

ON A NEW SUBCLASS OF MEROMORPHIC HARMONIC
FUNCTIONS WITH FIXED RESIDUE α

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ABSTRACT. In this paper, we obtain some properties such as coefficient conditions, distortion theorem and extreme points for a certain subclass of meromorphic harmonic functions with fixed residue α , $0 \leq \alpha < 1$.

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1. INTRODUCTION

The important study initiated by Clunie and Sheil-Small [3] on the class H consisting of complex valued harmonic sense-preserving univalent functions f in a simply connected domain $D \subseteq C$ defined on the open unit disc $\Delta = \{z : |z| < 1\}$ and normalized by $f(0) = f_z(0) - 1 = 0$ formed the basis for various studies related to different subclasses of harmonic univalent functions. It is known that [3] each function $f \in H$ can be expressed as $f = h + \bar{g}$, where h and g are analytic in D . In fact H reduces to g , the class of normalized univalent functions if the co-analytic part of f is zero. A compact survey on harmonic univalent functions is given by Ahuja [1].

Jinxi Ma [8] considered the class S_p of functions f which are meromorphic and univalent in the unit disc Δ normalized by $f(0) = 0$, $f'(0) = 1$ and $f(p) = \infty$, with $0 < p < 1$.

Observe that in the annulus $\{z : p < |z| < 1\}$ each function h in S_p admits an expansion written as

$$h(z) = \frac{\alpha}{z-p} + \sum_{n=1}^{\infty} a_n z^n \quad (1)$$

where $\alpha = Res(f, p)$ with $0 < \alpha \leq 1$, $z \in \Delta \setminus \{p\}$.

Jinxi Ma [8] and also Ghanim and Darus [4, 6] have made use of the function h given in (1) and studied some properties.

Let HS_p denote the class of functions $f = h + \bar{g}$ that are harmonic univalent and sense preserving in the punctured unit disk $\Delta \setminus \{p\}$.

For $f = h + \bar{g}$, we may write the analytic function h as in (1) and g as

$$g(z) = \sum_{n=1}^{\infty} b_n z^n.$$

Then we have

$$f(z) = h(z) + \overline{g(z)} = \frac{\alpha}{z-p} + \sum_{n=1}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n}, \quad (2)$$

where $\alpha = \text{Res}(f, p)$, with $0 < \alpha \leq 1$, $z \in \Delta \setminus \{p\}$.

Let \overline{HS}_p be a subclass of HS_p consisting of function of the form

$$f(z) = h(z) + \overline{g(z)} = \frac{\alpha}{z-p} + \sum_{n=1}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n}, \quad (a_n, b_n \geq 0) \quad (3)$$

where $\alpha = \text{Res}(f, p)$, with $0 < \alpha \leq 1$, $z \in \Delta \setminus \{p\}$, which are univalent harmonic in the punctured unit disk $\Delta \setminus \{p\}$. The functions h and g are analytic in $\Delta \setminus \{p\}$ and Δ respectively and h has a simple pole at the point p with residue α .

For $\alpha = 1$ and $p = 0$, the function f defined in (3) was studied by Bostanci, Yalcin and Öztürk [2].

In [4, 5, 6] Ghanim et al. defined the operator I^k on the class HS_p as follows:

$$\begin{aligned} I^0 f(z) &= f(z), \\ I^k f(z) &= I^k h(z) + \overline{I^k g(z)}, \quad k = 1, 2, 3, \dots, \end{aligned} \quad (4)$$

where,

$$I^k h(z) = z(I^{k-1} h(z))' + \frac{\alpha(2z-p)}{(z-p)^2} = \frac{\alpha}{z-p} + \sum_{n=1}^{\infty} n^k a_n z^n$$

and

$$I^k g(z) = z(I^{k-1} g(z))' = \sum_{n=1}^{\infty} n^k b_n z^n.$$

Motivated by the earlier works of [2, 6, 7, 9], we now introduce a new subclass $HS_p^*(k, \alpha, \beta, \mu)$ using the differential operator I^k .

Definition 1. A function $f \in HS_p^*(k, \alpha, \beta, \mu)$, if it satisfies

$$\left| \frac{H(z)}{\mu H(z) + (1 - \mu)} + 1 \right| \leq \left| \frac{H(z)}{\mu H(z) + (1 - \mu)} + 2\beta - 1 \right| \quad (5)$$

where $H(z) = \frac{z(I^k h(z))' + \overline{z(I^k g(z))'}}{I^k f(z)}$, ($k \in N_0 = N \cup \{0\}$), $0 \leq \beta < 1$, $0 \leq \mu < 1$ and for all z in $\Delta \setminus \{p\}$.

Remark 1. $HS_p^*(k, \alpha, \beta, 0) = SH_p^*(k, \alpha, \beta)$ [6].

Also

$$HS_p^*[k, \alpha, \beta, \mu] = HS_p^*(k, \alpha, \beta, \mu) \cap \overline{HS_p}.$$

We now obtain the coefficient estimates for the classes $HS_p^*(k, \alpha, \beta, \mu)$ and $HS_p^*[k, \alpha, \beta, \mu]$.

2. MAIN RESULTS

A sufficient coefficient condition for functions analytic in $\Delta \setminus \{p\}$ to be in $HS_p^*(k, \alpha, \beta, \mu)$ is now derived.

Theorem 1. Let $f(z) = h(z) + \overline{g(z)}$ be given by (2).

If

$$\sum_{n=1}^{\infty} n^k [(n + \beta) + \beta(n - 1)\mu] (1 - p)^2 (|a_n| + |b_n|) \leq \alpha(1 - \beta)[(1 - p) - \mu(2 - p)], \quad (6)$$

where $0 \leq \beta < 1$, $k \in N_0$, then f is sense preserving in $\Delta \setminus \{p\}$ and $f \in HS_p^*(k, \alpha, \beta, \mu)$.

Proof. Assume that (6) holds true for $0 \leq \beta < 1$. Then by (5) we have

$$\left| \frac{H(z)}{\mu H(z) + (1 - \mu)} + 1 \right| < \left| \frac{H(z)}{\mu H(z) + (1 - \mu)} + 2\beta - 1 \right|$$

This gives

$$\begin{aligned} & \left| [z(I^k h(z))' + \overline{z(I^k g(z))'}](1 + \mu) + I^k f(z)(1 - \mu) \right| \\ & < \left| [z(I^k h(z))' + \overline{z(I^k g(z))'}](1 + \mu(2\beta - 1)) + (2\beta - 1)(1 - \mu)I^k f(z) \right|. \end{aligned}$$

Let

$$M(z) = \left| [z(I^k h(z))' + \overline{z(I^k g(z))'}](1 + \mu) + I^k f(z)(1 - \mu) \right| \\ - \left| [z(I^k h(z))' + \overline{z(I^k g(z))'}](1 + \mu(2\beta - 1)) + (2\beta - 1)(1 - \mu)I^k f(z) \right|.$$

Then, for $|z| = r$, and since $|z - p| \geq |z| - p = r - p$, we have

$$M(z) = \left| (1 + \mu) \left[-\frac{\alpha z}{(z - p)^2} + z \sum_{n=1}^{\infty} n^{k+1} a_n z^{n-1} + z \sum_{n=1}^{\infty} n^{k+1} \overline{b_n z^{n-1}} \right] \right. \\ \left. + (1 - \mu) \left[\frac{\alpha}{z - p} + \sum_{n=1}^{\infty} n^k a_n z^n + \sum_{n=1}^{\infty} n^k \overline{b_n z^n} \right] \right| \\ - \left| (1 + \mu(2\beta - 1)) \left[-\frac{\alpha z}{(z - p)^2} + z \sum_{n=1}^{\infty} n^{k+1} a_n z^{n-1} + z \sum_{n=1}^{\infty} n^{k+1} \overline{b_n z^{n-1}} \right] \right. \\ \left. + (2\beta - 1)(1 - \mu) \left[\frac{\alpha}{z - p} + \sum_{n=1}^{\infty} n^k a_n z^n + \sum_{n=1}^{\infty} n^k \overline{b_n z^n} \right] \right|.$$

Also we notice that

$$M(r) \leq \frac{\alpha p + \mu(2\alpha r - \alpha p)}{(r - p)^2} + \sum_{n=1}^{\infty} n^k [(n + 1) + \mu(n - 1)] [|a_n| r^n + |b_n| r^n] \\ - \frac{[2\alpha r - 2\alpha\beta r + 2\alpha\beta p - \alpha p + \mu[4\alpha\beta r - 2\alpha r - 2\alpha\beta p + \alpha p]]}{(r - p)^2} \\ + \sum_{n=1}^{\infty} n^k [(n + 2\beta - 1) + \mu(2\beta - 1)(n - 1)] [|a_n| r^n + |b_n| r^n] \\ = -\frac{2\alpha(1 - \beta)}{r - p} + \frac{2\alpha\mu(2r - p)(1 - \beta)}{(r - p)^2} \\ + \sum_{n=1}^{\infty} n^k [2(n + \beta) + 2\beta(n - 1)\mu] [|a_n| + |b_n|] r^n.$$

In other words

$$(r - p)^2 M(r) \leq \sum_{n=1}^{\infty} n^k [2(n + \beta) + 2\beta(n - 1)\mu] [|a_n| + |b_n|] (r - p)^2 r^n \\ - 2\alpha(1 - \beta)(r - p) + 2\alpha\mu(2r - p)(1 - \beta) \quad (7)$$

The inequality in (7) holds true for all r ($0 \leq r < 1$). Therefore letting $r \rightarrow 1$ in (7), we obtain

$$(1-p)^2 M(r) \leq \sum_{n=1}^{\infty} 2n^k [(n+\beta) + \beta(n-1)\mu] (1-p)^2 [|a_n| + |b_n|] - 2\alpha(1-\beta)(1-p) + 2\alpha\mu(2-p)(1-\beta).$$

By the hypothesis (6), it follows that (5) holds, so that $f \in HS_p^*(k, \alpha, \beta, \mu)$. We observe that f is sense-preserving in $\Delta \setminus \{p\}$. This is because

$$\begin{aligned} |h'(z)| &\geq \frac{1}{|z-p|^2} - \sum_{n=1}^{\infty} n|a_n||z|^{n-1} \\ &\geq \frac{1}{|z|^2} - \sum_{n=1}^{\infty} n|a_n||z|^{n-1} \\ &\geq \frac{1}{r^2} - \sum_{n=1}^{\infty} n|a_n|r^{n-1} \\ &\geq 1 - \sum_{n=1}^{\infty} n|a_n| \\ &\geq 1 - \sum_{n=1}^{\infty} n[(n+\beta) + \beta(n-1)\mu](1-p)^2|a_n| \\ &\geq \sum_{n=1}^{\infty} n[(n+\beta) + \beta(n-1)\mu](1-p)^2|b_n| \\ &\geq \sum_{n=1}^{\infty} n|b_n| \geq \sum_{n=1}^{\infty} n|b_n| |z|^{n-1} \geq |g'(z)|. \end{aligned}$$

Hence the theorem.

Letting $k = \beta = 0$ and $p \rightarrow 0$ in Theorem 1, then we have the next corollary:

Corollary 2. *If $f(z) = h(z) + \overline{g(z)}$ is of the form (2) and satisfies the condition*

$$\sum_{n=1}^{\infty} n(|a_n| + |b_n|) \leq \alpha(1-2\mu)$$

then f is sense preserving in $\Delta \setminus \{0\}$ and $f \in HS_0^(0, \alpha, 0, \mu)$.*

Remark 2. *Let $k = \beta = \mu = 0$ and $p \rightarrow 0$ in Theorem 1, then we have a result obtained by Ghanim and Darus [6].*

Remark 3. Let $\mu = k = \beta = 0$, $\alpha = 1$ and $p \rightarrow 0$ in Theorem 1, then we have a result obtained by Bostanci, Yalcin and Öztürk [2].

Letting $k = 1$, $\beta = 0$ and $p \rightarrow 0$ in Theorem 1, then we have the next corollary:

Corollary 3. If $f(z) = h(z) + \overline{g(z)}$ is of the form (2) and satisfies the condition

$$\sum_{n=1}^{\infty} n^2(|a_n| + |b_n|) \leq \alpha(1 - 2\mu)$$

then f is sense preserving in $\Delta \setminus \{0\}$ and $f \in HS_0^*(1, \alpha, 0, \mu)$.

Remark 4. Let $k = 1$, $\beta = 0$, $\alpha = 1$, $\mu = 0$ and $p \rightarrow 0$ in Theorem 1, then we have a result due to Bostanci, Yalcin and Öztürk [2].

Next we obtain a necessary and sufficient condition for a function $f \in \overline{HS}_p$ given by (3) to be in $HS_p^*[k, \alpha, \beta, \mu]$.

Theorem 4. Let $f \in \overline{HS}_p$ be given by (3). Then $f \in HS_p^*[k, \alpha, \beta, \mu]$ if and only if

$$\sum_{n=1}^{\infty} n^k [(n + \beta) + \beta(n - 1)\mu](1 - p)^2 (a_n + b_n) \leq \alpha(1 - \beta)[(1 - p) - \mu(2 - p)] \quad (k \in N_0) \quad (8)$$

is satisfied. The estimate (8) is sharp and the equality is attained for the function

$$f(z) = \frac{\alpha}{z - p} + \frac{\alpha(1 - \beta)[(1 - p) - \mu(2 - p)]}{n^k [(n + \beta) + \beta(n - 1)\mu](1 - p)^2} z^n + \frac{\alpha(1 - \beta)[(1 - p) - \mu(2 - p)]}{n^k [(n + \beta) + \beta(n - 1)\mu](1 - p)^2} \overline{z}^n.$$

Proof. The if part follows from Theorem 1. Hence, it suffices to show that the ‘only if’ part is true.

Assume that $f \in HS_p^*[k, \alpha, \beta, \mu]$. Then

$$\begin{aligned} & \left| \frac{\frac{H(z)}{\mu H(z) + (1 - \mu)} + 1}{\frac{H(z)}{\mu H(z) + (1 - \mu)} + 2\beta - 1} \right| \\ &= \left| \frac{\frac{-\alpha p - \alpha\mu(2z - p)}{(z - p)^2} + \sum_{n=1}^{\infty} n^k [(n + 1) + \mu(n - 1)](a_n z^n + \overline{b_n z^n})}{\frac{-2\alpha z + 2\alpha\beta z - 2\alpha\beta p + \alpha p - \mu[4\alpha\beta z - 2\alpha z - 2\alpha\beta p + \alpha p]}{(z - p)^2} + \sum_{n=1}^{\infty} n^k [(n + 2\beta - 1) + \mu(2\beta - 1)(n - 1)] [a_n z^n + \overline{b_n z^n}]} \right| \\ &\leq 1, \end{aligned} \tag{9}$$

$z \in \Delta \setminus \{p\}$.

Since $Re(z) \leq |z|$ for all z it follows from (9) that

$$Re \left\{ \frac{\frac{-\alpha p - \alpha \mu (2z - p)}{(z - p)^2} + \sum_{n=1}^{\infty} n^k [(n + 1) + \mu(n - 1)] (a_n z^n + \overline{b_n z^n})}{\frac{-2\alpha [(1 - \beta)z + \beta p] + \alpha p - \mu [4\alpha \beta z - 2\alpha z - 2\alpha \beta p + \alpha p]}{(z - p)^2} + \sum_{n=1}^{\infty} n^k [(n + 2\beta - 1) + \mu(2\beta - 1)(n - 1)] [a_n z^n + \overline{b_n z^n}]} \right\} \leq 1, \quad z \in \Delta \setminus \{p\}. \quad (10)$$

Choosing the values z on the real axis and upon clearing the denominator in (10) and letting $z \rightarrow 1$ through real values, we obtain

$$\begin{aligned} & \sum_{n=1}^{\infty} n^k [(n + 1) + \mu(n - 1)] (1 - p)^2 (a_n + b_n) \\ & \leq 2\alpha(1 - \beta)(1 - p) - 2\alpha\mu(2 - p)(1 - \beta) \\ & - \sum_{n=1}^{\infty} n^k [(n + 2\beta - 1) + \mu(2\beta - 1)(n - 1)] (1 - p)^2 (a_n + b_n) \end{aligned}$$

which immediately yields the required condition (8).

3. DISTORTION THEOREM

We now prove the following distortion theorem for functions in the class $HS_p^*[k, \alpha, \beta, \mu]$.

Theorem 5. *If the function f defined by (3) is in the class $HS_p^*[k, \alpha, \beta, \mu]$, then for $|z| = r$, we have*

$$|f(z)| \leq \frac{\alpha}{r - p} + \frac{\alpha(1 - \beta)(1 - p) - \alpha\mu(2 - p)(1 - \beta)}{(1 + \beta)(1 - p)^2} r$$

Proof. Let $f \in HS_p^*[k, \alpha, \beta, \mu]$, taking the absolute value of f we obtain

$$\begin{aligned} |f(z)| &\leq \frac{\alpha}{r-p} + \sum_{n=1}^{\infty} (a_n + b_n)r^n \\ &\leq \frac{\alpha}{r-p} + \frac{[\alpha(1-\beta)(1-p) - \alpha\mu(2-p)(1-\beta)]}{(1+\beta)(1-p)^2} \\ &\quad \sum_{n=1}^{\infty} n^k \frac{[(n+\beta) + \beta(n-1)\mu](1-p)^2}{\alpha(1-\beta)(1-p) - \alpha\mu(2-p)(1-\beta)} (a_n + b_n)r^n \\ &\leq \frac{\alpha}{r-p} + \frac{\alpha(1-\beta)(1-p) - \alpha\mu(2-p)(1-\beta)}{(1+\beta)(1-p)^2} r. \end{aligned}$$

The functions

$$f(z) = \frac{\alpha}{z-p} + \frac{\alpha(1-\beta)(1-p) - \alpha\mu(2-p)(1-\beta)}{(1+\beta)(1-p)^2} z$$

and

$$f(z) = \frac{\alpha}{z-p} + \frac{\alpha(1-\beta)(1-p) - \alpha\mu(2-p)(1-\beta)}{(1+\beta)(1-p)^2} \bar{z}$$

for $0 \leq \alpha < 1$ and $0 \leq \beta < 1$, $0 \leq \mu < 1$ show that the bound given in Theorem 5 are sharp in $\Delta \setminus \{p\}$.

Theorem 6. *Let*

$$h_0(z) = \frac{\alpha}{z-p}, \quad g_0(z) = 0,$$

for $n = 1, 2, 3, \dots$,

$$h_n(z) = \frac{\alpha}{z-p} + \frac{\alpha(1-\beta)(1-p) - \alpha\mu(2-p)(1-\beta)}{n^k[(n+\beta) + \beta(n-1)\mu](1-p)^2} z^n \quad (11)$$

and

$$g_n(z) = \frac{\alpha(1-\beta)(1-p) - \alpha\mu(2-p)(1-\beta)}{n^k[(n+\beta) + \beta(n-1)\mu](1-p)^2} \bar{z}^n \quad (12)$$

Then $f \in HS_p^*[k, \alpha, \beta, \mu]$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=0}^{\infty} (\lambda_n h_n + \gamma_n g_n) \quad (13)$$

where $\lambda_n \geq 0$, $\gamma_n \geq 0$ and $\sum_{n=0}^{\infty} (\lambda_n + \gamma_n) = 1$. In particular, the extreme points of $HS_p^*[k, \alpha, \beta, \mu]$ are $\{h_n\}$ and $\{g_n\}$.

Proof. From (11), (12) and (13), we have

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} (\lambda_n h_n + \gamma_n g_n) \\ &= \sum_{n=0}^{\infty} (\lambda_n + \gamma_n) \frac{\alpha}{z-p} + \sum_{n=1}^{\infty} \frac{\alpha(1-\beta)(1-p) - \alpha\mu(2-p)(1-\beta)}{n^k[(n+\beta) + \beta(n-1)\mu](1-p)^2} \lambda_n z^n \\ &\quad + \sum_{n=0}^{\infty} \frac{\alpha(1-\beta)(1-p) - \alpha\mu(2-p)(1-\beta)}{n^k[(n+\beta) + \beta(n-1)\mu](1-p)^2} \gamma_n \bar{z}^n. \end{aligned}$$

Then

$$\begin{aligned} &\sum_{n=1}^{\infty} n^k[(n+\beta) + \beta(n-1)\mu](1-p)^2 \frac{\lambda_n}{n^k[(n+\beta) + \beta(n-1)\mu](1-p)^2} \\ &+ \sum_{n=0}^{\infty} n^k[(n+\beta) + \beta(n-1)\mu](1-p)^2 \frac{\gamma_n}{n^k[(n+\beta) + \beta(n-1)\mu](1-p)^2} \\ &= \sum_{n=1}^{\infty} (\lambda_n + \gamma_n) - \lambda_0 = 1 - \lambda_0 \leq 1 \end{aligned}$$

So $f \in HS_p^*[k, \alpha, \beta, \mu]$.

Conversely, suppose that $f \in HS_p^*[k, \alpha, \beta, \mu]$. Set

$$\lambda_n = \frac{n^k[(n+\beta) + \beta(n-1)\mu](1-p)^2}{\alpha(1-\beta)(1-p) - \alpha\mu(2-p)(1-\beta)} a_n, \quad n \geq 1$$

and

$$\gamma_n = \frac{n^k[(n+\beta) + \beta(n-1)\mu](1-p)^2}{\alpha(1-\beta)(1-p) - \alpha\mu(2-p)(1-\beta)} b_n, \quad n \geq 0$$

Then, by Theorem 4, $0 \leq \lambda_n \leq 1$ ($n = 1, 2, \dots$) and $0 \leq \gamma_n \leq 1$, ($n = 0, 1, 2, \dots$).

Define

$$\lambda_0 = 1 - \sum_{n=1}^{\infty} \lambda_n - \sum_{n=0}^{\infty} \gamma_n$$

and note that, by Theorem 4, $\lambda_0 \geq 0$.

Consequently, we obtain

$$f(z) = \sum_{n=0}^{\infty} (\lambda_n h_n + \gamma_n g_n),$$

as required.

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