]$. Let D be the closure in X of the set

$$\{x \in X : \ d_\alpha(x,0) \le r_\alpha + \delta \text{ for some } \alpha \in A\}.$$

We define $H : D \times [0,1] \to X$, by $H(x,\lambda) = \lambda A(x)$, where

$$A(x)(t) = \int_0^1 K(t,s,x(s))ds.$$

With this notation all the assumptions (i)-(vi) of Theorem 1 are satisfied and problem (1) has a solution.

For the detalies see the paper [2].

In Banach space, Theorem 3 becomes the following well-known result.

Corollary 4.*Let $(E, |.|)$ be a Banach space. Assume that the following conditions are satisfied:*

(1) $K : [0,1]^2 \times E \to E$ is continuous;

(2) there exists r > 0 *such that, any solution x of the equation*

$$x(t) = \lambda \int_0^1 K(t,s,x(s))ds, \qquad t \in [0,1], \tag{11}$$

for some $\lambda \in [0,1]$ satisfies $|x(t)| < r$, for all $t \in [0,1]$ and any $\lambda \in [0,1]$;

(3) there exists $L \in [0,1)$ *such that*

$$|K(t,s,x) - K(t,s,y)| \leq L\,|x-y| \tag{12}$$

for all $t,s \in [0,1]$ *and* $x,y \in E$ *with* $|x|\,,|y| \leq r$.

Then problem (1) has a solution.

Notice that an analogue result is true for Volterra integral equation

$$x(t) = \int_0^t K(t,s,x(s))ds, \qquad t \in [0,1]. \tag{13}$$

THEOREM 5.*Let* E *be a locally convex space, Hausdorff separated, complete by the sequences, with the topology defined by the saturated and sufficient set of semi-norms* $\{|.|_\alpha\,, \alpha \in A\}$ *and let* $\delta > 0$ *be a fixed number. Assume that the following conditions are satisfied:*

(1) $K : [0,1]^2 \times E \to E$ *is continuous;*

(2) there exists $r = \{r_\alpha\}_{\alpha \in A}$ *such that, any solution* x *of the equation*

$$x(t) = \lambda \int_0^t K(t,s,x(s))ds \qquad t \in [0,1]$$

for some $\lambda \in [0,1]$ *satisfies* $|x(t)|_\alpha \leq r_\alpha$*, for all* $t \in [0,1]$ *and* $\alpha \in A$*;*

(3) there exists $\{L_\alpha\}_{\alpha \in A} \in [0,1)^A$ *such that*

$$|K(t,s,x) - K(t,s,y)|_\alpha \leq L_\alpha\,|x-y|_{f(\alpha)}$$

whenever $\alpha \in A$*, for all* $t,s \in [0,1]$ *and* $x,y \in E_r$*;*

(4) $\sum_{n=0}^{\infty} L_\alpha L_{f(\alpha)} ... L_{f^n(\alpha)} < \infty, \qquad$ *for every* $\alpha \in A$*;*

(5) for every $\alpha \in A$ *and for each continuous function* $g : [0,1] \to E$*, one has*

$$\sup\{|g(t)|_{f^n(\alpha)} : t \in [0,1],\, n = 0,1,2,...\} < \infty;$$

(6) there exists C *with* $0 < C \leq \dfrac{1 - L_{f^n(\alpha)}}{M_{f^n(\alpha)}}$*, for all* $\alpha \in A$ *and* $n \in \mathbb{N}$*, where* $M_\alpha := \sup\limits_{\substack{t\in[0,1],\\ |x|_{f(\alpha)} \leq r_{f(\alpha)}}} |K(t,s,x)|_\alpha$.

Then, the problem (13) has a solution.

References

[1] A. Chis and R. Precup, *Continuation theory for general contractions in gauge spaces*, Fixed Point Theory and Applications **2004:**3 (2004), 173-185.

[2] A. Chis, *Continuation methods for integral equation in locally convex spaces,* to appear.

[3] J. Dugundji, Topology, Allyn and Bacon, Boston, 1966.

[4] M. Frigon, "*Fixed point results for generalized contractions in gauge space and applications",* Proc. Amer. Math. Soc. **128** (2000), 2957-2965.

[5] M. Frigon, *Fixed point results for generalized contractions in gauge space and applications,* Proc. Amer. Math. Soc. **128** (2000), 2957-2965.

[6] N. Gheorghiu, "*The contractions theorem in uniform spaces*", (Romanian), St. Cerc. Mat. **19** (1967), 131-135.

[7] N. Gheorghiu, "*Fixed point theorem in uniform spaces",* An. St. Univ. Al. I. Cuza Iasi **28** (1982), 17-18.

[8] N. Gheorghiu and M. Turinici , *Equation intégrales dans les espaces localment convexes,* Rev. Roumaine Math Pures Appl. **23** (1978), no. 1, 33-40 (French).

[9] D. O'Regan and R. Precup, *Theorems of Leray-Schauder Type and Applications*, Gordon and Breach Science Publishers, Amsterdam, 2001.

Author's Adela Chis
Department of Mathematics
University of Cluj-Napoca
Address:C. Daicoviciu no. 15, Cluj-Napoca, Romania
email:.Chis@math.utcluj.ro