TRIVARIATE APPROXIMATION OPERATORS ON CUBE BY PARAMETRIC EXTENSIONS

by
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Abstract: We obtain tridimensional approximation operators considering the boolean sum and the tensor product of the parametric extensions of some univariate operators. We extend univariate operators so that they can operate on functions of three variables.

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The interpolation by polynomials or other functions is a rather old method in applied mathematics. In the paper of M. Gasca and T. Sauer [9] it is mentioned that the word "interpolation" has apparently been introduced by J.Wallis as early as 1655. Compared to this, polynomial interpolation in several variables is a relatively new topic and it probably only started in the second half of 19^{th} century. In the same paper it is mentioned a statement of Andoyer regarding multivariate polynomial interpolation: "It is clear that the interpolation of function of several variables does not demand any new principles because the fact that the variable was unique has not played frequently any role." The time has contradicted this statement and multivariate polynomial interpolation has received constant attention and is today a basic subject in Approximation Theory and Numerical Analysis, with applications to many problems in Mathematics, Physics, Engineering, and so on.

An approximation problem consists in expressing a group of values in terms of another group of values. There are two main ways to approximate a function of several variables: to extend the known results from the univariate case or to use specific approximation procedures for multivariate functions.

Bivariate interpolation by the tensor product of univariate interpolation functions (that is, when the variables are treated separately) is the clasical approach to multivariate interpolation. This idea is impossible to use when the set of interpolation points is not a cartesian product grid.

In this paper we consider the cartesian product grid case. It is known that the tensorial product of two univariate interpolation operators is a bivariate interpolation operator. So, if A and B are interpolation operators corresponding to the nodes $s_1,\ldots,s_n$ and respectively $t_1,\ldots,t_m$ then $A \otimes B$ is an interpolation operator corresponding to the grid of nodes

$$\{(s_i, t_j) \mid i = 1,\ldots,n;\ j = 1,\ldots,m\}.$$

The interpolation on this cartesian grid is uniquely possible by a polynomial of the following form:

$$p(s,t)=\sum_{\nu=0}^{n}\sum_{\mu=0}^{m}a_{\nu\mu}s^{\nu}t^{\mu}\ .$$

W.J. Gordon has introduced the basic notions of the algebric theory of multivariate functions approximation, a theory which was studied and developed by F. J. Delvos and W. Schempp in [7]. The projectors are the basic tools of the boolean methods of approximation and interpolation. Let X be a linear space over $\mathbb{R}$ or $\mathbb{C}$.

Definition 1. The linear operator $P:X\to X$ is called a *projector* if P is idempotent:

$$P^2=P.$$

The identity operators on X, denoted I, is a projector. The operator $P^c=I-P$ is the remainder projector.

The boolean sum and tensor product operators yields new projectors. The variety of these projectors is a source for obtaining new interpolation and approximation projectors. The parametric extension method is a procedure for constructing linear operators on the spaces of multivariate functions, starting from linear operators defined on spaces of univariate functions, see for example [7]. J. Delvos and W. Schempp have shown that the parametric extensions of some bounded linear operators are also bounded linear operators.

Let $f\in\zeta(X\times Y)$ and $x\in X$. The function $f^x\in\zeta(X)$ is defined by

$$f^x(t)=f(x,t),\ \ t\in Y.$$

For $y\in Y$ the function ${}^yf\in\zeta(Y)$ is defined by

$${}^yf(s)=f(s,y),\ \ s\in X.$$

Let A be a bounded linear operator on $\zeta(X)$. The parametric extention A' of A is defined by

$$\mathrm{A(f)(x,y)=A(^y f)(x).}$$

Let B be a bounded linear operator on ζ(Y). The parametric extention B" of B is defined by

$$\mathrm{B''(f)(x,y)=B(f^x)(y).}$$

The parametric extentions A' and B" commute on $\zeta(X)\times\zeta(Y)$.

Let X and Y be and their cartesian product defined by

$$X\times Y=\{(x,y): x\in X, y\in Y\}.$$

We assume that the following set of nodes are given:

$$\{x_1,\ldots,x_n\}\subset X,\ \{y_1,\ldots,y_m\}\subset Y\ .$$

The cartesian grid generated by these sets of nodes is

$$N=\{x_1,\ldots,x_n\}\times\{y_1,\ldots,y_m\}=\{(x_i,y_j): i=1,\ldots,n;\ j=1,\ldots m\}$$

We consider the following interpolation operators:

$$(Pf)(x) = \sum_{i=1}^{n} f(x_i)u_i(x),$$
$$(Qg)(y) = \sum_{j=1}^{m} g(y_j)v_j(y),$$

with

$u_i(x_j) = \delta_{ij}$,

$v_i(x_j) = \delta_{ij}$.

We extend these operators in order to apply them to a bivariate function $F : X \times Y \to \mathbb{R}$:

$$(\overline{P}F)(x,y) = \sum_{i=1}^{n} F(x_i, y)u_i(x),$$
$$(\overline{Q}F)(x,y) = \sum_{j=1}^{m} F(x, y_j)v_j(y).$$

The function $\overline{P}F$ interpolates the function F on the set $\{x_1,\ldots,x_n\} \times Y$ while $\overline{Q}F$ interpolates the function F on the set $X\times \{y_1,\ldots,y_m\}$. The function $\overline{P}\overline{Q}F$ interpolates the function F on the set N, where

$$(\overline{\mathrm{P}}\overline{Q}F)(x,y) = \sum_{i=1}^{n}\sum_{j=1}^{m} F(x_i, y_j)u_i(x)v_j(y).$$

We note that $\overline{P}\overline{Q} = \overline{Q}\overline{P}$.

Concerning the theory presented here we mention the following theorem given by E. W. Cheney and W. Light:

Theorem 2. [3] Arbitrary data can be interpolated uniquely by the tensor product space $\prod_n(\mathbb{R}) \otimes \prod_m(\mathbb{R})$ on any set of nodes having the form $\{x_0,\ldots,x_n\} \times \{y_0,\ldots,y_m\}$.

The boolean sum operator is also very useful. With the previous notations we have that

$$(\overline{P} \oplus \overline{Q}F)(x,y) = \sum_{i=1}^{n} F(x_i, y)u_i(x) + \sum_{j=1}^{m} F(x, y_j)v_j(y) - \sum_{i=1}^{n}\sum_{j=1}^{m} F(x_i, y_j)u_i(x)v_j(y).$$

$\overline{P} \oplus \overline{Q}F$ interpolates F on the union

$(\{x_1,\ldots,x_n\} \times Y) \cup (X \times \{y_1,\ldots,y_m\})$.

Remark 3. We notice why we cannot use $\overline{P}+\overline{Q}$ and must use instead $\overline{P}+\overline{Q}-\overline{P}\overline{Q}$: the operator $\overline{P}+\overline{Q}$ will give $2F(x_i,y_j)$ at the nodes. Hence by substacting $\overline{P}\overline{Q}F$ we restore the correct values $F(x_i,y_j)$.

Remark 4. The interpolation processes described here are used in the automotive design.

The construction used for two-variables interpolation can be used for any number of variables. In the following we consider the case of three variables, see for example [3].

To interpolate a function of three variables, $(x,y,z)\mapsto F(x,y,z)$ we select the nodes $\{x_1,...,x_n\}$ in X, $\{y_1,...,y_m\}$ in Y, and $\{z_1,...,z_k\}$ in Z. We require functions (e.g., polynomials) such that

$$u_i(x_j)=v_i(y_j)=w_i(z_j)=\delta_{ij}.$$

The coresponding operators are:

$$(Pf)(x)=\sum_{i=1}^{n} f(x_i)u_i(x),$$

$$(Qg)(y)=\sum_{j=1}^{m} g(y_j)v_j(y),$$

$$(Rh)(z)=\sum_{l=1}^{k} g(z_l)w_l(z).$$

They can be extended to operate on functions of three variables. Their parametric extensions will be:

$$(\overline{P}F)(x,y,z)=\sum_{i=1}^{n} F(x_i,y,z)u_i(x),$$

$$(\overline{Q}F)(x,y,z)=\sum_{j=1}^{m} F(x,y_j,z)v_j(y),$$

$$(\overline{R}F)(x,y,z)=\sum_{l=1}^{k} F(x,y,z_l)w_l(z).$$

The tensor product operator $\overline{P}\overline{Q}\overline{R}$ interpolates F at every point of the three dimensional grid:

$\{(x_i,y_j,z_l): 1\le i\le n;\ 1\le j\le m; 1\le l\le k\}$.

The formula for $\overline{P}\overline{Q}\overline{R}F$ is

$$(\overline{\mathrm{P}}\overline{Q}\overline{R}F)(x,y,z)=\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{l=1}^{k} F(x_i,y_j,z_l)u_i(x)v_j(y)w_l(z).$$

The boolean sum operator $\overline{P} \oplus \overline{Q} \oplus \overline{R}$ interpolates F on the union
$(\{x_1,...,x_n\} \times X \times Z) \cup (X \times \{y_1,...,y_m\} \times Z) \cup (X \times Y \times \{z_1,...,z_k\})$.
The formula for boolean sum operator $\overline{P} \oplus \overline{Q} \oplus \overline{R}$ is
$\overline{P} \oplus \overline{Q} \oplus \overline{R} = \overline{P} + \overline{Q} + \overline{R} - \overline{P}\overline{Q} - \overline{P}\overline{R} - \overline{Q}\overline{R} + \overline{P}\overline{Q}\overline{R}$.
Therefore we have

$$\begin{aligned}(\overline{P} \oplus \overline{Q} \oplus \overline{R}F)(x,y,z) = {} & \sum_{i=1}^{n} F(x_i,y,z)u_i(x) + \sum_{j=1}^{m} F(x,y_j,z)v_j(y) + \sum_{l=1}^{k} F(x,y,z_l)w_l(z) \\ & -\sum_{i=1}^{n}\sum_{j=1}^{m} F(x_i,y_j,z)u_i(x)v_j(y) - \sum_{i=1}^{n}\sum_{l=1}^{k} F(x_i,y,z_l)u_i(x)w_l(z) \\ & -\sum_{j=1}^{m}\sum_{l=1}^{k} F(x,y_j,z_l)v_j(y)w_l(z) + \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{l=1}^{k} F(x_i,y_j,z_l)u_i(x)v_j(y)w_l(z\end{aligned}$$

Regarding these results we consider the following examples:
Example 1. We consider the cube $D_h=[0,h]^3$, $h>0$, and the function
$f:D_h \to \mathbb{R}$.
We have to find a polynomial which has the same values as f on the vertices V_i, $i=1,...,8$ of the cube, i.e.
$P(V_i)=f(V_i)$, $i=1,...,8$.
We consider the univariate Lagrange operators regarding the nodes $x_0=0$ and $x_1=h$, $y_0=0$ and $y_1=h$, $z_0=0$ and $z_1=h$:

$$(Pf_1)(x) = \frac{h-x}{h} f_1(0) + \frac{x}{h} f_1(h),$$

$$(Qf_2)(y) = \frac{h-y}{h} f_2(0) + \frac{y}{h} f_2(h),$$

$$(Rf_3)(z) = \frac{h-z}{h} f_3(0) + \frac{z}{h} f_3(h),$$

where f_1, f_2 and f_3 are univariate functions defined on $[0,h]$. The operators P,Q and R are projectors. According to the above presented theory, we extend these operators in order to operate on function f of three variables. The parametric extensions of these operators, applied to f, will be:

$$(\overline{P}f)(x,y,z)=\frac{h-x}{h}f(0,y,z)+\frac{x}{h}f(h,y,z),$$
$$(\overline{Q}f)(x,y,z)=\frac{h-y}{h}f(x,0,z)+\frac{y}{h}f(x,h,z),$$
$$(\overline{R}f)(x,y,z)=\frac{h-z}{h}f(x,y,0)+\frac{z}{h}f(x,y,h).$$

We have the set of nodes $\{x_0, x_1\}$, $\{y_0, y_1\}$ and $\{z_0, z_1\}$ and the three dimensional grid: $\{(x_i, y_j, z_l)$: $i=0,1;\ j=0,1;\ 1=0,1\}$
$=\{(0,0,0),\ (0,0,h),\ (0,h,0),\ (h,0,0),\ (0,h,h),\ (h,h,0),\ (h,0,h),\ (h,h,h)\}$.

The tensor product operator of the parametric extensions, $\overline{P}\overline{Q}\overline{R}$, interpolates f at every point of this three dimensional grid, i.e., it interpolates the function f on the vertices of the cube D_h. We have

$$\begin{aligned}(\overline{P}\overline{Q}\overline{R}f)(x,y,z)=&\frac{(h-x)(h-y)(h-z)}{h^3}f(0,0,0)+\frac{(h-x)y(h-z)}{h^3}f(0,h,0)\\&+\frac{x(h-y)(h-z)}{h^3}f(h,0,0)+\frac{(h-x)(h-y)z}{h^3}f(0,0,h)\\&+\frac{xy(h-z)}{h^3}f(h,h,0)+\frac{xz(h-y)}{h^3}f(h,0,h)\\&+\frac{yz(h-x)}{h^3}f(0,h,h)+\frac{xyz}{h^3}f(h,h,h).\end{aligned}$$

Example 2. We shall analyze now the approximation problem on the same cube D_h, but concerning the edges. This problem will be solved by boolean sum operator of parametric extensions of the operators P, Q and R, defined in Example 1. Therefore we have that $\overline{P}\oplus\overline{Q}\oplus\overline{R}$ interpolates f on the edges of the cube D_h.

The formula for boolean sum operator $\overline{P}\oplus\overline{Q}\oplus\overline{R}$ applied to f is

$$\begin{aligned}(\overline{P}\oplus\overline{Q}\oplus\overline{R}f)(x,y,z)&=\frac{h-x}{h}f(0,y,z)+\frac{x}{h}f(h,y,z)+\frac{h-y}{h}f(x,0,z)+\frac{y}{h}f(x,h,z)\\&+\frac{h-z}{h}f(x,y,0)+\frac{z}{h}f(x,y,h)-\frac{(h-x)(h-y)}{h^2}f(0,0,z)\\&-\frac{x(h-y)}{h^2}f(h,0,z)-\frac{(h-x)y}{h^2}f(0,h,z)-\frac{xy}{h^2}f(h,h,z)\\&-\frac{(h-x)(h-z)}{h^2}f(0,y,0)-\frac{(h-x)z}{h^2}f(0,y,h)-\frac{x(h-z)}{h^2}f(h,y,0)\\&-\frac{xz}{h^2}f(h,y,h)-\frac{(h-y)(h-z)}{h^2}f(x,0,0)-\frac{(h-y)z}{h^2}f(x,0,h)\\&-\frac{y(h-z)}{h^2}f(x,h,0)-\frac{yz}{h^2}f(x,h,h)+\frac{(h-x)(h-y)(h-z)}{h^3}f(0,0,0)\\&+\frac{(h-x)y(h-z)}{h^3}f(0,h,0)+\frac{x(h-y)(h-z)}{h^3}f(h,0,0)+\frac{(h-x)(h-y)z}{h^3}f(0,0,h)\\&+\frac{xy(h-z)}{h^3}f(h,h,0)+\frac{x(h-y)z}{h^3}f(h,0,h)+\frac{(h-x)yz}{h^3}f(0,h,h)+\frac{xyz}{h^3}f(h,h,h).\end{aligned}$$

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