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STABILITY ANALYSIS OF SHALLOW WAKE FLOWS WITH FREE SURFACE

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Shallow wake flows are flows behind obstacles where water depth is much smaller than the transverse scale of the flow. Limited water depth prevents three-dimensional instabilities and acts as a suppression factor thus limiting growth of perturbations. Hence the development of the shallow wakes is different from the wakes in deep water. Several authors have used the linear stability theory in order to understand when shallow flows become unstable. Two-dimensional depth-averaged Saint-Venant equations are usually used for the analysis under the following assumptions: 1) the independence of the velocity distribution on the vertical coordinate and 2) the independence of water depth h on the spatial coordinates x, y and time t (so-called "rigid-lid" assumption). However, errors introduced by both assumptions have not been evaluated. This paper presents the results of linear stability analysis of shallow wake flows with bottom friction performed with and without rigid-lid assumption. Momentum correction coefficients in the x and y directions are used in order to take the non-uniformity of the velocity distribution in the vertical direction into account. The stability of the flow is governed by the set of equations forming an eigenvalue problem. The linear stability of the base flow is determined by the real parts of the eigenvalues. If the real parts of all eigenvalues are positive, the flow is said to be linearly stable. On the other hand, if the real part of at least one eigenvalue is negative, the flow is said to be linearly unstable. The linear stability results are presented for the classic hyperbolic secant wake profile. The linear stability problem is solved by a pseudo spectral collocation method based on Chebyshev polynomials. It is shown that the stability boundary is quite sensitive to the variation of the momentum correction coefficients and the "rigid-lid" assumption.

In particular, neglecting of the momentum correction coefficients may lead to a systematic bias that underestimates critical transition values marking the boundary of instability.

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1. Introduction

Wake flows (flows behind obstacles, such as islands) are considered shallow if the transverse scale of the flow is much larger than the vertical scale (water depth). Experiments show that limited water depth has a strong influence on the development of flow instabilities. In particular, evolution of three-dimensional instabilities is prevented due to small vertical scale, but transverse growth of perturbations is hampered by bottom friction. As a result, development of wakes in shallow water is different from the ones in deep water.

However, vortex structures observed in shallow water in many cases resemble very much flow patterns in deep water. For example, photograph Nr. 173 by Van Dyke [6] shows formation of eddies organized into a vortex street behind an obstacle in shallow water although the Reynolds number for this case is 10^7 [6]. Note that vortex street pattern in unbounded flows is limited to much smaller Reynolds numbers.

The stability of shallow flows has been analyzed in literature both experimentally and theoretically [1], [2], [3], [5]. Two main assumptions are usually being made in order to facilitate the analysis under the shallow water model. The essence of the first assumption is that the free surface of the flow is not perturbed and acts as "rigid-lid" (so-called "rigid-lid" assumption). According to the second assumption, the velocity is supposed to be independent on vertical coordinate. This assumption results from the fact that the governing equations for shallow flow are the depth-averaged.

In some cases, however, the two assumptions may not be appropriate. Fluctuations of bottom friction coefficient and changes in flow geometry can result in appreciable deviation of the real flow from above-mentioned assumptions. In order to take into account the non-uniformity of velocity distribution, momentum correction coefficients were applied by several authors [8], [9].

The present paper makes an attempt to evaluate the influence of both "rigid-lid" and uniform velocity distribution assumptions on the stability analysis of shallow wake flows. Momentum correction coefficients are used in this paper in order to evaluate influence of non-uniformity of velocity distribution. The stability of the flow is analyzed for various values of the momentum correction coefficients. The Froude number is used in order to evaluate the influence of "rigid-lid" assumption on stability characteristics of shallow flows.

2.Problem Formulation

The governing equations for shallow flow can be obtained by integrating Euler equations with respect to vertical coordinate and have the form

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(uh) + \frac{\partial}{\partial y}(vh) = 0 \quad (1)$$

$$\frac{\partial u}{\partial t} + (2\beta_1 - 1)u\frac{\partial u}{\partial x} + [(\beta_1 - 1)\frac{u^2}{h} + g]\frac{\partial h}{\partial x} + (\beta_2 - 1)u\frac{\partial v}{\partial y} + $$
$$+(\beta_2 - 1)\frac{uv}{h}\frac{\partial h}{\partial y} + \beta_2 v\frac{\partial u}{\partial y} - S_{ox} + \frac{c_f u\sqrt{u^2+v^2}}{2h} - F(y) = 0 \quad (2)$$

$$\frac{\partial v}{\partial t} + \beta_2 u\frac{\partial v}{\partial x} + (\beta_2 - 1)v\frac{\partial u}{\partial x} + (\beta_2 - 1)\frac{uv}{h}\frac{\partial h}{\partial x} + (2\beta_3 - 1)v\frac{\partial v}{\partial y} + $$
$$+[(\beta_3 - 1)\frac{v^2}{h} + g]\frac{\partial h}{\partial y} - S_{oy} + \frac{c_f v\sqrt{u^2+v^2}}{2h} = 0 \quad (3)$$

where x and y are spatial coordinates, t is time, u and v are depth-averaged velocity components in the x and y directions respectively, h is water depth, g is acceleration due to gravity, $F(y)$ is the forcing function, $S_{0x} = -\partial z_b(x,y)/\partial x$ and $S_{0y} = -\partial z_b(x,y)/\partial y$ are the bed slopes, z_b is distance from the bottom, c_f is the friction coefficient defined by the equation

$$\frac{1}{\sqrt{c_f}} = A_s + B_s ln(R\sqrt{c_f})$$

where A_s and B_s are coefficients defined in [7]. Shear stress at the boundary is modelled by the Chezy formula $\tau_{wx} = \frac{1}{2}c_f\rho u\sqrt{u^2+v^2}$ and $\tau_{wy} = \frac{1}{2}c_f\rho v\sqrt{u^2+v^2}$, where ρ is density, τ_{wx} and τ_{wy} are wall shear stresses along the x and y directions respectively.

The coefficients β_1, β_2, and β_3 in equations (1-3) are the momentum correction coefficients which are introduced in order to take into account non-uniformity of velocity distribution in the vertical direction. The momentum correction coefficients are defined as follows:

$$\beta_1 = \frac{1}{hU^2}\int_{z_1}^{z_2} u^2 dz \qquad (4)$$

$$\beta_2 = \frac{1}{hUV}\int_{z_1}^{z_2} uv dz \qquad (5)$$

$$\beta_3 = \frac{1}{hV^2}\int_{z_1}^{z_2} v^2 dz \qquad (6)$$

where u and v are velocity components, but U and V are depth-averaged velocity components in the x and y directions respectively.

Introducing characteristic length b and the characteristic velocity U_0, we choose the measure of time in the form b/U_0. Rewriting the equations in dimensionless form, we obtain:

$$\frac{\partial h_d}{\partial t_d} + \frac{\partial}{\partial x_d}(u_d h_d) + \frac{\partial}{\partial y_d}(v_d h_d) = 0 \qquad (7)$$

$$\frac{\partial u_d}{\partial t_d} + (2\beta_1 - 1)u_d\frac{\partial u_d}{\partial x_d} + (\beta_1 - 1)\frac{u_d^2}{h_d}\frac{\partial h_d}{\partial x_d} + \frac{1}{Fr^2}\frac{\partial h_d}{\partial x_d} + \beta_2 v_d\frac{\partial u_d}{\partial y_d} + (\beta_2 - 1)u_d\frac{\partial v_d}{\partial y_d} + (\beta_2 - 1)\frac{u_d v_d}{h_d}\frac{\partial h_d}{\partial y_d} + \frac{c_f u_d\sqrt{u_d^2 + v_d^2}}{2h_d} - \tilde{F}(y_d) = 0 \qquad (8)$$

$$\frac{\partial v_d}{\partial t_d} + \beta_2 u_d\frac{\partial v_d}{\partial x_d} + (\beta_3 - 1)\frac{v_d^2}{h_d}\frac{\partial h_d}{\partial y_d} + \frac{1}{Fr^2}\frac{\partial h_d}{\partial y_d} + (\beta_2 - 1)v_d\frac{\partial u_d}{\partial x_d} + (\beta_2 - 1)\frac{u_d v_d}{h_d}\frac{\partial h_d}{\partial x_d} + (2\beta_3 - 1)v_d\frac{\partial v_d}{\partial y_d} + \frac{c_f v_d\sqrt{u_d^2 + v_d^2}}{2h_d} = 0 \qquad (9)$$

where $u_d = u/U_0$, $t_d = tU_0/b$, $x_d = x/b$, $h_d = h/H_0$, $y_d = y/b$, $\tilde{F}(y_d) = bF(y_d)/U_0^2$, H_0 is undistributed water depth and Fr is Froude number, representing the ratio of inertia and gravity forces, that is defined by the expression $Fr = U_0/\sqrt{gH_0}$.

Dropping the subscript "d", we seek a perturbed solution for equations (7-9) in the form:

$$u = U(y) + \hat{u}(y)e^{-\lambda t + ikx} \qquad (10)$$

$$v = \hat{v}(y)e^{-\lambda t + ikx} \qquad (11)$$

$$h = \frac{H_0}{b} + \hat{h}(y)e^{-\lambda t + ikx} \qquad (12)$$

where k is a wavenumber and $\lambda = \lambda_r + i\lambda_i$ is a complex eigenvalue.

Substituting (10-12) into (7-9), and performing linearization in the neighborhood of the base flow, we obtain a system of ordinary differential equations:

$$\frac{H_0}{b} ik\hat{u} + ikU\hat{h} + \frac{H_0}{b}\frac{d\hat{v}}{dy} - \lambda\hat{h} = 0 \tag{13}$$

$$[(2\beta_1-1)ikU+sU]\hat{u}+[(\beta_1-1)\frac{ikU^2b}{H_0}+\frac{ik}{Fr^2}-\frac{sU^2b}{2H_0}]\hat{h}+(\beta_2-1)U\frac{d\hat{v}}{dy}+\beta_2 vU_y-\lambda\hat{u} = 0 \tag{14}$$

$$\frac{1}{Fr^2}\frac{d\hat{h}}{dy} + (ik\beta_2 U + \frac{s}{2}U)\hat{v} - \lambda\hat{v} = 0 \tag{15}$$

with the boundary conditions

$$v(\pm\infty) = 0 \tag{16}$$

where $s = \frac{c_f b}{H_0}$, $u = u(y)$, $v = v(y)$ and $h = h(y)$.

3.Solution Method

Using a substitution

$x = \frac{2}{\pi}\arctan(y)$; $y \in (-\infty; +\infty)$; $x \in [-1; 1]$,

we represent the functions $v(x)$, $u(x)$ and $h(x)$ in the form of fundamental interpolation polynomials:

$$u(x) = \sum_{k=1}^{n} a_k \frac{T_n(x)}{(x - x_k)T'_n(x_k)} \tag{17}$$

$$v(x) = \sum_{k=1}^{n} b_k \frac{(1 - x^2)}{(1 - x_k^2)} \frac{T_n(x)}{(x - x_k)T'_n(x_k)} \tag{18}$$

$$h(x) = \sum_{k=1}^{n} c_k \frac{T_n(x)}{(x - x_k)T'_n(x_k)} \tag{19}$$

where a_k, b_k and c_k are unknown constants, but $T_n(x)$ is an n-order Chebyshev polynomial that has the form $T_n(x) = \cos(n * \arccos(x))$. The points x_k, defined by the expression $x_k = \cos\frac{(2k-1)\pi}{2n}$, are the zeroes of the Chebyshev polynomial of order n, that is, $(T_n(x_k) = 0)$. It is obvious that the term

$\frac{T_n(x)}{(x-x_k)T'_n(x_k))}$ is equal to zero, if $x = x_j$, where x_j is a zero of an n-order Chebyshev polynomial, and $j \neq k$. If $x = x_k$ then using the Taylor series expansion of $T_n(x)$ about the point $x = x_k$ we obtain:

$$\frac{T_n(x)}{(x-x_k)T'_n(x_k))} = \frac{T_n(x_k) + (x-x_k)T'_n(x_k) + \frac{(x-x_k))^2}{2}T''_n(x_k) + < ... >}{(x-x_k)T'_n(x_k)} = \\ = 1 + \frac{(x-x_k)}{2}\frac{T''_n(x_k)}{T'_n(x_k)} + < ... > \quad (20)$$

Hence

$$\frac{T_n(x)}{(x-x_k)T'_n(x_k))} = \begin{cases} 0, & \text{if } x = x_j,\ j \neq k \\ 1, & \text{if } x = x_j,\ j = k \end{cases} \quad (21)$$

Using the collocation method and choosing zeroes of Chebyshev polynomials as the collocation points we obtain

$$(A - \lambda B)d = 0 \quad (22)$$

where A and B are two complex-valued matrices. Vector d has the form
$d = a_1, a_2, ..., a_n, b_1, b_2, ..., b_n, c_1, c_2, ..., c_n$

Solving the generalized eigenvalue problem (22), for given s and k we obtain a set of eigenvalues λ.

The real parts of eigenvalues λ determine linear stability of the base flow. The flow is said to be linearly stable, if real parts of all λ are positive. If the real part of the eigenvalue λ of at least one mode is negative then a perturbation grows exponentially with time and the flow is said to be linearly unstable. Numerical methods can be used in order to find for a given wavenumber k a value of s, for which one mode has λ_r equal to zero, while all other λ have positive real parts. That enables to build the neutral stability curve, that is defined as a set of points in the (k,s)-plane for which one λ has the real part equal to zero, while real parts of all other λ are positive. The critical value, s_c of the parameter s is defined as the coordinate of the highest point of the curve, or $s_c = max(s^{(n)}(k))$.

The s_c parameter is very important in linear stability analysis. The flow is stable for all k if the value of s is higher than s_c, and flow is unstable for some k if $s < s_c$.

4. Results and discussion

This paper presents an attempt to evaluate the influence of "rigid-lid" assumption and momentum correction coefficients on the value of s_c parameter. The rigid-lid assumption is evaluated by solving problems (13-15) numerically for different values of Froude number as well as b/H_0 parameter that is the ratio of the characteristic width of the wake and water depth and comparing the critical values, s_c, of the parameter s. The assumption of uniform velocity distribution across the vertical coordinate is evaluated by solving problems (13-15) for different values of momentum correction coefficients β_1 and β_2.

The values of s_c have been calculated for the following values of the parameters Fr, b/H_0, β_1, and β_2:

$Fr = 0.0001, 0.1, 0.2.$

$b/H_0 = 5, 50.$

$\beta_1 = 1.00, 1.05, 1.10.$

$\beta_2 = 1.00, 1.05, 1.10.$

The critical values of the stability parameter for finite Froude number Fr and various values of parameter b/H_0 are compared with those obtained under "rigid-lid" assumption by Kolyshkin&Nazarovs [4].

The results (in terms of percentage difference) are shown in Figure 1.

The two values of the parameter b/H_0 are chosen since the condition $b/H_0 \gg 1$ is consistent with the shallow water approximation. It is seen that although the stability boundary is quite sensitive to variations of Froude number, the error in determining the s_c parameter is below 6% if Froude number is less than 0.2 for the case b/H_0=5, and if the Froude number is less than 0.1 for the case b/H_0=50. The Froude number Fr_H (based on the undisturbed water depth) is related to Fr by means of the formula $Fr_H = Fr\sqrt{b/H_0}$. The parameter Fr_H for real island wakes is in the range 0.1-0.2. So, the "rigid-lid" assumption is precise enough for calculation of the s_c parameter for the range of Froude numbers typical for shallow flows.

Figure 2 presents results of the comparison of the s_c parameter calculated for different values of momentum correction coefficients β_1 and β_2. The results are compared to the values of s_c calculated for β_1=1.00 and β_2=1.00 that corresponds to the case when the velocity non-uniformity across the vertical coordinate is not taken into account. As it can be seen, for some combination of the values of β_1 and β_2 the relative error can reach 10%. The increase of β_1 leads to growth of s_c, so the flow becomes more unstable. The β_2 coefficient

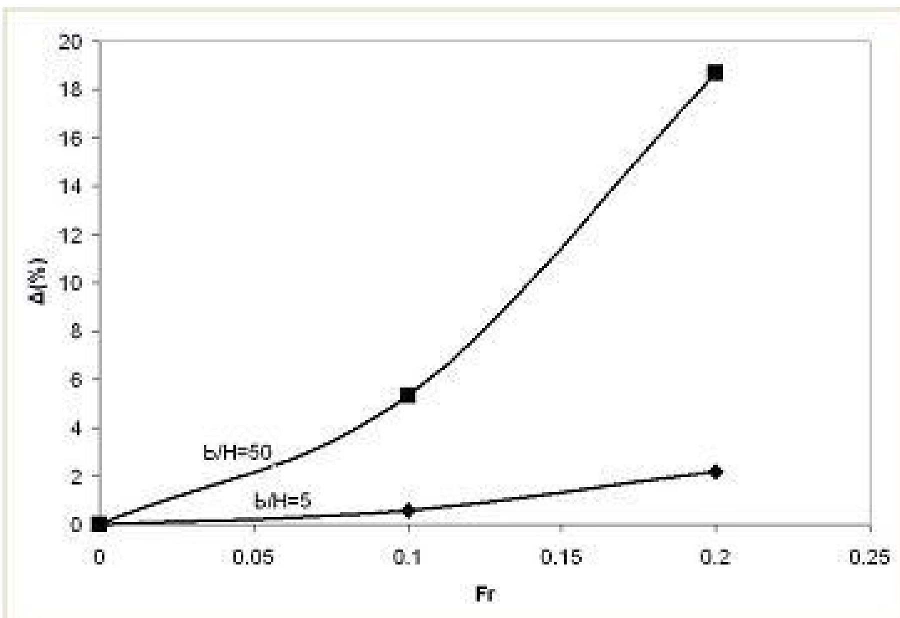

Figure 1: The percentage difference Δ between the values of the s_c with and without the rigid-lid assumption for the case b/H_0=5 and b/H_0=50.

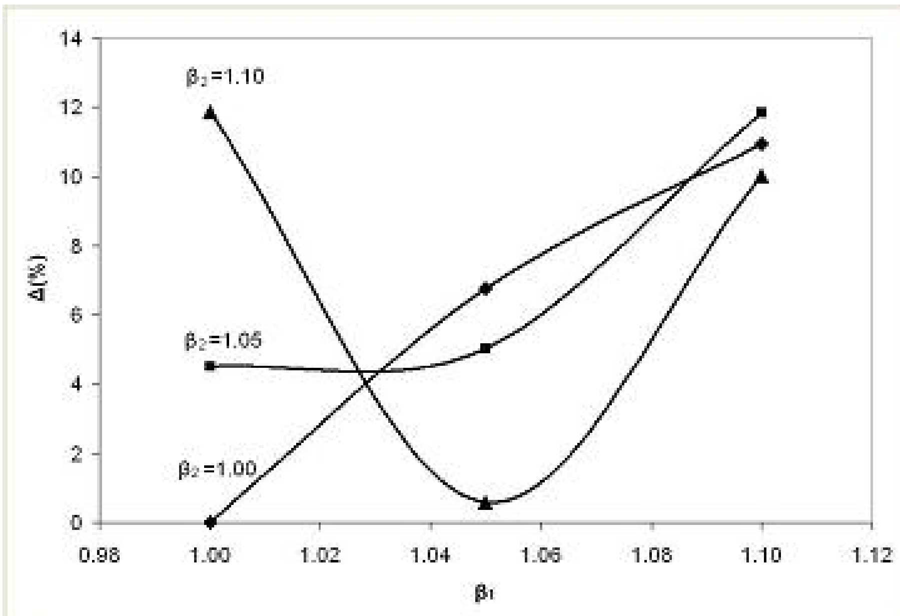

Figure 2: The percentage difference Δ between the values of the s_c for depth-averaged equations ($\beta_1 = 1$, $\beta_2 = 1$) and equations with correction factors ($\beta_1 > 1$, $\beta_2 > 1$).

has, in its turn, stabilizing effect on the flow, but its influence diminishes with the growth of β_1. Unfortunately, the values of coefficients β_1 and β_2 for real island wakes are not known. However as the error in determining the s_c parameter may grow with increased values of β_1 (the stability boundary can be underestimated with increase of β_1) it might be important to know the values of β_1 and β_2 for analyzed shallow flows.

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