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NOTE ON SOME APPLICATIONS OF SRIVASTAVA-ATTIYA OPERATOR TO P-VALENT STARLIKE FUNCTIONS. II

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Abstract. In this note we re-proof Theorems 2.3 and 2.5 in [1], considering the generalized Srivastava-Attiya operator $J_{s,b}(f)(z)$ with $b \in \mathbb{C}\diagdown\mathbb{Z}^- = \{-1,-2,...\}$; $s \in \mathbb{C}; p \in \mathbb{N}$.

1. Introduction

Let $A(p)$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \qquad (p \in \mathbb{N} = \{1,2,...\}) \tag{1.1}$$

which are analytic and p-valent in the unit disc $U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. In [1] the authors used the generalized Srivastava-Attiya operator $J_{s,p}f(z)$ defined by Liu (see [2]) as follows:

$$J_{s,b}(f)(z) = z^p + \sum_{n=1}^{\infty} \left(\frac{1+b}{n+1+b}\right)^s a_{n+p} z^{n+p} (b \in \mathbb{C}\backslash Z^- = \{-1,-2,...\}; s \in \mathbb{C}; p \in \mathbb{N}; z \in U),$$

to introduce the following classes:

$$S^*_{p,s,b}(\gamma) = \left\{f : f \in A(p) \text{ and } J_{s,b}(f)(z) \in S^*_p(\gamma), 0 \le \gamma < p, p \in \mathbb{N}\right\},$$

$$C_{p,s,b}(\gamma) = \{f : f \in A(p) \text{ and } J_{s,b}(f)(z) \in C_p(\gamma), 0 \le \gamma < p, p \in \mathbb{N}\},$$

$$K_{p,s,b}(\beta,\gamma) = \{f : f \in A(p) \text{ and } J_{s,b}(f)(z) \in K_p(\beta,\gamma), 0 \le \beta,\gamma < p, p \in \mathbb{N}\},$$

$$K^*_{p,s,b}(\beta,\gamma) = \left\{f : f \in A(p) \text{ and } J_{s,b}(f)(z) \in K^*_p(\beta,\gamma), 0 \le \beta,\gamma < p, p \in \mathbb{N}\right\},$$

where the classes $S^*_p(\gamma), C_p(\gamma), K_p(\beta,\gamma)$ and $K^*_p(\beta,\gamma)$ are, respectively, p-valent starlike of order γ, p-valent convex of order γ, p-valent close-to-convex of order β and type γ and p-valent quasi-convex of order β and type γ.

In this note we re-proof Theorems 2.3 and 2.5 in [1], considering the generalized Srivastava-Attiya operator $J_{s,b}(f)(z)$ with $b \in \mathbb{C}\diagdown\mathbb{Z}^- = \{-1,-2,...\}$; $s \in \mathbb{C}; p \in \mathbb{N}$.

2. Main Results

To prove our main results we shall need the following lemma.

Lemma 1. *[3]. Let $\theta(u,v)$ be a complex-valued function such that*

$$\theta : D \to \mathbb{C},\ D \subset \mathbb{C} \times \mathbb{C} \quad (\mathbb{C} \text{ is the complex plane})$$

and let $u = u_1 + iu_2$ and $v = v_1 + iv_2$. Suppose that $\theta(u,v)$ satisfies the following conditions :

(i) $\theta(u,v)$ is continuous in D;
(ii) $(1,0) \in D$ and $\Re\{\theta(1,0)\} > 0$;
(iii) for all $(iu_2, v_1) \in D$ such that

$$v_1 \leq -\frac{1}{2}(1+u_2^2) \quad , \quad \Re\{\theta(iu_2, v_1)\} \leq 0\,.$$

Let

$$q(z) = 1 + q_1 z + q_2 z^2 + ...$$

be analytic in U such that $(q(z), zq^{'}(z)) \in D\,(z \in U)$. If

$$\Re\left\{\theta(q(z), zq^{'}(z))\right\} > 0 \quad (z \in U)\,,$$

then

$$\Re\{q(z)\} > 0 \quad (z \in U)\,.$$

Theorem 2. *$S^*_{p,s,b}(\gamma) \subset S^*_{p,s+1,b}(\gamma)$ for $s,\ b \in \mathbb{C}$ and b satisfying $\Re\{b\} = b_1 > p - \gamma - 1$.*

Proof. Let $f(z) \in S^*_{p,s,b}(\gamma)$ and set

$$\frac{z(J_{s+1,b}f(z))^{'}}{J_{s+1,b}f(z)} = \gamma + (p-\gamma)h(z) \tag{2.1}$$

where $h(z) = 1 + c_1 z + c_2 z^2 + ...$ By using the identity:

$$z(J_{s+1,b}f(z))^{'} = [p-(1+b)]J_{s+1,b}f(z) + (1+b)J_{s,b}f(z), \tag{2.2}$$

we have

$$\frac{J_{s,b}f(z)}{J_{s+1,b}f(z)} = \frac{1}{(b+1)}\left\{\gamma + (p-\gamma)h(z) - [p-(1+b)]\right\} \tag{2.3}$$

Differentiating (2.3) logarithmically with respect to z, we obtain

$$\frac{z(J_{s,b}f(z))'}{J_{s,b}f(z)} - \gamma = (p-\gamma)h(z) + \frac{(p-\gamma)zh'(z)}{(p-\gamma)h(z)+\gamma-p+b+1}\,. \tag{2.4}$$

Let

$$\theta(u,v) = (p-\gamma)u - \frac{(p-\gamma)v}{(p-\gamma)u+\gamma-p+b+1} \tag{2.5}$$

with $u = h(z) = u_1 + iu_2$, $v = zh'(z) = v_1 + iv_2$ and $b = b_1 + ib_2$. Then

(i) $\theta(u,v)$ is continuous in $D = \left(\mathbb{C}\backslash\left\{\frac{\gamma-p+b+1}{\gamma-p}\right\}\right) \times \mathbb{C}$;

(ii) $(1,0) \in D$ with $\{\theta(1,0)\} = p - \gamma > 0$;

(iii) for all $(iu_2, v_1) \in D$ such that $v_1 \leq -\frac{1}{2}\left(1+u_2^2\right)$ we have

$$\begin{aligned} \Re\{\theta(iu_2,v_1)\} &= \Re\left\{\frac{(p-\gamma)v}{(p-\gamma)iu_2+\gamma-p+b+1}\right\} \\ &= \frac{(p-\gamma)\left[\gamma-p+b_1+1\right]v_1}{((p-\gamma)u_2+b_2)^2+(\gamma-p+b_1+1)^2} \\ &\leq -\frac{(p-\gamma)(1+u_2^2)(\gamma-p+b_1+1)}{2\left(\left[(p-\gamma)u_2+b_2\right]^2+(\gamma-p+b_1+1)^2\right)} \\ &< 0 \end{aligned} \tag{2.6}$$

which shows that $\theta(u,v)$ satisfies the hypotheses of Lemma 1. Consequently, we have, $f(z) \in S^*_{p,s+1,b}(\gamma)$. This completes the proof of Theorem 1.

Theorem 3. *$K_{p,s,b}(\beta,\gamma) \subset K_{p,s+1,b}(\beta,\gamma)$ for $s, b \in \mathbb{C}$ and b satisfying $\Re\{b+(p-\gamma)H(z)\} > p-\gamma-1$ and $\Re\{H(z)\} > 0$ $(z \in U)$.*

Proof. Let $f(z) \in K_{p,s,b}(\beta,\gamma)$. Then there exists a function $g(z) \in S^*_p(\gamma)$ such that

$$\Re\left(\frac{z(J_{s,b}f(z))'}{g(z)}\right) > \beta \quad (z \in U)\,. \tag{2.7}$$

We put

$$J_{s,b}k(z) = g(z)\ ,$$

so that we have

$$\Re\left(\frac{z(J_{s,b}f(z))'}{J_{s,b}k(z)}\right) > \beta \quad (z \in U)\,.$$

We next put

$$\frac{z(J_{s+1,b}f(z))'}{J_{s+1,b}k(z)}=\beta+(p-\beta)h(z)\,, \tag{2.7}$$

where $h(z)=1+c_1z+c_2z^2+....$ Thus, by using the identity (2.2), we obtain

$$\begin{aligned}\frac{z(J_{s,b}f(z))'}{J_{s,b}k(z)} &= \frac{(J_{s,b}(zf'(z))}{J_{s,b}k(z)}\\ &= \frac{z\left[J_{s+1,b}(zf'(z))\right]'-(p-1-b)J_{s+1,b}(zf'(z))}{z(J_{s+1,b}k(z))'-(p-1-b)J_{s+1,b}k(z)}\\ &= \frac{\frac{z\left[J_{s+1,b}(zf'(z))\right]'}{J_{s+1,b}k(z)}-(p-1-b)\frac{J_{s+1,b}(zf'(z))}{J_{s+1,b}k(z)}}{\frac{z(J_{s+1,b}k(z))'}{J_{s+1,b}k(z)}-(p-1-b)}\,. \end{aligned} \tag{2.8}$$

Since $k(z)\in S^*_{p,s,b}(\gamma)$ then, by using Theorem 1, we can put

$$\frac{z(J_{s+1,b}k(z))'}{J_{s+1,b}k(z)}=\gamma+(p-\gamma)H(z)\,,$$

where,
$H(z)=h_1(x,y)+ih_2(x,y)$ and $\Re((H(z))=h_1(x,y)>0 \quad (z\in U)$.
Then

$$\frac{z(J_{s,b}f(z))'}{J_{s,b}k(z)}=\frac{\frac{z\left[J_{s+1,b}(zf'(z))\right]'}{J_{s+1,b}k(z)}-(p-1-b)\left[\beta+(p-\beta)h(z)\right]}{(p-\gamma)H(z)+\gamma-(p-1-b)}\,. \tag{2.9}$$

We thus find from (2.8) that

$$z(J_{s+1,b}f(z))'=J_{s+1,b}k(z)\left[\beta+(p-\beta)h(z)\right]\,. \tag{2.10}$$

Differentiating both sides of (2.11) with respect to z, and multiplying by z, we obtain

$$\frac{z\left[J_{s+1,b}(zf'(z))\right]'}{J_{s+1,b}k(z)}=(p-\beta)zh'(z)+\left[\beta+(p-\beta)h(z)\right]\left[\gamma+(p-\gamma)H(z)\right]\,. \tag{2.11}$$

By substituting (2.12) into (2.10), we have

$$\frac{z(J_{s,b}f(z))'}{J_{s,b}k(z)}-\beta=\left\{(p-\beta)h(z)+\frac{(p-\beta)zh'(z)}{(p-\gamma)H(z)+\gamma-(p-1-b)}\right\}\,.$$

Taking $u = h(z) = u_1 + iu$, $v = zh^{'}(z) = v_1 + iv_2$ and $b = b_1 + ib_2$, we define the function $\Phi(u, v)$ by

$$\Phi(u,v) = (p-\beta)u + \frac{(p-\beta)v}{(p-\gamma)H(z) + \gamma - (p-1-b)}\,, \tag{2.12}$$

where $(u, v) \in D = \mathbb{C} \times \mathbb{C}$ and
Then it follows from (2.13) that
(i) $\Phi(u, v)$ is continuous in D;
(ii) $(1, 0) \in D$ and $\Re\{\Phi(1, 0)\} = p - \beta > 0$;
(iii) for all $(iu_2, v_1) \in D$ such that $v_1 \le -\frac{1}{2}(1 + u_2^2)$, we have

$$\begin{aligned}
\Re\{\Phi(iu_2, v_1)\} &= \Re\left\{\frac{(p-\beta)v}{(p-\gamma)H(z)+\gamma-(p-1-b)}\right\} \\
&= \frac{(p-\beta)v_1\left[(p-\gamma)h_1(x,y)+\gamma-(p-1-b_1)\right]}{\left[(p-\gamma)h_1(x,y)+\gamma-(p-1-b_1)\right]^2+\left[(p-\gamma)h_2(x,y)+b_2\right]^2} \\
&\le -\frac{(p-\beta)(1+u_2^2)\left[(p-\gamma)h_1(x,y)+\gamma-(p-1-b_1)\right]}{2\left(\left[(p-\gamma)h_1(x,y)+\gamma-(p-1-b_1)\right]^2+\left[(p-\gamma)h_2(x,y)+b_2\right]^2\right)} \\
&< 0\,,
\end{aligned}$$

which shows that $\Phi(u, v)$ satisfies the hypotheses of Lemma 1. Consequently, we have, $f(z) \in K_{p,s+1,b}(\gamma)$. This completes the proof of Theorem 2.

References

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