APPROXIMATION NEW ERROR BOUNDED BY SPLINE DEGREE SIX

Karwan H-F, Faraidun K. H-S, and Radha K. G

Abstract. In this work, we obtain best error bounds for six degree lacunary spline function which interpolate to the data (0, 1, 3), and also two illustrate example discussed in the finial section .

Keywords: best error bounds, spline function, CityplaceTaylor's expansion

1. Introduction

The subject of lacunary interpolation by polynomials has a rich history .For complete background refer to the survey article [2] .The g-splines (i.e, general splines) where introduced by Ahlberg, Nilson and Walsh [6] as a devise of lacunary interpolation of derivatives through order $n-1 \geq 0$ at a finite state of points on an interval $[a, b]$.In a paper, Varma [1] obtained the error bounds for some classes of deficient sixtic splines which interpolate to (0,1,3) data on equidistant knots. In the present work we modified the same lacunry data and we showed that the error bounded better than Varma theoretically and practically. Several authors used the same technique but for different lacunary data, for example see [3-5].

This work is organized as follows: In section two, spline function of degree six is defined which interpolates the lacunary data (0, 1, 3) .Some theoretical results about existence and uniqueness of the spline function of degree six are introduced in section three .In section four, convergence analysis and stability are studied . To demonstrate the convergence of the prescribed lacunary spline function, numerical examples are presented in section five.
we present a six degree spline interpolation for one dimensional and given sufficiently smooth function $f(x)$ defined on I=[0,1] denote the uniform partition of $I$with knots $x_i = \frac{i}{2m}$, i=0, 2, . . . , $2m$ and n=2m+1.

We define the class of spline function $Sp(6,3,n)$ where $Sp(6,3,n)$ denotes the class of all splines of degree six which belongs to $C^3[0,1]$, and n is the number of knots , as follow :

Any element $S_\Delta(x) \in Sp(6,3,n)$ if the following two conditions are satisfied:

$$\left[\begin{array}{l} (i)\ \ S_\Delta(x) \in C^3[0,1] \\ (ii)\ \ S_\Delta(x)\ \ is\, a\, polynomi\, al\, of\ \deg ree\ \ six\, in\, each\ [x_{2i}, x_{2i+2}]\ , \\ \ i = 0,\ 1, ...,\ \ m-1 \end{array}\right. \quad (1)$$

Theorem 1 *For given arbitrary numbers*$f(x_{2i})$, $f^{(r)}(t_{2i})$, *i=0,1,. . . ,m-1; r=0, 1,3 and*$f^4(x_0)$, $f^4(x_{2m})$ *,There exists a unique spline*$S_\Delta(x) \in Sp(6,3,n)$ *such that*

$$\left[\begin{array}{l} S_n(x_{2i}) = f(x_{2i})\ \ ,\ i = 0,\ 1, ..., m \\ S_n^{(r)}(t_{2i}) = f^{(r)}(t_{2i})\, , i = 0,\ 1, ..., m-1;\ r = 0,\ 1, 3 \\ S_n^{(4)}(x_0) = f^{(4)}(x_0)\, , S_n^{(4)}(x_{2m}) = f^{(4)}(x_{2m}) \end{array}\right. \quad (2)$$

Theorem 2 *Let* $f \in C^6[0,1]$ *and* $S_\Delta(x) \in Sp(6,3,n)$ *be a unique spline satisfying the conditions of Theorem 1.1, then*

$$\left\| S_n^{(r)}(x) - f^{(r)}(x) \right\| \le 34.32668942 \quad m^{r-6} w(f^{(6)}; \frac{1}{m}) +\ 3m^{r-6} || f^{(6)}|,$$

$$\mathrm{r} = 0, 1, 2, 3,\ 4\ ,\ 5.$$

Where $(f^{(6)}; \frac{1}{m})$*denotes the modules of continuity of* y^6 *and*

$$\left\| f^{(6)} \right\| = \max\left\{ \left| f^{(6)}(x) \right| ; 0 \le x \le 1 \right\}$$

2. Construction of the Spline Function

If *P (x)* is a polynomial of degree six on $[0,1]$, then we have

$$P(x) = P(0)A_0(x) + P(\frac{1}{3})A_1(x) + P(1)A_2(x) + p'(\frac{1}{3})A_3(x) + P^{(3)}(\frac{1}{3})A_4(x)$$

$$+P^{(4)}(0)A5(x)+P^{4}(1)A_6(x). \qquad (3)$$

Where

$$\begin{cases} A_0(x)=\frac{1}{34}(243\,x^6-729\,x^5+630\,x^3-21\,x^2-157\,x+34),\\ A_1(x)=\frac{1}{136}(-729\ x^6+2187\,x^5-1890x^3-243x^2+675x),\ ,\\ A_2(x)=\frac{1}{136}(-243\,x^6+729\,x^5-630x^3+327\,x^2-47x),\\ A_3(x)=\frac{1}{68}(243\ x^6-729\,x^5+1630x^3-123\,x^2-12\,x),\\ A_4(x)=\frac{1}{204}(27\ x^6-81\,x^5+104x^3-59\,x^2-9x),\\ A_5(x)=\frac{1}{110160}(567\ x^6-2619\ x^5+4590x^4-3630x^3+1243x^2-151\,x),\\ A_6(x)=\frac{1}{55080}(324\,x^6-513\,x^5+330x^3-162x^2+23x). \end{cases} \qquad (4)$$

For $f\in C^6[0,1]$ we have the following expansions

$$\begin{cases} f(x_{2i+2})=f(x_{2i})+2hf'(x_{2i})+\ 2h^2f''(x_{2i})+\frac{4}{3}h^3f'''(x_{2i})+\\ \quad \frac{2}{3}h^4f^{(4)}(x_{2i})+\frac{4}{15}h^5f^{(5)}(x_{2i})+\frac{4}{45}h^6f^{(6)}(\lambda_{1,2i}),\ \ x_{2i}<\lambda_{1,2i}<x_{2i+2}\\ f(x_{2i-2})=f(x_{2i})-2hf'(x_{2i})+2h''(x_{2i})-\frac{4}{3}h^3f'''(x_{2i})+\frac{2}{3}h^4f^{(4)}(x_{2i})\\ \quad -\frac{4}{15}h^5f^{(5)}(x_{2i})+\frac{4}{45}h^6f^{(6)}(\lambda_{2,2i})\ ,\ x_{2i-2}<\lambda_{2,2i}<x_{2i}\\ f(t_{2i})=f(x_{2i})+\frac{2}{3}hf'(x_{2i})+\frac{2}{9}h^2f''(x_{2i})+\frac{4}{81}h^3f'''(x_{2i})+\\ \frac{2}{243}h^4f^{(4)}(x_{2i})+\frac{4}{3645}h^5f^{(5)}(x_{2i})+\frac{4}{32805}h^6f^{(6)}(\lambda_{3,2i}),\ x_{2i}<\lambda_{3,2i}<t_{2i}\\ f(t_{2i-2})=f(x_{2i})-\frac{4}{3}hf'(x_{2i})+\ \frac{8}{9}h^2f''(x_{2i})-\frac{32}{81}h^3f'''(x_{2i})+\ \ \frac{32}{243}h^4f^{(4)}(x_{2i})\\ \quad -\frac{128}{3645}h^5f^{(5)}(x_{2i})+\frac{256}{32805}h^6f^{(6)}(\lambda_{4,2i}),\ t_{2i-2}<\lambda_{4,2i}<x_{2i}\\ f'(t_{2i})=f'(x_{2i})+\frac{2}{3}hf''(x_{2i})+\ \frac{2}{9}h^2f'''(x_{2i})+\frac{4}{81}h^3f^{(4)}(x_{2i})+\\ \quad \frac{2}{243}h^4f^{(5)}(x_{2i})+\frac{4}{3645}h^5f^{(6)}(\lambda_{5,2i}),x_{2i}<\lambda_{5,2i}<t_{2i}\\ f'(t_{2i-2})=f'(x_{2i})-\frac{4}{3}hf''(x_{2i})+\ \frac{8}{9}h^2f'''(x_{2i})-\frac{32}{81}h^3f^{(4)}(x_{2i})+\\ \quad \frac{32}{243}h^4f^{(5)}(x_{2i})-\frac{128}{3645}h^5f^{(6)}(\lambda_{6,2i}),t_{2i-2}<\lambda_{6,2i}<x_{2i}\\ f'''(t_{2i})=f'''(x_{2i})+\frac{2}{3}h\ f^{(4)}(x_{2i})+\ \frac{2}{9}h^2f^{(5)}(x_{2i})+\frac{4}{81}h^3f^{(6)}(\lambda\ _{7,\ 2i}),\\ \quad \mathrm{x_{2i}}<\lambda_{7,2i}<t_{2i}\\ f'''(t_{2i-2})=f'''(x_{2i})-\frac{4}{3}h\ f^{(4)}(x_{2i})+\ \frac{8}{9}h^2f^{(5)}(x_{2i})-\frac{32}{81}h^3f^{(6)}(\lambda\ _{8,\ 2i}),\\ \quad \mathrm{t_{2i-2}}<\lambda_{8,2i}<x_{2i}\\ f^{(4)}(x_{2i+2})=f^{(4)}(x_{2i})+2hf^{(5)}(x_{2i})+\ 2h^2f^{(5)}\lambda_{9,2i}),\ \ \mathrm{x_{2i}}<\lambda_{9,2i}<x_{2i}\\ f^{(4)}(x_{2i-2})=f^{(4)}(x_{2i})-2hf^{(5)}(x_{2i})+2h^3f^{(6)}(\lambda_{10,2i}),x_{2i-2}<\lambda_{10,2i}<x_{2i}\\ f^{(4)}(t_{2i})=f^{(4)}(x_{2i})+\frac{2}{3}h\ f^{(5)}(x_{2i})\ \ +\frac{2}{9}h^2f^{(6)}(\lambda_{11H2i}),x_{2i}<\lambda\ _{11,2i}<t_{2i}\\ f^{(5)}(t_{2i})=f^{(5)}(x_{2i})+\frac{2}{3}h\ f^{(6)}(\lambda_{12H2i}),x_{2i}<\lambda\ _{12,2i}<t_{2i} \end{cases} \qquad (5)$$

Proof of the theorem 1.1:
The proof depends on the following representations of $S_n(x)$., for *2ih* ≤x≤ *(2i+2) h*,
i=0,1, . . . , m-1, we have

$$S_n(x) = f(x_{2i})A_0(\tfrac{x-2ih}{2h}) + f(t_{2i})A_1(\tfrac{x-2ih}{2h}) + f(x_{2i+2})A_2(\tfrac{x-2ih}{2h}) +$$
$$+2hf'(t_{2i})A_3(\tfrac{x-2ih}{2h}) + 8h^3 f^{(3)}(t_{2i})A_4(\tfrac{x-2ih}{2h}) + 16h^4 S_n^{(4)}(x_{2i})A_5(\tfrac{x-2ih}{2h}) \quad (6)$$
$$+16h^4\, S_n^{(4)}(x_{2i+2})A_6(\tfrac{x-2ih}{2h}).$$

On using equation 6 and the condition

$$S_n^{(4)}(0) = f^{(4)}(0)\ ,\ \ S_n^{(4)}(0) = f^{(4)}(0). \quad (7)$$

We see that $S_n(x)$. as given by (6) satisfies (1) and is sixtic in $[x2i,\ x2i{+}2]$, *i=0,1, . . . , m-1.*
We also need to show that whether it is possible to determine$\mathrm{S_n^{(12)}(x_{2i})}$, $\mathrm{i} = 1,\ 2,\ ...,\ \mathrm{m}-1$ uniquely. For this purpose we use the fact that $s_n(x) \in C^3[0,1]$ and therefore the conditions:

$$\mathrm{S_n^{(3)}(x_{2i+})} = \mathrm{S_n^{(3)}\,(x_{2i-})}\ ,\mathrm{i} = 1,\ 2,\ ...,\ \mathrm{m}-1$$

$$-\tfrac{2}{153}h^4 S_n^{(4)}(x_{2i-2}) + \tfrac{233}{306}h^4 S_n^{(4)}(x_{2i}) - \tfrac{11}{153}h^4 S_n^{(4)}(x_{2i+2}) = \tfrac{675}{17}f(x_{2-2i}) +$$
$$\tfrac{135}{34}f(x_{2i}) - \tfrac{945}{272}f(x_{2+2}) - \tfrac{2835}{592}f(t_{2i}) - \tfrac{2025}{68}f(t_{2i-2}) - \tfrac{675}{17}h\, f'(t_{2i-2}) + \quad (8)$$
$$\tfrac{945}{68}h\, f'(t_{2i}) + \tfrac{83}{17}\, h^3\, f'''\,(t_{2i-2}) + \tfrac{52}{17}\, h^3\, f'''(t_{\ 2i}\ )\,,$$

$$for \quad i = 1, 2, ..., m-1.$$

Equation (8) is a strictly tri-diagonal dominant system which has a unique solution [7] .Thus$S_n^{(4)}(x_{2i})$, i=1, 2 . . . , m-1 can be obtained uniquely by the system (8) which established Theorem 1.

3. Convergence and Error Bounds

In this section, the upper bounds for errors studied first help results of the following:

Lemma 1 *Let us write 2i=* $\left|S_n^{(4)}(x_{2i}) - f^{(4)}(x_{2i})\right|$ *, , then for f ∈C6[0,1], we have*

$Max_{\,2i} \leq \frac{10646}{1863} h^2 w(f^{(6)}; \frac{1}{m})$ *for i=1,2,. . . ,m-1*

Proof 1 *From (8) we have*

$$-\tfrac{2}{153}h^4(S_n^{(4)}(x_{2i-2}) - f^{(4)}(x_{2i-2})) + \tfrac{233}{306}h^4(S_n^{(4)}(x_{2i}) - f^{(4)}(x_{2i})) - \tfrac{11}{153}h^4$$

$$(S_n^{(4)}(x_{2i+2}) - f^{(4)}(x_{2i+2})) = \tfrac{675}{17}f(x_{2i-2}) + \tfrac{135}{34}f(x_{2i}) - \tfrac{945}{272}f(x_{2i+2}) - \tfrac{2835}{592}f(t_{2i})$$

$$-\tfrac{2025}{68}f(t_{2i-2}) + \tfrac{675}{17}h\,f'(t_{2i-2}) + \tfrac{945}{68}h\,f'(t_{2i}) - \tfrac{83}{17}h^3f'''(t_{2i-2}) + \tfrac{52}{17}h^3f'''(t_{2i})$$

$$-\tfrac{2}{153}h^4f^{(4)}(x_{2i-2}) - \tfrac{233}{306}h^4f^{(4)}(x_{2i}) + \tfrac{11}{153}h^4f^{(4)}(x_{2i+2}) = \tfrac{60}{17}h^6f^{(6)}(\lambda_{2,2i})$$

$$-\tfrac{21}{68}h^6f^{(6)}(\lambda_{1,2i}) - \tfrac{7}{5508}h^6f^{(6)}(\lambda_{3,2i}) - \tfrac{320}{1377}h^6f^{(6)}(\lambda_{4,2i}) - \tfrac{640}{459}h^6f^{(6)}(\lambda_{6,2i}) +$$

$$\tfrac{7}{459}h^6f^{(6)}(\lambda_{5,2i}) \;-\tfrac{2656}{1377}h^6f^{(6)}(\lambda_{11,2i})\tfrac{208}{1377}h^6f^{(6)}(\lambda_{8,2i}) + \tfrac{4}{153}h^6f^{(6)}(\lambda_{11,2i})$$

$$+\tfrac{22}{153}h^6f^{(6)}(\lambda_{10,2i}) = \tfrac{5323}{1377}\,h^5\,\alpha_1\,w(f^{(6)};\tfrac{1}{m}), \quad |\,\alpha_1\,| \leq 1$$

The result 1 follows on using the property of diagonal dominant [7, p.398].

Lemma 2 *Let* $f \in C^5[0.1]$*,then*

$$\left|S_n^{(6)}(t_{2i}) - f^{(6)}(t_{2i})\right| \leq \frac{49755}{3519}\,w(f^{(6)};\frac{1}{m}), \tag{9}$$

$$\left|S_n^{(5)}(x_{2i+}) - f^{(5)}(x_{2i})\right| \leq \frac{15412}{1173}hw(f^{(6)};\frac{1}{m}), \tag{10}$$

$$\left|S_n^{(5)}(x_{2i-}) - f^{(5)}(x_{2i})\right| \leq \frac{45383}{2346}hw(f^{(6)};\frac{1}{m}), \tag{11}$$

$$\left|S_n^{(5)}(t_{2i}) - f^{(5)}(t_{2i})\right| \leq \frac{201253}{31671}hw(f^{(6)};\frac{1}{m}), \tag{12}$$

$$\left|S_n^{(4)}(t_{2i}) - f^{(4)}(t_{2i})\right| \leq \frac{288682}{95013} h^2\, w(f^{(6)}; \frac{1}{m}), \quad (13)$$

$$|S_n''(t_{2i}) - f''(t_{2i})| \leq \frac{615394}{4275585} h^4\, w(f^{(6)}; \frac{1}{m}), \quad (14)$$

Proof 2 *From 6 we have*

$$h^6(S_n^{(6)}(t_{2i}) - f^{(6)}(t_{2i}) = -\tfrac{1}{136} h^6 f^{(6)}(\lambda_{3,2i}) - \tfrac{243}{136} h^6 f^{(6)}(\lambda_{1,2i}) + \tfrac{12}{136} h^6 f^{(6)}(\lambda_{5,2i})$$
$$+\tfrac{80}{136} h^6 f^{(6)}(\lambda_{8,2i}) + \tfrac{288}{136} h^6 f^{(6)}(\lambda_{10,2i}) - h^6 f^{(6)}(t_{2i}) + \tfrac{18}{17} h^4 (S_n^{(4)}(x_{2i+2)} - f^{(4)}(x_{2i+2}))$$
$$+\tfrac{63}{68} h^4 (S_n^{(4)}(x_{2i)} - f^{(4)}(x_{2i})) = \tfrac{95}{34}\, h^6 \alpha_2 w(f^{(6)}; \tfrac{1}{m}) + \tfrac{18}{17}\, h^4 (S_n^{(4)}(x_{2i+2}) - f^{(4)}(x_{2i+2}))$$
$$+\tfrac{63}{68} h^4 (S_n^{(4)}(x_{2i}) - f^{(4)}(x_{2i}))\,,\ |\alpha_2|\ \leq 1$$

Hence

$$h^6(S_n^{(6)}(t_{2i}) - f^{(6)}(t_{2i})) = -\tfrac{1}{136} h^6 f^{(6)}(\lambda_{3,_{2i}}) - \tfrac{243}{136} h^6 f^{(6)}(\lambda_{1,_{2i}}) + \tfrac{12}{136} h^6 f^{(6)}(\lambda_{5,_{2i}})$$
$$+\tfrac{80}{136} h^6 f^{(6)}(\lambda_{8,_{2i}}) + \tfrac{288}{136} h^6 f^{(6)}(\lambda_{10,_{2i}}) - h^6 f^{(6)}(t_{2i}) + \tfrac{18}{17}\, h^4 (S_n^{(4)}(x_{2i+2}) - f^{(4)}(x_{2i+2}))$$
$$+\tfrac{63}{68} h^4 (S_n^{(4)}(x_{2i}) - f^{(4)}(x_{2i})) = \tfrac{95}{34}\, h^6 \alpha_2 w(f^{(6)}; \tfrac{1}{m}) + \tfrac{18}{17}\, h^4 (S_n^{(4)}(x_{2i+2}) - f^{(4)}(x_{2i+2}))$$
$$+\tfrac{63}{68} h^4 (S_n^{(4)}(x_{2i}) - f^{(4)}(x_{2i}))\,,\ |\alpha_2|\ \leq 1$$

By using 1, proof of lemma 4.1 (9) *and the proofs of lemma 4.1(2-6) are similar, and we only*

$$h^5 S_n^{(5)}(x_{2i+}) = -\tfrac{10935}{136} f(x_{2i}) + \tfrac{32805}{544} f(t_{2i}) + \tfrac{10935}{544} f(x_{2i+2}) - \tfrac{10935}{136} h f'(t_{2i}) - \tfrac{405}{34} h^3$$
$$f'''(t_{2i}) - \tfrac{97}{68} h^4 S_n^{(4)}(x_{2i}) - \tfrac{19}{34} h^4 S_n^{(4)}(x_{2i+2}),$$

$$h^5 S_n^{(5)}(x_{2i-}) = \tfrac{10935}{136} f(x_{2i-2}) - \tfrac{32805}{544} f(t_{2i-2}) - \tfrac{10935}{544} f(x_{2i}) + \tfrac{10935}{136} h f'(t_{2i-2}) + \tfrac{405}{34} h^3$$
$$f'''(t_{2i-2}) + \tfrac{29}{68} h^4 S_n^{(4)}(x_{2i-2}) + \tfrac{53}{34} h^4 S_n^{(4)}(x_{2i}),$$

$$h^5 S_n^{(5)}(t_{2i}) = -\tfrac{3645}{136} f(x_{2i}) + \tfrac{10935}{544} f(t_{2i}) + \tfrac{3645}{544} f(x_{2i+2}) - \tfrac{3645}{136} h f'(t_{2i}) - \tfrac{135}{34} h^3$$
$$f'''(t_{2i}) - \tfrac{55}{68} h^4 S_n^{(4)}(x_{2i}) + \tfrac{3}{34} h^4 S_n^{(4)}(x_{2i+2}),$$

$$h^4 S_n^{(4)}(t_{2i}) = -\tfrac{1215}{34} f(x_{2i}) + \tfrac{3645}{136} f(t_{2i}) + \tfrac{1215}{136} f(x_{2i+2}) - \tfrac{1215}{34} h f'(t_{2i}) - \tfrac{90}{17} h^3$$
$$f'''(t_{2i}) + \tfrac{13}{51} h^4 S_n^{(4)}(x_{2i}) - \tfrac{7}{51} h^4 S_n^{(4)}(x_{2i+2}),$$

And

$$h^2 S_n''(t_{2i}) = \tfrac{96}{17} f(x_{2i}) - \tfrac{729}{136} f(t_{2i}) - \tfrac{39}{136} f(x_{2i+2}) + \tfrac{141}{34} h f'(t_{2i}) + \tfrac{20}{51} h^3 f'''(t_{2i})$$
$$-\tfrac{4}{153} h^4 S_n^{(4)}(x_{2i}) + \tfrac{4}{765} h^4 S_n^{(4)}(x_{2i+2}),$$

Proof of the theorem 1.2:

For $0 \le t \le 1$we obtain

$$A_0(t) + A_1(t) + A_2(t) = 1. \tag{15}$$

Let $x2i{\le}x{\le}x2i{+}2$ on using 15 and 6 we get

$$S_n^{(5)}(x) - f^{(5)}(x) = (S_n^{(5)}(x_{2i+}) - f^{(5)}(x))A_0(\frac{x-2ih}{2h}) + (S_n^{(5)}(x_{2i+2}) - f^{(5)}(x))$$

$$A_2(\frac{x-2ih}{2h}) + (S_n^{(5)}(t_{2i}) - f^{(5)}(x) A_1(\frac{x-2ih}{2h}) + 2h\, S_n^{(6)}(t_{2i})\, A_3(\frac{x-2ih}{2h})$$

$$= L1 + L2 + L3 + L4 \tag{16}$$

From 4 it follows that:

$$|A_0(x)| \le 1 \;\;, \;\; |A_1(x)| \le 1 \;\;, \;|\, A_2(x)| \le 1 \, and \;\; |\, A_3(x)| \le 1 \tag{17}$$

Since $f^{(13)}(x) = f^{(13)}(x_{2i}) + (x - x_{2i}) f^{(14)}(\lambda)$, $x2i{<}\lambda{<}x$,

To

$$L_1 = (S_n^{(5)}(x_{2i+}) - f^{(5)}(x))A_0(\frac{x-2ih}{2h})$$

$$= (S_n^{(5)}(x_{2i}) - f^{(5)}(x_{2i}) - (x - x_{2i})f^{(6)}(\lambda))A_0(\frac{x-2ih}{2h}).$$

Therefore,On using lemma2 (2-3),(18) and $|x - x_{2i}| \leq 2h$. We obtain

$$|L_1| \leq \frac{15412}{1173} hw(f^{(6)}; \frac{1}{m}) + 2h \left\| f^{(6)} \right\| \quad (18)$$

$$|L_2| \leq \frac{45383}{2346} hw(f^{(6)}; \frac{1}{m}) + 2h \left\| f^{(6)} \right\| \quad (19)$$

$$L_3 = (S_n^{(5)}(t_{2i}) - f^{(5)}(x))A_1(\frac{x-2ih}{2h}) = \ (S_n^{(5)}(t_{2i}) - f^{(5)}(t_{2i}) + f^{(5)}(t_{2i})$$

$$-f^{(5)}(x_{2i}) - (x - x_{2i})f^{(6)}(\lambda))A_1(\frac{x-2ih}{2ih}) \quad (20)$$

From (4) we obtain

$$f^{(4)}(t_{2i}) - f^{(4)}(x_{2i}) = \frac{2}{3} h f^{(5)}(\lambda_{10,2i}) \quad (21)$$

Therefore by using 4.1(12) in (21) we obtain

$$|L_3| \leq \frac{264595}{31671} hw(f^{(6)}; \frac{1}{m}). \quad (22)$$

$$L_4 = S_n^{(6)}(t_{2i})A_1(\frac{x-2ih}{2h}) = (S_n^{(6)}(t_{2i}) - f^{(6)}(t_{2i}) + f^{(6)}(t_{2i}))A_3(\frac{x-2ih}{2h})$$

Therefore by using 4.1(9)and (18) and $|x - x_{2i}| \leq 2h$ we obtain

$$|L_4| \leq \frac{99510}{3519} hw(f^{(6)}; \frac{1}{m}) + 2h \left\| f^{(6)} \right\| \quad (23)$$

using (19), (20) (22)and (23) in (16) we obtain

$$\left| S_n^{(5)}(x) - f^{(5)}(x) \right| \leq \frac{4251275}{63342} hw(f^{(6)}; \frac{1}{m}) + \ 6h \left\| f^{(6)} \right\|. \quad (24)$$

This proves Theorem 2 for r=5. To Prove the Theorem 2 for r=4:
Since $S_n^{(4)}(x) - f^{(4)}(x) = \int_{t_{2i}}^{x}(S_n^{(5)}(t) - f^{(5)}(t))dt + S_n^{(4)}(t_{2i}) - f^{(4)}(t_{2i})$.
On using lemma 2(13)and (24) we obtain

$$\left|S_n^{(4)}(x) - f^{(4)}(x)\right| \leq \frac{13042507}{95013}\, h^2 w(f^{(6)}; \frac{1}{m}) + \; 12\,\mathrm{h}^2 \left\|f^{(6)}\right\|. \qquad (25)$$

This proves Theorem 2 for r=4. To Prove the Theorem 2 for r=3:
We get

$$S_n^{(3)}(x) - f^{(3)}(x) = \int_{t_{2i}}^{x}(S_n^{(4)}(t) - f^{(4)}(t))dt + S_n^{(3)}(t_{2i}) - f^{(3)}(t_{2i}).$$

Since $S_n^{(11)}(t_{2i}) - f^{(11)}(t_{2i}) = 0.$. and using (25) we obtain

$$\left|S_n^{(3)}(x) - f^{(3)}(x)\right| \leq \frac{26085014}{95013} h^3 w(f^{(6)}; \frac{1}{m}) + \; 24\mathrm{h}^3 \left\|f^{(6)}\right\|. \qquad (26)$$

This proves Theorem 2 for r=3. To Prove the Theorem 2 for r=2:
We get $S_n''(x) - f''(x) = \int_{t_{2i}}^{x}(S_n^{(3)}(t) - f^{(3)}(t))dt + S_n''(t_{2i}) - f''(t_{2i})$
On using lemma 2 (14)and (26) we obtain

$$|S_n''(x) - f''(x)| \leq \frac{2348266654}{4275585}\, h^{\,4} w(f^{(6)}; \frac{1}{m}) + \; 48\,\mathrm{h}^4 \left\|f^{(6)}\right\|. \qquad (27)$$

This proves Theorem 2 for r=2. To Prove the Theorem 2 for r=1:
We obtain $S_n'(x) - f'(x) = \int_{t_{2i}}^{x}(S_n^{(2)}(t) - f^{(10)}(t))dt + S_n'(t_{2i}) - f'(t_{2i})$
Since $S_n(t_{2i}) - f'(t_{2i}) = 0.$ and using (27) we obtain

$$|S_n'(x) - f'(x)| \;\leq\; \frac{4696533308}{4275585} h^5 w(f^{(6)}; \frac{1}{m}) + 96h^5 \left\|f^{(6)}\right\|. \qquad (28)$$

This proves Theorem 2 for r=1. To Prove the Theorem (**??**) for r=0:
W get $S_n(x) - f(x) = \int_{t_{2i}}^{x}(S_n'(t) - f'(t))dt + S_n(t_{2i}) - f(t_{2i})$
Since $S_n(t_{2i}) - f(t_{2i}) = 0.$ and using (28) we obtain

$$|S_n(x) - f(x)| \;\leq\; \frac{9393066616}{4275585}\, h^6\, w(f^{(6)}; \frac{1}{m}) + 192\, h^6 \;\left\|f^{(6)}\right\|.$$

Take $h = \frac{1}{2m}$ we get

$$|S_n(x) - f(x)| \;\leq\; 34.3266864942\; m^{-6}\, w(f^{(6)}; \frac{1}{m}) + 3m^{-6} \;\left\|f^{(6)}\right\|.$$

This completes the proof of theorem 2

References

[1] A.K.Varma ,Lacunary interpolation splines.II.(0,4) and (0,1,3) cases,Acta Math .Acad. Sci.Hungar., 31(3-4)(1978):193-203.

[2] B.K.Shwartz and S.R.Varga, Error Bounds for Spline and L-Spline Interpolation, Journal Of Approximation Theory, 6(1972):6-49.

[3] K.H-F. Jwamer and R.K.Saeed, Minimizing Error Bounds in Lacunary Interpolation By Spline Function (0, 1, 4) Case. Journal Of CityplaceAl-Nahrain University, country-regionIraq, 8(10)(2005): 114-119.

[4] K.H-F.Jwamer, Minimizing Error Bounds In (0, 2, 3) Lacunary Interpolation By Sixtic Spline , Journal Mathematics and CityplaceStatistics, country-regionAmerica, 3(12) (2007): 249-256.

[5] K.H-F. Jwamer and F.H-S.Kadir, Minimizing Error Bounds in Lacunary Interpolation By Spline Function For (0,2) Case,Journal Of Kirkuk University-Scientific Studies, country-regionplaceIraq ,4(2)(2009):117-124. [6] J.H. Ahlberg ,E.N. Nilson and J.L .Walsh, The Theory of Spline and Their Application, Academic press,StateNew York and CityplaceLondon,(1967).

[7] R.L.Burden and J.D. Faires, Numerical Analysis 7th Edition,Brooks Cole online at www.brookscole.com I, (2001).

Karwan H-F Jwamer
Department of Mathematics
University of Sulaimani
Kurdistan Region
e-mail:*jwameri1973@gmail.com*

Faraidun K. Hamasalh
Department of Mathematics
University of Sulaimani
PM Box 334, Sulaimani, Kurdistan Region
Email:*faraidunsalh@gmail.com*

Radha K.G
Department of Mathematics and Computers
University of Sulaimani
PM Box 334, Sulaimani, Kurdistan Region
Email:*raza82_kh@yahoo.com*