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SUFFICIENT CONDITIONS FOR HAMILTONIANCITY OF CERTAIN SPECIAL GRAPHS

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Abstract. In this article, Ore-type conditions for certain class of graphs, G, to be Hamiltonian are established. It involves partitioning vertex set $V(G)$ of G into two subvertices, with specific conditions on the degrees of their of vertices such that for several distance-2 vertices $v, u \in V(G)$, $d(v) + d(u)$ can be much less than the order of G, particularly as $|V(G)| \to \infty$.

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1. Preliminaries

Here we present the existing results and some definitions needed in the work.

Theorem 1. *(Dirac [1]): If G is a simple graph with n vertices, where $n \geq 3$ and $\delta(G) \geq \frac{n}{2}$, then G is Hamiltonian.*

This result by Dirac was improved by Ore in the next result

Theorem 2. *(Ore [4]). Let G be a simple graph with n vertices and u, v be distinct nonadjacent vertices of G with $d(u) + d(v) \geq n$, then G is Hamiltonian.*

More recently, Li et, al.[3] presented a result that improved Ore's result for certain graphs.

Theorem 3. *Let G be a 2-connected graph with $n \geq 3$ vertices. If $d(u)+d(v) \geq n-1$ for every pair of vertices u and v with $d(u, v) = 2$, then G is Hamiltonian unless n is odd and $G \in L_n$*

For the definition of L_n, see [3].
We define $[a, b]$ as the set of integers $\{a, a + 1, a + 2, ..., b\}$

2. Main Results

We begin with the next lemma.

Lemma 4. *Let G be a simple connected graph with $|V(G)| \leq \infty$. If G is a 2-regular graph, then G is a cycle.*

Proof. Let $n \geq 3$ be a positive integer and $|V(G)| = n$. For $u_0, u_1 \in V(G)$, let $u_0u_1 \in E(G)$. Since G is a simple and 2-regular, then there exists $u_2 \in V(G)$ such that $u_2 \neq u_0$, such that $u_1u_2 \in E(G)$. Since G is connected and 2-regular, and with an iteration based on the last statement, there exist a path $P_n = v_0v_1v_2...v_{n-1}$ in G and it consists of all the vertices in G. Now, for all $v_i \in V(P_n), i \neq 0, n-1$, $d(v_i) = 2$. Now, since G is 2-regular then $u_0u_i \notin E(G)$ for all $i \neq 1, n-1$. Since $u_{n-2}u_{n-1} \in E(G)$, then $u_0u_{n-1} \in E(G)$ and thus, G is a cycle.

Using the lemma, we obtain the main results:

Theorem 5. *Let G be a simple connected graph with $|V(G)| \geq 3$, and $|V(G)| \equiv 0 \mod 3$. suppose $V(G)$ is partitioned into V and U with $|U| = \frac{|V|}{2}$ for each $u_i \in U$, $V \subseteq N_G(u_i)$. Suppose further that for $V = u_0, u_1, ..., u_{n+1}$, there exist a $E(V) = \{u_0u_1, u_2u_3, ..., u_{m-2}u_{m-1}\} \subset E(G)$, such that $|E(V)| = |U|$, then G is Hamiltonian.*

Proof. From the hypothesis, $d(u_i) \geq |V|$ for all $u_i \in U$ since $N_G(u_i) = |V|$. Therefore each $u_i \in U$ is incident to all $e_i \in E(V)$. Thus, suppose $U = \{u_0, u_1, ..., u_{n-1}\}$ and $V = \{v_0, v_1, ..., v_{m-1}\}$. For each $u_i \in U, i \in [1, n-2]$, let $u_iv_{2i-1} \in E(G)$ and also $u_iv_{2(i+1)} \in E(G)$. Likewise, for some $u_0 \in U$, $u_0v_0, u_0v_2 \in E(G)$ and $u_{n-1}v_{m-3}, u_{n-1}v_{m-1} \in E(G)$. Thus each vertex on every member of $E(V)$ is incident to some vertex in U and suppose every other edge in G is deleted, the resultant graph say, G', remains connected and for every $v \in V(G'), d(v) = 2$. Thus by Lemma 4, G' is a spanning cycle of G and hence, G is Hamiltonian.

Since $|V(G)| \equiv 0 \mod 3$ in Theorem 5 above, it is easy to see that $|V(G)| - |U|$ is even. Thus, the vertices in V can be paired into edges in $E(V)$. The next results take care of situations that are different.

Theorem 6. *Let G be a connected graph of order $|V(G)|$ with $|V(G)| \equiv 1 \mod 3$ and let $V(G)$ be partitioned into U and V with $|U| = \left\lfloor \frac{|V(G)|}{3} \right\rfloor$ with $V \subseteq N_G(u_i)$ for all $u_i \in U$. Suppose there exists a path $P_3 \subset G$ such that $V(P_3) \subseteq V$, and suppose V' is defined as $V' = \{v_0, v_1, ..., v_{k-1}\} = V \backslash V(P_3)$. If for all $v_i \in V'$, there exists $E(V') = \{v_0v_1, v_2v_3...v_{k-2}v_{k-1}\} \subset E(G)$, then G is Hamiltonian.*

We should note that since for any positive integer p, $|V(G)| = 3p + 1$ then, $|U| = \left\lfloor \frac{|V(G)|}{3} \right\rfloor = p$. Therefore $|V(G)| - |U|$ is odd. However, $|V(P_3)| = 3$ and since $V' = V \backslash V(P_3)$, then k is even and thus members of V' can be paired.

Now we proceed to proof Theorem 6.

Proof. It is easy to see from the hypothesis that $|V'| = |U| - 1$. Now, suppose that path $P_3 = v_i v_{i+1} v_{i+2}$, where $\{v_{i+j}\}_{j=0}^{2} \subseteq V$ is a set of arbitrary vertices in V. Obviously, since $U \subset N_G(v_{i+1})$, $d(v_{i+1}) \geq 2 + |U|$. Let $E(v_{i+1})$ be the set of all edges associated with v_{i+1} and let $E'(v_{i+1}) = E(v_{i+1} \backslash \{v_i v_{i+1}, v_{i+1} v_{i+2}\})$. Now suppose we delete $E'(v_{i+1})$ then $d(v_{i+1}) = 2$. Thus, if there exists a spanning cycle $C_{|V(G)|}$ in $G \backslash E'(v_{i+1})$, then $P_3 \subseteq C_{|V(G)|}$. Thus, we 'shunt' P_3 into edge $v_i v_{i+2}$ such that $E(V') \cup v_i v_{i+2} = E(V")$. Clearly, $|E(V")| = |U|$. Thus the claim follows from Lemma 4 and Theorem 5.

It should be noted, however, that there is an interesting relationship between the length of the path and the order of $|U|$ in 6. This is expressed in the following corollary.

Corollary 7. *Let G be as in 6. If $|U|$ is reduced to $|U| - r$ as $r \to |U| - 2$, and path P_3 extends to P_{3+r} also $r \to |U| - 2$, then G is Hamiltonian. Furthermore, if $|U| = 2$, then G is a cycle.*

In the next theorem, we consider the second situation where $|V(G)| \equiv 2 \mod 3$.

Theorem 8. *Let G be a connected graph of order $|V(G)|$ with $|V(G)| \equiv 2 \mod 3$ and let $V(G)$ be partitioned into U and V with $|U| = \left\lceil \frac{|V(G)|}{3} \right\rceil$ with $V \subseteq N_G(u_i)$ for all $u_i \in U$. Suppose that, except for some $v_k \in V$, for all v_i in $V' = \{v_0, v_1, ..., v_{m-1}\} = V \backslash v_k$, there exist $E(V') = \{v_0 v_1, v_2 v_3, ..., v_{m-2} v_{m-1}\} \subset E(G)$. Then G is Hamiltonian.*

Clearly, for any positive integer q, $|V(G)| = 3q + 2$ and thus, $\left\lceil \frac{|V(G)|}{3} \right\rceil = q + 1$. Therefore, $|V(G)| - |U|$ is odd and thus, $|V'| = |V \backslash v_k|$ is even. By this then, the members of V' can be paired to form $E(V')$.

Proof. It is easily verifiable that $|E(V')| = |U| - 1$. Now, let $v_k \in V$ such that there is no such vertex $v_j \in U$ such that $v_k v_j \in E(G)$. Since $V \subseteq N_G(u_i)$ for all $u_i \in U$, then there exist $u_a, u_b \in U$ such that $u_a v_k u_b$ form a path $P_3 \in G$. Clearly of all the $q + 1$ vertices in U, two vertices u_a, u_b are already incident to v_k. Now, we can imagine 'fusing' v_a, v_b into a single vertex $u_{ab} \in U$ and therefore for the new U, say, U' $|U'| = q$. So for $E(V')$, with $|E(V')| = q$. The claim follows directly from Lemma 4 and Theorem 5.

3. Implication of the Results

It is clear from Theorem 5 that for every pair $v_1, v_2 \in V$, $d(v_1, v_2) \leq 2$ and in many cases, equality holds. Since $U \subset N_G(v_i)$, $v_i \in V$, and $v_i, v_{i+1} \in E(V)$, then $d(v_i) \geq \frac{|V(G)|}{3}+1$. Thus $d(v')+d(v'') \geq \frac{2}{3}(|V(G)|+1)$ for cases where $d(v', v'' = 2)$. Likewise, in Theorems 6 and 8, $d(v')+d(v'') \geq \frac{2}{3}(|V(G)|+2)$ and $d(v')+d(v'') \geq \frac{2}{3}(|V(G)|+4)$ respectively. This is a significant improvement over the result in Theorem 2 for this class of graphs. It is especially obvious as the order of G increases.

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