Acta Universitatis Apulensis No. 35/2013
ISSN: 1582-5329 pp. 251-257

$\mathfrak{B}-$TANGENT DEVELOPABLE SURFACES OF BIHARMONIC $\mathfrak{B}-$SLANT HELICES IN $\phi-$RICCI SYMMETRIC PARA-SASAKIAN MANIFOLD

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Abstract. In this paper, we study $\mathfrak{b}-$tangent developable surfaces of biharmonic $\mathfrak{b}-$slant helices in the special three-dimensional $\phi-$Ricci symmetric para-Sasakian manifold $\mathbb{P}$. Finally, we find out explicit parametric equations of $\mathfrak{b}-$tangent developable surfaces of biharmonic $\mathfrak{b}-$slant helices in the special three-dimensional $\phi-$Ricci symmetric para-Sasakian manifold $\mathbb{P}$.

2000 *Mathematics Subject Classification*: 53C41.

Keywords: biharmonic curve, para-Sasakian manifold, Bishop frame.

1. Introduction

Paper, sheet metal, and many other materials are approximately unstretchable. The surfaces obtained by bending these materials can be flattened onto a plane without stretching or tearing. More precisely, there exists a transformation that maps the surface onto the plane, after which the length of any curve drawn on the surface remains the same. Such surfaces, when sufficiently regular, are well known to mathematicians as developable surfaces. While developable surfaces have been widely used in engineering, design and manufacture, they have been less popular in computer graphics, despite the fact that their isometric properties make them ideal primitives for texture mapping, some kinds of surface modelling, and computer animation.

In this paper, we study $\mathfrak{b}-$tangent developable surfaces of biharmonic $\mathfrak{b}-$slant helices in the special three-dimensional $\phi-$Ricci symmetric para-Sasakian manifold $\mathbb{P}$. Finally, we find out explicit parametric equations of $\mathfrak{b}-$tangent developable surfaces of biharmonic $\mathfrak{b}-$slant helices in the special three-dimensional $\phi-$Ricci symmetric para-Sasakian manifold $\mathbb{P}$.

2. Biharmonic $\mathfrak{b}-$Slant Helices in the Special Three-Dimensional $\phi-$Ricci Symmetric Para-Sasakian Manifold $\mathbb{P}$

Let us consider biharmonicity of curves according to Bishop frame in the special three-dimensional $\phi-$Ricci Symmetric para-Sasakian manifold $\mathbb{P}$. Let $\{\mathbf{t},\mathbf{n},\mathbf{b}\}$ be the Frenet frame field along γ. Then, the Frenet frame satisfies the following Frenet–Serret equations:

$$\begin{aligned} \nabla_{\mathbf{t}}\mathbf{t} &= \kappa\mathbf{n}, \\ \nabla_{\mathbf{t}}\mathbf{n} &= -\kappa\mathbf{t}+\tau\mathbf{b}, \\ \nabla_{\mathbf{t}}\mathbf{b} &= -\tau\mathbf{n}, \end{aligned} \tag{2.1}$$

where $\kappa=|\mathcal{T}(\gamma)|=|\nabla_{\mathbf{t}}\mathbf{t}|$ is the curvature of γ and τ its torsion and

$$\begin{aligned} g(\mathbf{t},\mathbf{t}) &= 1,\ g(\mathbf{n},\mathbf{n})=1,\ g(\mathbf{b},\mathbf{b})=1, \\ g(\mathbf{t},\mathbf{n}) &= g(\mathbf{t},\mathbf{b})=g(\mathbf{n},\mathbf{b})=0. \end{aligned}$$

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. The Bishop frame is expressed as

$$\begin{aligned} \nabla_{\mathbf{t}}\mathbf{t} &= k_1\mathbf{m}_1+k_2\mathbf{m}_2, \\ \nabla_{\mathbf{t}}\mathbf{m}_1 &= -k_1\mathbf{t}, \\ \nabla_{\mathbf{t}}\mathbf{m}_2 &= -k_2\mathbf{t}, \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} g(\mathbf{t},\mathbf{t}) &= 1,\ g(\mathbf{m}_1,\mathbf{m}_1)=1,\ g(\mathbf{m}_2,\mathbf{m}_2)=1, \\ g(\mathbf{t},\mathbf{m}_1) &= g(\mathbf{t},\mathbf{m}_2)=g(\mathbf{m}_1,\mathbf{m}_2)=0. \end{aligned}$$

Here, we shall call the set $\{\mathbf{t},\mathbf{m}_1,\mathbf{m}_2\}$ as Bishop trihedra, k_1 and k_2 as Bishop curvatures and $\zeta(s)=\arctan\frac{k_2}{k_1}$, $\tau(s)=\zeta'(s)$ and $\kappa(s)=\sqrt{k_1^2+k_2^2}$.

Theorem 1. *Let $\gamma: I\longrightarrow\mathbb{P}$ be a unit speed non-geodesic biharmonic $\mathfrak{b}-$slant* *helix. Then, the parametric equations of* γ *are*

$$\begin{aligned} x(s) &= \sin\mathcal{E}s+\mathcal{A}_1, \\ y(s) &= \frac{e^{\sin\mathcal{E}s+\mathcal{A}_1}}{\mathcal{A}_0^2+\sin^2\mathcal{E}}(\sin\mathcal{E}-\mathcal{A}_0)\cos\mathcal{E}\cos[\mathcal{A}_0s+\mathcal{A}] \\ &\quad+\frac{e^{\sin\mathcal{E}s+\mathcal{A}_1}}{\mathcal{A}_0^2+\sin^2\mathcal{E}}(\sin\mathcal{E}+\mathcal{A}_0)\cos\mathcal{E}\sin[\mathcal{A}_0s+\mathcal{A}])+\mathcal{A}_2, \\ z(s) &= \frac{e^{\sin\mathcal{E}s+\mathcal{A}_1}}{\mathcal{A}_0^2+\sin^2\mathcal{E}}\mathcal{A}_0\cos\mathcal{E}\cos[\mathcal{A}_0s+\mathcal{A}] \\ &\quad-\frac{e^{\sin\mathcal{E}s+\mathcal{A}_1}}{\mathcal{A}_0^2+\sin^2\mathcal{E}}\sin\mathcal{E}\cos\mathcal{E}\sin[\mathcal{A}_0s+\mathcal{A}])+\mathcal{A}_3, \end{aligned} \tag{2.3}$$

where $\mathcal{A},\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3$ *are constants of integration and*

$$\mathcal{A}_0=(\frac{k_1^2+k_2^2-\cos^2\mathcal{E}}{\cos^2\mathcal{E}})^{\frac{1}{2}}.$$

3. $\mathfrak{b}-$Tangent Developable Surfaces of Biharmonic $\mathfrak{b}-$Slant Helices in the Special Three-Dimensional $\phi-$Ricci Symmetric Para-Sasakian Manifold $\mathbb{P}$

To separate a tangent developable according to Bishop frame from that of Frenet-Serret frame, in the rest of the paper, we shall use notation for this surface as $\mathfrak{b}-$tangent developable.

The purpose of this section is to study $\mathfrak{b}-$tangent developable surfaces of $\mathfrak{b}-$slant helices in $\mathbb{P}$.

The $\mathfrak{b}-$tangent developable of γ is a ruled surface

$$\mathcal{R}(s,u)=\gamma(s)+u\gamma'(s). \tag{3.1}$$

Theorem 2. *Let* $\mathcal{R}$ *be* $\mathfrak{b}-$*tangent developable of a unit speed non-geodesic biharmonic* $\mathfrak{b}-$*slant helix in* $\mathbb{P}$*. Then, the parametric equations of* $\mathfrak{b}-$*tangent developable are*

$$\begin{aligned}
\boldsymbol{x}_{\mathcal{R}}(s,u) &= \sin\mathcal{E}s + u\sin\mathcal{E} + \mathcal{A}_1, \\
\boldsymbol{y}_{\mathcal{R}}(s,u) &= \frac{e^{\sin\mathcal{E}s+\mathcal{A}_1}}{\tan^2\mathcal{E}+\sin^2\mathcal{E}}(\sin\mathcal{E}-\tan\mathcal{E})\cos\mathcal{E}\cos[\tan\mathcal{E}s+\mathcal{A}] \\
&\quad + \frac{e^{\sin\mathcal{E}s+\mathcal{A}_1}}{\tan^2\mathcal{E}+\sin^2\mathcal{E}}(\sin\mathcal{E}+\tan\mathcal{E})\cos\mathcal{E}\sin[\tan\mathcal{E}s+\mathcal{A}]) \\
&\quad + ue^{\sin\mathcal{E}s+\mathcal{A}_1}(\cos\mathcal{E}\cos[\mathcal{A}_0 s+\mathcal{A}]+\cos\mathcal{E}\sin[\tan\mathcal{E}s+\mathcal{A}])+\mathcal{A}_2, \qquad (3.2) \\
\boldsymbol{z}_{\mathcal{R}}(s,u) &= \frac{e^{\sin\mathcal{E}s+\mathcal{A}_1}}{\tan^2\mathcal{E}+\sin^2\mathcal{E}}\tan\mathcal{E}\cos\mathcal{E}\cos[\tan\mathcal{E}s+\mathcal{A}] \\
&\quad - \frac{e^{\sin\mathcal{E}s+\mathcal{A}_1}}{\tan^2\mathcal{E}+\sin^2\mathcal{E}}\sin\mathcal{E}\cos\mathcal{E}\sin[\tan\mathcal{E}s+\mathcal{A}] \\
&\quad - ue^{\sin\mathcal{E}s+\mathcal{A}_1}\cos\mathcal{E}\sin[\tan\mathcal{E}s+\mathcal{A}]+\mathcal{A}_3,
\end{aligned}$$

where $\mathcal{A},\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3$ are constants of integration.

Proof. By the Bishop formula, we have the following equation

$$\mathbf{t} = \cos\mathcal{E}\cos[\mathcal{A}_0 s+\mathcal{A}]\,\mathbf{e}_1 + \cos\mathcal{E}\sin[\mathcal{A}_0 s+\mathcal{A}]\,\mathbf{e}_2 - \sin\mathcal{E}\mathbf{e}_3. \qquad (3.3)$$

Using (3.3), we obtain

$$\begin{aligned}
\mathbf{t} &= (\sin\mathcal{E}, e^{\sin\mathcal{E}s+\mathcal{A}_1}(\cos\mathcal{E}\cos[\mathcal{A}_0 s+\mathcal{A}]+\cos\mathcal{E}\sin[\mathcal{A}_0 s+\mathcal{A}]), \\
&\quad -e^{\sin\mathcal{E}s+\mathcal{A}_1}\cos\mathcal{E}\sin[\mathcal{A}_0 s+\mathcal{A}]),
\end{aligned}$$

where

$$\mathcal{A}_0 = (\frac{k_1^2+k_2^2-\cos^2\mathcal{E}}{\cos^2\mathcal{E}})^{\frac{1}{2}}.$$

Consequently, the parametric equations of $\mathcal{R}$ can be found from (3.1), (3.3). This concludes the proof of Theorem.

We can prove the following interesting main result.

Theorem 3. . *Let $\mathcal{R}$ be $\mathfrak{b}-$tangent developable of a unit speed non-geodesic biharmonic $\mathfrak{b}-$slant helix in $\mathbb{P}$. Then the equation of $\mathcal{B}-$tangent developable is given* *by*

$$\begin{aligned}
\mathcal{R}(s,u) &= e^{-(\sin\mathcal{E}s+u\sin\mathcal{E}+\mathcal{A}_1)}[\frac{e^{\sin\mathcal{E}s+\mathcal{A}_1}}{\tan^2\mathcal{E}+\sin^2\mathcal{E}}(\sin\mathcal{E}-\tan\mathcal{E})\cos\mathcal{E}\cos[\tan\mathcal{E}s+\mathcal{A}] \\
&+\frac{e^{\sin\mathcal{E}s+\mathcal{A}_1}}{\tan^2\mathcal{E}+\sin^2\mathcal{E}}(\sin\mathcal{E}+\tan\mathcal{E})\cos\mathcal{E}\sin[\tan\mathcal{E}s+\mathcal{A}]) \\
&+ue^{\sin\mathcal{E}s+\mathcal{A}_1}(\cos\mathcal{E}\cos[\mathcal{A}_0s+\mathcal{A}]+\cos\mathcal{E}\sin[\tan\mathcal{E}s+\mathcal{A}])+\mathcal{A}_2 \\
&+\frac{e^{\sin\mathcal{E}s+\mathcal{A}_1}}{\tan^2\mathcal{E}+\sin^2\mathcal{E}}\tan\mathcal{E}\cos\mathcal{E}\cos[\tan\mathcal{E}s+\mathcal{A}] \qquad (3.4) \\
&-\frac{e^{\sin\mathcal{E}s+\mathcal{A}_1}}{\tan^2\mathcal{E}+\sin^2\mathcal{E}}\sin\mathcal{E}\cos\mathcal{E}\sin[\tan\mathcal{E}s+\mathcal{A}] \\
&-ue^{\sin\mathcal{E}s+\mathcal{A}_1}\cos\mathcal{E}\sin[\tan\mathcal{E}s+\mathcal{A}]+\mathcal{A}_3]\mathbf{e}_1 \\
&-e^{-(\sin\mathcal{E}s+u\sin\mathcal{E}+\mathcal{A}_1)}[\frac{e^{\sin\mathcal{E}s+\mathcal{A}_1}}{\tan^2\mathcal{E}+\sin^2\mathcal{E}}\tan\mathcal{E}\cos\mathcal{E}\cos[\tan\mathcal{E}s+\mathcal{A}] \\
&-\frac{e^{\sin\mathcal{E}s+\mathcal{A}_1}}{\tan^2\mathcal{E}+\sin^2\mathcal{E}}\sin\mathcal{E}\cos\mathcal{E}\sin[\tan\mathcal{E}s+\mathcal{A}] \\
&-ue^{\sin\mathcal{E}s+\mathcal{A}_1}\cos\mathcal{E}\sin[\tan\mathcal{E}s+\mathcal{A}]+\mathcal{A}_3]\mathbf{e}_2 \\
&-[\sin\mathcal{E}s+u\sin\mathcal{E}+\mathcal{A}_1]\mathbf{e}_3,
\end{aligned}$$

where $\mathcal{A},\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3$ are constants of integration.

Proof. We assume that γ is a unit speed $\mathfrak{b}-$slant helix.

Substituting basis to (3.2), we have (3.4). Thus, the proof is completed.

Thus, we proved the following:

Theorem 4. *Let $\mathcal{R}$ be $\mathfrak{b}-$tangent developable of a unit speed non-geodesic biharmonic $\mathfrak{b}-$slant helix in $\mathbb{P}$. Then, normal of $\mathfrak{b}-$tangent developable of γ is*

$$\begin{aligned}
\mathbf{n}_{\mathcal{R}} &= [uk_1\sin[\tan\mathcal{E}s+\mathcal{A}]-uk_2\sin\mathcal{E}\cos[\tan\mathcal{E}s+\mathcal{A}]]\mathbf{e}_1 \\
&+[-uk_1\cos[\tan\mathcal{E}s+\mathcal{A}]-uk_2\sin\mathcal{E}\sin[\tan\mathcal{E}s+\mathcal{A}]]\mathbf{e}_2 \\
&-uk_2\cos\mathcal{E}\mathbf{e}_3.
\end{aligned}$$

where $\mathcal{A}$ is constants of integration.

Proof. Assume that $\mathbf{n}_{\mathcal{R}}$ be the normal vector field on $\mathfrak{b}-$tangent developable defined by

$$\mathbf{n}_{\mathcal{R}}=\mathcal{R}_s\wedge\mathcal{R}_u.$$

This concludes the proof of theorem.

Acknowledgements. The authors would like to express their sincere gratitude to the referees for the valuable suggestions to improve the paper.

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